Some criteria for higher order Turán inequalities in the spirit of Mařík's theorem

Bo Wang **Candice X.T. Zhang**

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China

Email: *bowang@nankai.edu.cn* *zhang_xutong@mail.nankai.edu.cn*

Abstract. By using Mařík's theorem and the theory of multiplier sequences, we give some sufficient conditions for proving the higher order Turán inequalities for nonnegative sequences. As an immediate consequence, we present a simple proof of the higher order Turán inequalities for the Boros-Moll sequence, which were recently established by Jeremy Guo and then reproved by James Zhao.

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1 Introduction

Recently, the higher order Turán inequalities for real sequences have been widely studied; see Chen, Jia and Wang [6], Guo [15], Hou and Li [17], Wang [31] and references therein. Recall that a sequence $\{a_k\}_{k=0}^{\infty}$ of real numbers is said to satisfy the higher order Turán inequalities if for $k \geq 1$,

$$4\left(a_k^2 - a_{k-1}a_{k+1}\right)\left(a_{k+1}^2 - a_k a_{k+2}\right) - \left(a_k a_{k+1} - a_{k-1}a_{k+2}\right)^2 \geq 0.$$

The main objective of this paper is to provide some sufficient conditions to prove the higher order Turán inequalities for nonnegative sequences.

A basic tool to prove the higher order Turán inequalities is Mařík's theorem [24], which can be stated as follows.

Theorem 1.1 ([24])**.** *If the real polynomial*

$$f(x) = \sum_{k=0}^{n} \frac{a_k}{k!(n-k)!} x^k$$

of degree $n \geq 3$ has only real zeros, then the sequence $\{a_k\}_{k=0}^{n}$ satisfies the higher order Turán inequalities.

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The above theorem bulids a connection between the real-rootedness of a polynomial and the higher order Turán inequalities for its coefficient sequence.

An interesting application of Theorem 1.1 is to prove the higher order Turán inequalities for the Maclaurin coefficients of a real entire function in the Laguerre-Pólya class; see Dimitrov [12]. Recall that a real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$$

is said to belong to the Laguerre-Pólya class, denoted by $\psi(x) \in \mathcal{LP}$, if

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

where c, β, x_k are real numbers, $\alpha \geq 0$, m is a nonnegative integer and $\sum x_k^{-2} < +\infty$. Those functions in the $\mathcal{LP}$ class have many essential properties. For more information, see [27, 26, 21, 9, 11, 29]. Jensen [18] showed that $\psi(x)$ belongs to the $\mathcal{LP}$ class if and only if for any positive integer m, the m-th associated Jensen polynomial

$$J_m(x) = \sum_{k=0}^{m} \binom{m}{k} \gamma_k x^k$$

has only real zeros. Thus, Jensen's theorem and Mařík's theorem imply that if $\psi(x) \in \mathcal{LP}$, then $\{\gamma_k\}_{k=0}^n$ for any $n \geq 1$, and hence $\{\gamma_k\}_{k\geq 0}$, satisfies the higher order Turán inequalities.

As noted by Dimitrov and Lucas [13], the main interest for the $\mathcal{LP}$ class is the fact that it is closely related to the celebrated Riemann hypothesis. Recall that the Riemann Xi function is defined by

$$\Xi(z) = \frac{1}{2}\left(-z^2 - \frac{1}{4}\right) \pi^{\frac{iz}{2} - \frac{1}{4}} \Gamma\left(-\frac{iz}{2} + \frac{1}{4}\right) \zeta\left(-iz + \frac{1}{2}\right),$$

where $\zeta(z)$ is the Riemann ζ-function and $\Gamma(z)$ is the gamma function. It is known that [28] the Riemman hypothesis is equivalent to $\Xi(z) \in \mathcal{LP}$, or equivalently, for any nonnegative integers m and n, the Jensen polynomial of degree m and shift n associated to the Maclaurin coefficients $\{\hat{\gamma}_k\}_{k=0}^{\infty}$ of $\frac{1}{8}\xi\left(\frac{i\sqrt{x}}{2}\right)$, which is defined by

$$\hat{J}_{m,n}(x) = \sum_{k=0}^{m} \binom{m}{k} \hat{\gamma}_{n+k} x^k,$$

has only real zeros. Though the Riemman hypothesis is widely open, some progress has been made. Dimitrov and Lucas [13] proved that $\{\hat{\gamma}_k\}_{k=0}^{\infty}$ satisfies the higher order Turán inequalities, or equivalently, the polynomial $\hat{J}_{3,n}(x)$ has only real zeros. Griffin, Ono, Rolen, and Zagier [14] proved that for each $m \geq 1$, $\hat{J}_{m,n}(x)$ has only real zeros for sufficiently large n.

It is worth mentioning that the work of Griffin, Ono, Rolen, and Zagier [14] was motivated by the study of the higher order Turán inequalities for the partition function in [6]. Chen, Jia and Wang [6] confirmed a conjecture of Chen [4], which illustrates that for $n \geq 95$, the partition function $p(n)$ satisfy the higher order Turán inequalities. Chen, Jia and Wang [6] further conjectured that for every integer $m \geq 1$, there exists a positive integer $N(m)$ such that for any $n \geq N(m)$ the Jensen polynomial of degree m and shift n associated to the partition function $p(n)$, which is defined by

$$J_{m,n}(x) = \sum_{k=0}^{m} \binom{m}{k} p(n+k)x^k,$$

has only real zeros. In [14], Griffin, Ono, Rolen and Zagier confirmed this conjecture for all $m \geq 1$. Furthermore, Larson and Wagner [20] gave the precise values $N(3) = 94$, $N(4) = 206$ and $N(5) = 381$, as well as the optimal upper bound $N(m) \leq (3m)^{24m}(50m)^{3d^2}$.

Except for the partition function, the higher order Turán inequalities for other sequences of combinatorial significance were also studied. Wang [31] gave a unified approach to proving the higher order Turán inequalities for the sequence $\{a_n/n!\}_{n\geq 0}$ where a_n are the Motzkin numbers, the fine numbers, the Franel numbers of order 3 and the Domb numbers. Hou and Li [17] gave a sufficient condition for determining the higher order Turán inequalities asymptotically, and they further applied their criterion to a P-recursive sequence $\{a_n\}_{n\geq 0}$ and gave a method to find a lower bound N such that $\{a_n\}_{n\geq N}$ satisfies the higher order Turán inequalities.

In the past two decades, the Boros-Moll sequence was extensively studied; for instance see [25, 19, 7, 3, 5, 8]. The higher order Turán inequalities for the Boros-Moll sequence was first proved by Guo [15], and then reproved by Zhao [32] by using Hou and Li's sufficient condition [17]. We would like to point out that Mařík's theorem was not used in Zhao's proof. In Guo's proof, Mařík's theorem was used to prove the higher order Turán inequalities of a variation of the Boros-Moll sequence, but not directly for the original Boros-Moll sequence. Both proofs rely on some techniques for proving inequalities. It is certainly desirable to derive the higher order Turán inequalities for the Boros-Moll sequence directly from Mařík's theorem. This motivated us to look for some criteria for higher order Turán inequalities in the spirit of Mařík's theorem.

Note that, to prove the higher order Turán inequalities for a finite sequence $\{a_k\}_{k=0}^{n}$ by using Mařík's theorem, the key is to prove the real-rootedness of $f(x) = \sum_{k=0}^{n} \frac{a_k}{k!(n-k)!} x^k$, which could be very difficult sometimes. However, in some cases this difficulty can be overcome by proving the real-rootedness of its certain variations. One choice for our purpose is the polynomial $\sum_{k=0}^{n} \frac{a_k}{k!} x^k$, and the other is $\sum_{k=0}^{n} \frac{a_k}{(n-k)!} x^k$. The main result of this paper is as follows.

Theorem 1.2. *Let $n \geq 3$, and $\{a_k\}_{k=0}^{n}$ be a sequence of positive integers. If one of the following polynomials*

(1) $f_1(x) = \sum_{k=0}^{n} a_k x^k$,

(2) $f_2(x) = \sum_{k=0}^{n} \frac{a_k}{k!} x^k$,

(3) $f_3(x) = \sum_{k=0}^{n} \frac{a_k}{(n-k)!} x^k$,

has only real zeros, then the sequence $\{a_k\}_{k=0}^{n}$ satisfies the higher order Turán inequalities.

In the next section we will give a proof of the above theorem. In Section 3, we will present two applications of Theorem 1.2, one of which provides a new and simple proof of the higher order Turán inequalities for the Boros-Moll sequence.

2 Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2. The basic idea of the proof is to prove the real-rootedness of $\sum_{k=0}^{n} \frac{a_k}{k!(n-k)!} x^k$ and then to derive the desired higher order Turán inequalities from Mařík's theorem. In order to prove the real-rootedness of $\sum_{k=0}^{n} \frac{a_k}{k!(n-k)!} x^k$, we need the theory of multiplier sequences.

Let us now recall some definitions and results on multiplier sequences. A sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is a *multiplier sequence* if whenever any real polynomial

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

has only real zeros, so does the polynomial

$$\Gamma[f(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k.$$

We first recall two fundamental results in the theory of distribution of zeros of polynomials, which produce several classical and important multiplier sequences.

Theorem 2.1 (Laguerre's Theorem [26]). *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitary real polynomial of degree n, and let $\psi(x) \in \mathcal{LP}$. Suppose that none of the zeros of ψ lie in the interval $(0, n)$. Then we have*

$$Z_c\left(\sum_{k=0}^{n} \psi(k) a_k x^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k x^k\right) = Z_c\left(f(x)\right),$$

where $Z_c(p(x))$ denotes the number of non-real zeros, counting multiplicities, of a given polynomial $p(x)$.

Theorem 2.2 (The Malo-Schur Composition Theorem [26, 30]). *Let*

$$f(x) = \sum_{k=0}^{m} a_k x^k \text{ and } g(x) = \sum_{k=0}^{n} b_k x^k$$

be polynomials with only real zeros. Besides, the zeros of polynomial $g(x)$ are of the same sign. Then the polynomials

$$\sum_{k=0}^{t} k! a_k b_k x^k \text{ and } \sum_{k=0}^{t} a_k b_k x^k$$

have only real zeros, where $t = \min\{m, n\}$.

The following result is a consequence of the Malo-Schur Composition Theorem and Laguerre's Theorem, and its proof was also hinted in [10] immediately after stating these two theorems. Since this result is critical for our main theorem, we include the proof here for the sake of completeness.

Lemma 2.3 ([10]). *For any positive n, the sequences*

$$\left\{\frac{1}{k!}\right\}_{k=0}^{\infty}, \left\{\frac{1}{(n-k)!}\right\}_{k=0}^{\infty} \text{ and } \left\{\frac{1}{k!(n-k)!}\right\}_{k=0}^{\infty}$$

are multiplier sequences, where we set $\frac{1}{(n-k)!} = \frac{1}{k!(n-k)!} = 0$ for $k > n$.

Proof. Apply Theorem 2.1 to the function $\psi(x) = \frac{1}{\Gamma(x+1)}$, which belongs to the $\mathcal{LP}$ class and has non-positive zeros. We see that the sequence $\{1/k!\}_{k=0}^{\infty}$ will not increase the number of non-real zeros when applied to any real polynomials, and therefore it is a multiplier sequence.

For the remaining two sequences, set $g(x) = (x+1)^n$ in Theorem 2.2. Then both

$$\sum_{k=0}^{m} k! \binom{n}{k} a_k x^k \text{ and } \sum_{k=0}^{m} \binom{n}{k} a_k x^k$$

have only real zeros, for any positive integer m. By definition, we obtain that the sequences

$$\left\{\frac{1}{(n-k)!}\right\}_{k=0}^{\infty}, \text{ and } \left\{\frac{1}{k!(n-k)!}\right\}_{k=0}^{\infty}$$

are multiplier sequences. This completes the proof. ∎

Now we are in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Let

$$f(x)=\sum_{k=0}^{n}\frac{a_k}{k!(n-k)!}x^k.$$

Note that

$$f(x)=\Gamma_1[f_1(x)]=\Gamma_2[f_2(x)]=\Gamma_3[f_3(x)],$$

where

$$\Gamma_1=\left\{\frac{1}{k!(n-k)!}\right\}_{k=0}^{\infty},\ \Gamma_2=\left\{\frac{1}{(n-k)!}\right\}_{k=0}^{\infty} \text{ and } \Gamma_3=\left\{\frac{1}{k!}\right\}_{k=0}^{\infty}.$$

By Lemma 2.3, if one of $f_1(x)$, $f_2(x)$ and $f_3(x)$ has only real zeros, so does $f(x)$. Then by Theorem 1.1 we get the desired result. This completes the proof. ∎

Theorem 1.2 enables us to give more sufficient conditions for proving higher order Turán inequalities. We would like to point out that these conditions might be easier to verify than those conditions in Theorem 1.2, though the former are stronger than the latter. By using (1) and (2) of Theorem 1.2, we obtain the following result.

Corollary 2.4. *Let $n\geq 3$, and $\{a_k\}_{k=0}^n$ be a sequence of positive integers. If one of the following polynomials:*

(1) $h_1(x)=\sum_{k=0}^n \frac{a_k}{(k+1)\cdots(k+m)}x^k$, for some $m\geq 1$,

(2) $h_2(x)=\sum_{k=0}^n \frac{a_k}{(k+m)!}x^k$, for some $m\geq 1$,

has only real zeros, then the sequence $\{a_k\}_{k=0}^n$ satisfies the higher order Turán inequalities.

Proof. We may first assume that $h_1(x)$ has only real zeros. Let

$$g(x)=x^m h_1(x)=\sum_{k=0}^{n}\frac{a_k}{(k+1)\cdots(k+m)}x^{k+m}.$$

It is clear that $g(x)$ has only real zeros. Since the derivative operator preserves real-rootedness of polynomials, the m-th derivative of $g(x)$,

$$g^{(m)}(x)=\sum_{k=0}^{n}a_k x^k,$$

has only real zeros. Then by (1) of Theorem 1.2, the sequence $\{a_k\}_{k=0}^n$ satisfies the higher order Turán inequalities.

Now assume that $h_2(x)$ has only real zeros. We set

$$p(x) = x^m h_2(x) = \sum_{k=0}^{n} \frac{a_k}{(k+m)!} x^{k+m}.$$

Then the m-th derivative of $p(x)$,

$$p^{(m)}(x) = \sum_{k=0}^{n} \frac{a_k}{k!} x^k,$$

has only real zeros. Then by (2) of Theorem 1.2, we obtain the desired result. ∎

Finally, we give another application of (2) of Theorem 1.2.

Corollary 2.5. *Let $n \geq 3$, and $\{a_k\}_{k=0}^n$ be a sequence of positive integers. If for $1 \leq m \leq n-3$ the polynomial*

$$h_3(x) = \sum_{k=m}^{n} \frac{a_k}{(k-m)!} x^k$$

has only real zeros, then the sequence $\{a_k\}_{k=m}^n$ satisfies the higher order Turán inequalities.

Proof. Since $h_3(x)$ can be written in the form of

$$x^m \sum_{k=0}^{n-m} \frac{a_{k+m}}{k!} x^k,$$

and the polynomial

$$\sum_{k=0}^{n-m} \frac{a_{k+m}}{k!} x^k$$

has only real zeros for $m \leq n$. Then by (2) of Theorem 1.2, the sequence $\{a_{k+m}\}_{k=0}^{n-m}$ satisfies the higher order Turán inequalities. ∎

3 Applications

In this section, we give two applications of Theorem 1.2. Note that, to prove the higher order Turán inequalities for a real sequence $\{a_k\}_{k=0}^n$, it is sufficient to prove the real-rootedness of $f(x) = \sum_{k=0}^n \frac{a_k}{k!(n-k)!} x^k$ by Mařík's theorem. However, sometimes it is much easier to establish the real-rootedness of $f_1(x)$, $f_2(x)$ or $f_3(x)$ in Theorem 1.2 or the real-rootedness of $h_1(x)$, $h_2(x)$ or $h_3(x)$ in Corollary 2.4 and Corollary 2.5. In general it is a subtle question which polynomial should be used for a specific sequence. As one can see below, it is natural to choose $f_2(x)$ for the Boros-Moll sequence, while it is more appropriate to choose $f_3(x)$ for a sequence involving Stirling numbers of the second kind.

3.1 Boros-Moll sequences

The Boros-Moll sequence $\{d_k(n)\}_{k=0}^n$ was introduced in [2], and defined by

$$d_k(n) = 2^{-2n} \sum_{i=k}^{n} 2^i \binom{2n-2i}{n-i} \binom{n+i}{n} \binom{i}{k}.$$

As mentioned in the introduction, Guo [15] obtained the following result.

Theorem 3.1 ([15]). *The Boros-Moll sequence $\{d_k(n)\}_{k=0}^n$ satisfies the higher order Turán inequalities.*

His proof is based on the result that the sequence $\{(n-k)!d_k(n)\}_{k=0}^n$ satisfies the higher order Turán inequalities and this result is an immediate consequence of Mařík's theorem and the real-rootedness of

$$Q_n(x) = \sum_{k=0}^{n} \frac{d_k(n)}{k!} x^k.$$

Note that the real-rootedness of $Q_n(x)$ was first conjectured by Brändén [3], and then was proved by Chen, Dou and Yang [5] by establishing the recurrence relation

$$\begin{aligned} Q_{n+1}(x) = &\left(\frac{(2n+1)x}{(n+1)^2} + \frac{8n^2+8n+3}{2(n+1)^2} \right) Q_n(x) \\ &- \frac{(4n-1)(4n+1)}{4(n+1)^2} Q_{n-1}(x) + \frac{x}{(n+1)^2} Q_n'(x), \quad \text{for } n \geq 1 \end{aligned}$$

and using the real-rootedness criterion due to Liu and Wang [22]. In fact, Chen, Dou and Yang [5] also proved that

$$R_n(x) = \sum_{k=0}^{n} \frac{d_k(n)}{(k+2)!} x^k$$

has only real zeros for any n by proving the following recurrence relation

$$\begin{aligned} R_{n+1}(x) = &\left(\frac{(2n+1)x}{(n+1)(n+3)} + \frac{8n^2+8n+7}{2(n+1)(n+3)} \right) R_n(x) \\ &- \frac{(4n-1)(4n+1)(n-2)}{4n(n+1)(n+3)} R_{n-1}(x) + \frac{5x}{(n+1)(n+3)} R_n'(x), \quad \text{for } n \geq 1. \end{aligned}$$

Keeping in mind of Mařík's theorem, it is natural to consider whether

$$T_n(x) = \sum_{k=0}^{n} \frac{d_k(n)}{k!(n-k)!} x^k$$

has only real zeros for any n. However, it seems that $T_n(x)$ does not satisfy a simple recurrence relation as $Q_n(x)$ or $R_n(x)$. Here we give a new proof of Theorem 3.1 by using Theorem 1.2.

Proof of Theorem 3.1. Combining (2) of Theorem 1.2 with the fact that the polynomial $Q_n(x)$ has only real zeros for any $n \geq 0$, we can obtain the desired result. ∎

Remark 3.2. *By (2) of Corollary 2.4, we can also prove Theorem 3.1 by using the real-rootedness of $R_n(x)$. However, it seems impossible to prove Theorem 3.1 by using (1) and (3) of Theorem 1.2. Boros and Moll [1] proved that the polynomial*

$$\sum_{k=0}^{n} d_k(n)x^k$$

has no real zeros for n even and a single real zero for n odd. We check that the polynomial

$$\sum_{k=0}^{n} \frac{d_k(n)}{(n-k)!}x^k$$

has non-real zeros for $6 \leq n \leq 100$, and conjecture that it has non-real zeros for any $n \geq 6$.

3.2 Stirling numbers of the second kind

In this subsection we aim to use the Stirling numbers of the second kind to construct a sequence $\{a_k\}_{k=0}^{n}$ satisfying the higher order Turán inequalities such that the corresponding polynomial $f_3(x)$ in Theorem 1.2 has only real zeros.

Recall that the Stirling number of the second kind, denoted by $S(n,k)$, satisfies the following recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k),$$

for $1 \leq k \leq n$. Equivalently, the associated generating function $B_n(x) = \sum_{k=0}^{n} S(n,k)x^k$, called the n-th Bell polynomial, satisfies

$$B_n(x) = xB_{n-1}(x) + xB'_{n-1}(x)$$

with $B_0(x) = 1$. By using the above recurrence relation, Harper [16] first proved that the Bell polynomial $B_n(x)$ has only real zeros for each $n \geq 1$. As an application, Ma and Wang [23] obtained the following result.

Theorem 3.3 ([23]). *For any $n \geq 1$, the polynomial*

$$S_n(x) = \sum_{k=0}^{n} (k+1)! S(n+1, k+1) x^k$$

has only real zeros, which are all in the interval $[-1, 0]$.

In fact, the above result can be also directly proved by using a recurrence relation of $S_n(x)$, namely,

$$S_{n+1}(x) = (1+2x)S_n(x) + x(x+1)S_n'(x).$$

Based on the above theorem, we obtain the following result.

Corollary 3.4. *For $n \geq 2$, the sequence $\left\{ \frac{S(n+1,k+1)}{\binom{n+1}{k+1}} \right\}_{k=0}^{n}$ satisfies the higher order Turán inequalities, respectively.*

In order to use Mařík's theorem to prove the above result, it is natural to consider whether

$$P_n(x) = \sum_{k=0}^{n} \frac{S(n+1, k+1)}{\binom{n+1}{k+1}} \Big/ \left(k!(n-k)!\right) x^k$$

has only real zeros for any n. Nevertheless, it seems that $P_n(x)$ does not satisfy a simple recurrence relation as $S_n(x)$. Hence we give a proof of Corollary 3.4 based on Theorem 1.2.

Proof of Corollary 3.4. In Theorem 1.2 take

$$a_k = \frac{S(n+1, k+1)}{\binom{n+1}{k+1}}. \tag{3.1}$$

Note that

$$f_3(x) = \sum_{k=0}^{n} \frac{a_k}{(n-k)!} x^k = (n+1)! S_n(x),$$

which has only real zeros by Theorem 3.3. Therefore, by (3) of Theorem 1.2, the sequence $\{a_k\}_{k=0}^{n}$ satisfies the higher order Turán inequalities, as desired. ∎

Remark 3.5. *Comparing with Remark 3.2, we note that it seems impossible to prove Corollary 3.4 by using (1) and (2) of Theorem 1.2. Letting a_k be as given by (3.1), one can check that the polynomial*

$$f_1(x) = \sum_{k=0}^{n} a_k x^k$$

has non-real zeros for $2 \leq n \leq 100$, and the polynomial

$$f_2(x) = \sum_{k=0}^{n} \frac{a_k}{k!} x^k$$

has non-real zeros for $3 \leq n \leq 100$. We conjecture that neither of these two polynomials has only real zeros for $n \geq 3$.

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