On critical graphs for the chromatic edge-stability number

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Abstract

The *chromatic edge-stability number* $es_\chi(G)$ of a graph G is the minimum number of edges whose removal results in a spanning subgraph with the chromatic number smaller than that of G. A graph G is called $(3,2)$*-critical* if $\chi(G) = 3$, $es_\chi(G) = 2$ and for any edge $e \in E(G)$, $es_\chi(G-e) < es_\chi(G)$. In this paper, we characterize $(3,2)$-critical graphs which contain at least five odd cycles. This answers a question proposed by Brešar, Klavžar and Movarraei in [Critical graphs for the chromatic edge-stability number, *Discrete Math.* **343**(2020) 111845].

Keywords: chromatic edge-stability number; critical graphs; odd cycles

1 Introduction

Let $G = (V(G), E(G))$ be a graph. A function $c : V(G) \to [k] = \{1, \dots, k\}$ is called a *proper coloring* of G, if $c(u) \neq c(v)$ for any $uv \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G admits a proper coloring using k colors. The *chromatic edge-stability number* of G, denoted by $\mathrm{es}_\chi(G)$, is the minimum number of edges

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of G whose removal results in a graph with the chromatic number smaller than that of G. The chromatic edge-stability number was first studied by Staton [7], which provided upper bounds of es_χ for regular graphs in terms of the size of a given graph. The invariant was subsequently investigated in [2, 1, 6]. For a graph G with $\chi(G) = 3$, the chromatic edge-stability number is equal to the bipartite edge frustration [5], which is defined as the smallest number of edges that have to be deleted from G to obtain a bipartite spanning subgraph.

For any $u, v \in V(G)$, let $d_G(u, v)$ denote the length of the shortest (u, v)-path. For any $A \subseteq E(G)$, let $G - A$ be the graph obtained from G by deleting all the edges in A. If $A = \{e\}$, we simply write $G-e$ instead of $G-\{e\}$. We say a graph G is *edge-stability critical* if $es_\chi(G - e) < es_\chi(G)$ holds for every edge $e \in E(G)$. A graph G is called (k, ℓ)*-critical*, if G is an edge-stability critical graph with $\chi(G) = k$ and $es_\chi(G) = \ell$, for $k, \ell \geq 2$. Naturally, a graph G is $(k, 2)$-critical if and only if for every edge $e \in E(G)$, $\chi(G - e) = k$, and there exists an edge $f \in E(G - e)$ such that $\chi(G - \{e, f\}) = k - 1$. In this paper we focus on $(3, 2)$-critical graphs and the graphs we consider are simple.

In [4], the authors proved the following theorem.

Theorem 1.1 ([4]) *$\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ is the family of $(3, 2)$-critical graphs (without isolated vertices) that contain at most four odd cycles.*

These four graph families are defined as follows. Let $G + H$ denote the disjoint union of graphs G and H. We use C_n to denote the cycle on n vertices. A path or a cycle is odd if it has an odd number of edges, otherwise, we say it even. Then the families of $(3, 2)$-critical graphs mentioned in [4] are as follows. Let $\mathcal{A} = \{C_{2k+1} + C_{2\ell+1} \mid k, \ell \geq 1\}$ and let $\mathcal{B}$ be the family of graphs that are obtained from $C_{2k+1} + C_{2\ell+1}, k, \ell \geq 1$, by identifying a vertex of C_{2k+1} and a vertex of $C_{2\ell+1}$. Let x_i, y_i be the end vertices of the paths $Q_i, i \in [4]$, exactly two of the Q_i are odd, and at most one of them is of length one. The family $\mathcal{C}$ consists of the graphs that are obtained from such four paths, by identifying the vertices x_1, x_2, x_3, and x_4 and also identifying the vertices y_1, y_2, y_3, and y_4. The family $\mathcal{D}$ consists of the following subdivisions of the graph K_4: (i) all the subdivided paths are of odd length, (ii) exactly three of the paths are odd, and these three paths induce an odd cycle or a path, (iii) exactly two of the paths are odd, and these two paths are vertex disjoint, and (iv) exactly two of the paths are even and these two paths have a common vertex.

At the end of [4], Brešar, Klavžar, and Movarraei defined the family $\mathcal{E}$, which is obtained from the disjoint union of k even cycles $C_{2n_1}, \ldots, C_{2n_k}$ as follows. For each $i \in [k]$, let x_i and y_i be any two distinct vertices of C_{2n_i}, where they only require that $\sum_{i=1}^{k} d_{C_{2n_i}}(x_i, y_i)$ is odd. A graph $G \in \mathcal{E}$ is obtained by identifying y_i and x_{i+1} for $i \in [k-1]$, and identifying y_k and x_1. They proposed the following problem and suspected it has a positive answer.

Problem 1.2 ([4]) *Is it true that* $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$ *is the family of* $(3, 2)$*-critical graphs (without isolated vertices)?*

We answer this problem by giving a positive proof.

Theorem 1.3 $\mathcal{E}$ *is the family of* $(3, 2)$*-critical graphs (without isolated vertices) which contain at least five odd cycles.*

2 Properties of $(3, 2)$-critical graphs

In this section, we establish some structural results on $(3, 2)$-critical graphs. The following lemmas and propositions were proved in [3, 4] and will be used in this paper.

Lemma 2.1 ([4]) *If* G *is a* $(3, 2)$*-critical graph that contains at least three odd cycles, then every two distinct odd cycles intersect in more than one vertex.*

Let $\mathcal{G}_i$ $(i \in [7])$ be the family of graphs as shown in Figure 1. For $i \in [7]$ and $G_i \in \mathcal{G}_i$, three internally disjoint (x, y)-paths of G_i formed by solid lines from left to right are denoted as Q_1, Q_2, Q_3. Let $D_1 = Q_1 \cup Q_2$ and $D_2 = Q_2 \cup Q_3$. Every graph in $\mathcal{G}_i$ $(i \in [7])$ satisfies that D_1 and D_2 are odd cycles, and the dotted line is internally disjoint from these solid lines. Let $\mathcal{G} = \bigcup_{i\in[7]} \mathcal{G}_i$.

Proposition 2.2 ([4]) *If* G *is a* $(3, 2)$*-critical graph that contains at least three odd cycles, then there exists an* $H \in \mathcal{G}$ *such that* $H \subseteq G$*.*

A graph G is *connected* if there is a (u, v)-path in G for any $u, v \in V(G)$. A *separation* of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. This common vertex is called a *separating vertex* of the graph. A graph is *nonseparable* if it is connected and has no separating vertices. Let F be a nontrivial proper subgraph of a graph G. An *ear* of F in G is a nontrivial path in G whose endpoints lie in F but whose internal vertices do not. An ear is an *open ear* if the endpoints of the path are distinct. For completeness, we present the proof of the following proposition from [3].

Proposition 2.3 ([3]) *Let* F *be a nontrivial proper subgraph of a nonseparable graph* G*. Then* F *has an open ear in* G*.*

Proof. If F is a spanning subgraph of G, then $E(G) \setminus E(F)$ is nonempty because, by hypothesis, F is a proper subgraph of G. Any edge in $E(G) \setminus E(F)$ is then an ear of F in G. We may suppose, therefore, that F is not spanning. Since G is connected, there is an edge xy of G with $x \in V(F)$ and $y \in V(G) \setminus V(F)$. Because G is nonseparable, $G - x$ is

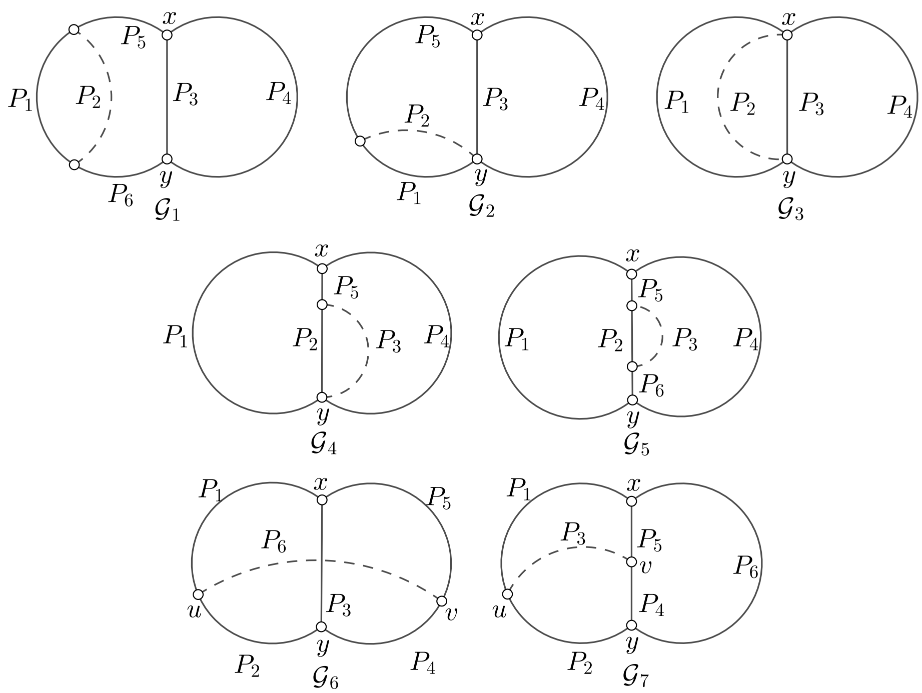

Figure 1: Seven families of subgraphs of (3,2)-critical graphs.

connected. So there is a $(y, F - x)$-path Q in $G - x$. The path $P := xyQ$ is an open ear of F. ■

We first prove the following lemma.

Lemma 2.4 *If G is a $(3,2)$-critical graph that contains at least three odd cycles (without isolated vertices), then G is nonseparable.*

Proof. We claim that if G is $(3,2)$-critical, then every edge of G is contained in at least one odd cycle. Suppose $e \in E(G)$ and e is not contained in any odd cycle. By the definition of $(3,2)$-critical graph, there exists at least one edge $f \in E(G)\backslash\{e\}$ such that $\chi(G-\{e,f\}) = 2$. Since e is not contained in any odd cycle, we have $\chi(G - f) = 2$, contradicting the fact that G is $(3,2)$-critical.

Let G be a $(3,2)$-critical graph that contains at least three odd cycles (without isolated vertices). Suppose G is not a connected graph or G contains a separating vertex v. Let $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \emptyset$ or $\{v\}$, and there is at least one edge in G_i $(i \in [2])$. By Lemma 2.1, one of G_1 and G_2 contains all odd cycles. Thus there exists at least one edge that is not contained in any odd cycle, a contradiction. Hence, G is nonseparable. ■

For an edge e_i, let $\mathcal{F}_i = \{f_i \in E(G) \mid \chi(G - \{e_i, f_i\}) = 2\}$.

Theorem 2.5 *Let G be a $(3,2)$-critical graph with at least three odd cycles. Suppose there are two odd cycles D_1 and D_2 in G satisfying the following three conditions.*

(1) *The intersection of D_1 and D_2 is a nontrivial path;*

(2) *There are two edges e_1 and e_2 in G such that $e_1 \in E(D_1) \backslash E(D_2)$ and $e_2 \notin E(D_2)$;*

(3) *$\mathcal{F}_1 \subseteq E(D_1) \cap E(D_2)$ and $\mathcal{F}_2 \cap (E(D_1) \cap E(D_2)) \neq \emptyset$.*

Then $\mathcal{F}_1 \subseteq \mathcal{F}_2$. In particular, if $e_2 \in E(D_1) \backslash E(D_2)$ and $\mathcal{F}_2 \subseteq E(D_1) \cap E(D_2)$, then $\mathcal{F}_1 = \mathcal{F}_2$.

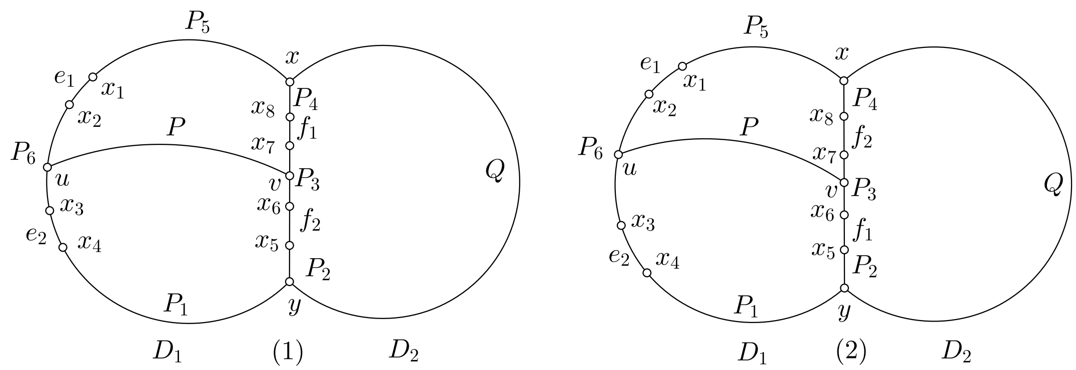

Figure 2: Subgraphs of G.

Proof. Since G is $(3,2)$-critical, $\mathcal{F}_1$ and $\mathcal{F}_2$ are non-empty. Suppose that there are two odd cycles D_1 and D_2 in G satisfying the above three conditions, but $\mathcal{F}_1 \nsubseteq \mathcal{F}_2$. Then there exists an edge $f_1 \in \mathcal{F}_1 \setminus \mathcal{F}_2$ such that $\chi(G - \{e_2, f_1\}) = 3$. This implies that there exists at least one odd cycle C which is distinct from D_1 and D_2 in G such that $e_2, f_1 \notin E(C)$, since G is a $(3,2)$-critical graph with at least three odd cycles. Moreover, we have $(\{e_1\} \cup \mathcal{F}_2) \subseteq E(C)$ since $e_2, f_1 \notin E(C)$. Next we show that this will lead to a contradiction.

Denote by x and y the two endpoints of the path which is the intersection of D_1 and D_2. Suppose $e_1 = x_1x_2$ and $e_2 = ab$. Since $\mathcal{F}_2 \cap (E(D_1) \cap E(D_2)) \neq \emptyset$, let $f_2 \in \mathcal{F}_2 \cap (E(D_1) \cap E(D_2))$. Since $\chi(G - \{e_2, f_1\}) = 3$, we have $f_2 \neq f_1$. Let x_5, x_6, x_7, x_8 be the endpoints of f_1 and f_2 of which assignment depends on their position in $D_1 \cap D_2$ (see Figure 2, where each of the two cases has a separate figure). Let P_2, P_3, P_4, P_5 be the (y, x_5)-path, (x_6, x_7)-path, (x_8, x)-path, (x, x_1)-path of D_1 and Q be the (y, x)-path of D_2 in a counter clockwise direction, respectively, as shown in Figure 2. If $e_2 \in E(D_1)$, then let $e_2 = ab = x_3x_4$ and P_1, P_6 be the (x_4, y)-path, (x_2, x_3)-path of D_1 in a counter clockwise direction, respectively. If $e_2 \notin E(D_1)$, then let $P_6 = P_1$ be the (x_2, y)-path of D_1 in a counter clockwise direction and $x_2 = x_3 = x_4$. We first prove the following claim.

Claim: Let $u \in (V(D_1) \cup V(D_2)) \setminus V(P_3)$ and $v \in V(P_3)$. Then there is no (u, v)-path P such that $V(P) \cap (V(D_1) \cup V(D_2)) = \{u, v\}$, where $P \neq f_1, f_2$.

Proof. Suppose there is a (u, v)-path P such that $V(P) \cap (V(D_1) \cup V(D_2)) = \{u, v\}$ and $P \neq f_1, f_2$. It suffices to consider the two structures as shown in Figure 2. The difference between two graphs in Figure 2 is the position relation of the three edge e_1, f_1 and f_2. In the following, we will consider the two structures simultaneously. Let M_1 and M_2 be the (x_6, v)-path and (v, x_7)-path of D_1 in a counter clockwise direction, respectively.

First, suppose $u \in V(P_1)$. Let M_3 and M_4 be the (x_4, u)-path and (u, y)-path of D_1 in a counter clockwise direction, respectively. If $|E(P \cup M_4)|$ and $|E(M_1 \cup x_5x_6 \cup P_2)|$ have different parity, then $P \cup M_4 \cup P_2 \cup x_5x_6 \cup M_1$ is an odd cycle. Otherwise $P \cup M_4 \cup Q \cup P_4 \cup x_8x_7 \cup M_2$ is an odd cycle since D_2 is an odd cycle. If $P \cup M_4 \cup P_2 \cup x_5x_6 \cup M_1$ is an odd cycle, then $\chi(G - \{e_1, f_1\}) = 3$ in Figure 2 (1) and $\chi(G - \{e_2, f_2\}) = 3$ in Figure 2 (2). If $P \cup M_4 \cup Q \cup P_4 \cup x_8x_7 \cup M_2$ is an odd cycle, then $\chi(G - \{e_2, f_2\}) = 3$ in Figure 2 (1) and $\chi(G - \{e_1, f_1\}) = 3$ in Figure 2 (2). This contradicts $\chi(G - \{e_i, f_i\}) = 2$ for $i \in [2]$. Similarly, if $u \in V(P_5)$, then we also can get a contradiction.

Now suppose $u \in V(P_2)$. Let M_5 and M_6 be the (y, u)-path and (u, x_5)-path of D_1 in a counter clockwise direction, respectively. If $|E(P)|$ and $|E(M_1 \cup x_6x_5 \cup M_6)|$ have different parity, then $P \cup M_1 \cup x_6x_5 \cup M_6$ is an odd cycle. Otherwise $M_5 \cup P \cup M_2 \cup x_7x_8 \cup P_4 \cup Q$ is an odd cycle since D_2 is an odd cycle. If $P \cup M_1 \cup x_6x_5 \cup M_6$ is an odd cycle, then $\chi(G - \{e_1, f_1\}) = 3$ in Figure 2 (1) and $\chi(G - \{e_2, f_2\}) = 3$ in Figure 2 (2). If $M_5 \cup P \cup M_2 \cup x_7x_8 \cup P_4 \cup Q$ is an odd cycle, then $\chi(G - \{e_2, f_2\}) = 3$ in Figure 2 (1) and $\chi(G - \{e_1, f_1\}) = 3$ in Figure 2 (2), a contradiction. Similarly, if $u \in V(P_4 \cup Q)$, then we also can get a contradiction.

Finally, we only need to consider the case of $e_2 \in E(D_1)$ and $u \in V(P_6)$. In this case, let M_7 and M_8 be the (x_2, u)-path and (u, x_3)-path of D_1 in a counter clockwise direction, respectively. In Figure 2 (1), since D_1 is an odd cycle, either $P \cup M_7 \cup e_1 \cup P_5 \cup P_4 \cup f_1 \cup M_2$ or $P \cup M_8 \cup e_2 \cup P_1 \cup P_2 \cup f_2 \cup M_1$ is an odd cycle. Then $\chi(G - \{e_2, f_2\}) = 3$ or $\chi(G - \{e_1, f_1\}) = 3$, a contradiction. In Figure 2 (2), since D_1 and D_2 are odd cycles, we have either $|E(P \cup M_8 \cup e_2 \cup P_1)|$ and $|E(P_2 \cup f_1 \cup M_1)|$ have the same parity or $|E(P \cup M_7 \cup e_1 \cup P_5)|$ and $|E(P_4 \cup f_2 \cup M_2)|$ have the same parity. So either $P \cup M_8 \cup e_2 \cup P_1 \cup Q \cup P_4 \cup f_2 \cup M_2$ or $P \cup M_7 \cup e_1 \cup P_5 \cup Q \cup P_2 \cup f_1 \cup M_1$ is an odd cycle. Then $\chi(G - \{e_1, f_1\}) = 3$ or $\chi(G - \{e_2, f_2\}) = 3$, a contradiction. Hence the claim holds. □

Let $w = x_6$ if $f_2 = x_5x_6$ and $w = x_7$ if $f_2 = x_7x_8$. Since $(\{e_1\} \cup \mathcal{F}_2) \subseteq E(C)$, we have $e_1, f_2 \in E(C)$. Let P_0 be the (x_1, w)-path contained in C with $f_2 \notin E(P_0)$. Let $P \subseteq P_0$

be the (u,v)-path where $u \in \{V(P_0) \cap (V(D_1) \cup V(D_2)\} \setminus V(P_3)$ and $v \in V(P_0) \cap V(P_3)$ such that $d_{P_0}(u,v)$ is as small as possible. Since $f_1 \notin E(C)$ and $f_2 \notin E(P_0)$, we have $P \neq f_1, f_2$. By the choice of u and v, we know that $V(P) \cap (V(D_1) \cup V(D_2)) = \{u,v\}$. So there is a (u,v)-path P such that $V(P) \cap (V(D_1) \cup V(D_2)) = \{u,v\}$ and $P \neq f_1, f_2$, where $u \in (V(D_1) \cup V(D_2)) \setminus V(P_3)$ and $v \in V(P_3)$. By **Claim**, we get a contradiction. Hence $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

In particular, if $e_2 \in E(D_1)$ and $\mathcal{F}_2 \subseteq (E(D_1) \cap E(D_2))$, then we have $\mathcal{F}_1 = \mathcal{F}_2$ by the symmetry of e_1 and e_2. This completes the proof of Theorem 2.5. ■

Theorem 2.6 *Let G be a $(3,2)$-critical graph with at least five odd cycles and $H \in \mathcal{G}$ with $H \subseteq G$. Then (i) $H \notin \mathcal{G} \setminus \{\mathcal{G}_4 \cup \mathcal{G}_5\}$, and (ii) $H \in \mathcal{G}_4 \cup \mathcal{G}_5$ (see Figure 1) satisfying that $P_2 \cup P_3$ is an even cycle in H.*

Proof. By Proposition 2.2, there exists an $H \in \mathcal{G}$ such that $H \subseteq G$. Let D_1 and D_2 be as stated when we introduce the definition of $\mathcal{G}_i$ for $i \in [7]$. Clearly we have known that D_1 and D_2 are odd cycles. By the definition of $(3,2)$-critical, the following observation holds directly.

Observation: If G is $(3,2)$-critical, then for any $e \in E(G)$, all odd cycles share one edge in $G - e$.

It suffices to prove the following three claims.

Claim 1. $H \notin \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Proof. Suppose $H \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. Let $P = P_5 \cup P_6$, $P = P_5$ and $P = \emptyset$ if $H \in \mathcal{G}_1$, $H \in \mathcal{G}_2$ and $H \in \mathcal{G}_3$, respectively. Then $D_1 = P_1 \cup P_3 \cup P$ and $D_2 = P_3 \cup P_4$ are odd cycles.

We first claim $P_1 \cup P_2$ is an even cycle. Suppose $H \in \mathcal{G}_1$ or $H \in \mathcal{G}_2$. Let $e \in E(P_5)$. By **Observation**, all odd cycles share one edge in $G - e$. Since there is no edge in $(P_1 \cup P_2) \cap D_2$, $P_1 \cup P_2$ is an even cycle. Suppose $H \in \mathcal{G}_3$. If $P_1 \cup P_2$ is an odd cycle, then H contains exactly four odd cycles $D_1, D_2, P_1 \cup P_2$ and $P_2 \cup P_4$. Since G contains at least five odd cycles, there exists an edge $e \in E(G) \setminus E(H)$. By **Observation**, all odd cycles share one edge in $G - e$, contradicting the fact that $E(P_1 \cup P_2) \cap E(D_2) = \emptyset$. So $P_1 \cup P_2$ is an even cycle. This means that $D_1' = P_2 \cup P_3 \cup P$ is an odd cycle since D_1 is an odd cycle. For any $e_1, e_2 \in E(P_1) \cup E(P_2)$, we have $\emptyset \neq \mathcal{F}_1, \mathcal{F}_2 \subseteq E(P_3)$. Without loss of generality, suppose that $e_1, e_2 \in E(P_1)$, or $e_1 \in E(P_1)$ and $e_2 \in E(P_2)$. If $e_1, e_2 \in E(P_1)$, then D_1, D_2 and e_1, e_2 satisfy the conditions in Theorem 2.5 since $D_1 \cap D_2 = P_3$. By the symmetry of e_1 and e_2, we have $\mathcal{F}_1 = \mathcal{F}_2 \neq \emptyset$ by Theorem 2.5. If $e_1 \in E(P_1)$ and $e_2 \in E(P_2)$, we have D_1, D_2, e_1, e_2 and D_1', D_2, e_1, e_2 satisfy the conditions in Theorem 2.5 since $D_1 \cap D_2 = D_1' \cap D_2 = P_3$. In this case again we have $\mathcal{F}_1 = \mathcal{F}_2 \neq \emptyset$ by Theorem 2.5. Therefore, for any $e_1, e_2 \in E(P_1 \cup P_2)$, $\mathcal{F}_1 = \mathcal{F}_2$. For any edge $f \in \mathcal{F}_1 = \mathcal{F}_2$ and $e \in E(P_1) \cup E(P_2)$, we have $\chi(G - f) = 3$ and $\chi(G - \{e, f\}) = 2$ as G is $(3,2)$-critical. So there is at least one odd cycle C in $G - f$ such that $e \in E(C)$. By the arbitrariness of e, we have $P_1 \cup P_2 \subseteq C$. Hence $C = P_1 \cup P_2$ is an odd cycle, contradicting the fact that $P_1 \cup P_2$ is an even cycle. $\square$

Claim 2. $H \notin \mathcal{G}_6 \cup \mathcal{G}_7$.

Proof. Suppose $H \in \mathcal{G}_6$. Let P_1, P_2, P_3 be the (x,u)-path, (u,y)-path, (y,x)-path of D_1 and P_4, P_5 be the (y,v)-path, (v,x)-path of D_2 in a counter clockwise direction, respectively, as shown in Figure 1. Then $D_1 = P_1 \cup P_2 \cup P_3$ and $D_2 = P_4 \cup P_5 \cup P_3$. Let P_6 denote the (u,v)-path that is internally disjoint from $D_1 \cup D_2$. Since both D_1 and D_2 are odd cycles, $P_1 \cup P_5 \cup P_4 \cup P_2$ is an even cycle. So the two cycles $D_3 = P_6 \cup P_2 \cup P_4$ and $D_4 = P_6 \cup P_5 \cup P_1$ have the same parity. If D_3 and D_4 are odd cycles, then H contains exactly four odd cycles. Since G contains at least five odd cycles, there exists an edge $e \in E(G) \setminus E(H)$. By **Observation**, all odd cycles share one edge in $G - e$, contradicting the fact that $E(D_1) \cap E(D_2) \cap E(D_3) \cap E(D_4) = \emptyset$. So D_3 and D_4 are even cycles. Then $D_5 = P_5 \cup P_6 \cup P_2 \cup P_3$ and $D_6 = P_1 \cup P_6 \cup P_4 \cup P_3$ are odd cycles. For $e_1 = w_1v \in E(P_6)$, $e_2 = w_2v \in E(P_5)$ and $e_3 = w_3v \in E(P_4)$, let $\mathcal{F}_i = \{f_i \in E(G) \mid \chi(G - \{e_i, f_i\}) = 2\}$ for $i \in [3]$. Then we have $\emptyset \neq \mathcal{F}_1 \subseteq E(P_3)$, $\emptyset \neq \mathcal{F}_2 \subseteq E(P_1 \cup P_3)$ and $\emptyset \neq \mathcal{F}_3 \subseteq E(P_2 \cup P_3)$. Note that $D_1 \cap D_6 = P_1 \cup P_3$ with $e_1 \in E(D_6) \setminus E(D_1)$ and $e_2 \notin E(D_1) \cup E(D_6)$, and $D_1 \cap D_5 = P_2 \cup P_3$ with $e_1 \in E(D_5) \setminus E(D_1)$ and $e_3 \notin E(D_1) \cup E(D_5)$. We have that D_6, D_1, e_1, e_2 and D_5, D_1, e_1, e_3 satisfy the conditions in Theorem 2.5. Therefore by Theorem 2.5, we have $\emptyset \neq \mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\emptyset \neq \mathcal{F}_1 \subseteq \mathcal{F}_3$. For any edge $f \in \mathcal{F}_1$ and e_i ($i \in [3]$), we have $\chi(G - f) = 3$ and $\chi(G - \{e_i, f\}) = 2$. So there is at least one odd cycle C in $G - f$ such that $e_1, e_2, e_3 \in E(C)$. Then the degree of v in C is three, a contradiction.

Suppose $H \in \mathcal{G}_7$. Let P_1, P_2, P_4, P_5 be the (x,u)-path, (u,y)-path, (y,v)-path, (v,x)-path of D_1 and P_6 be the (y,x)-path of D_2 in a counter clockwise direction, respectively, as shown in Figure 1. Then $D_1 = P_1 \cup P_2 \cup P_4 \cup P_5$ and $D_2 = P_4 \cup P_5 \cup P_6$. Let P_3 denote the (u,v)-path that is internally disjoint from $D_1 \cup D_2$. Since both D_1 and D_2 are odd cycles, we have $P_1 \cup P_2 \cup P_6$ is an even cycle and either $P_1 \cup P_3 \cup P_5$ or $P_2 \cup P_3 \cup P_4$ is an odd cycle. Without loss of generality, we assume $D_3 = P_2 \cup P_3 \cup P_4$ is an odd cycle. Then $D_4 = P_3 \cup P_1 \cup P_6 \cup P_4$ is an odd cycle. For $e_1 = v_1x \in E(P_1)$, $e_2 = v_2x \in E(P_5)$ and $e_3 = v_3x \in E(P_6)$, let $\mathcal{F}_i = \{f_i \in E(G) \mid \chi(G - \{e_i, f_i\}) = 2\}$ for $i \in [3]$. Then we have $\emptyset \neq \mathcal{F}_1 \subseteq E(P_4)$, $\emptyset \neq \mathcal{F}_2 \subseteq E(P_3 \cup P_4)$, and $\emptyset \neq \mathcal{F}_3 \subseteq E(P_2 \cup P_4)$. Note that $D_3 \cap D_4 = P_3 \cup P_4$ with $e_1 \in E(D_4) \setminus E(D_3)$ and $e_2 \notin E(D_3) \cup E(D_4)$, and $D_1 \cap D_3 = P_4 \cup P_2$ with $e_1 \in E(D_1) \setminus E(D_3)$ and $e_3 \notin E(D_1) \cup E(D_3)$. We have that D_3, D_4, e_1, e_2 and D_3, D_1, e_1, e_3 satisfy the conditions in Theorem 2.5. Therefore, we have $\emptyset \neq \mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\emptyset \neq \mathcal{F}_1 \subseteq \mathcal{F}_3$ by Theorem 2.5. For any edge $f \in \mathcal{F}_1$ and e_i ($i \in [3]$), we have $\chi(G - f) = 3$ and $\chi(G - \{e_i, f\}) = 2$. So there is at least one odd cycle C in $G - f$ such that $e_1, e_2, e_3 \in E(C)$. Then the degree of x in C is three, a contradiction. $\square$

Claim 3. $H \in \mathcal{G}_4 \cup \mathcal{G}_5$ and $P_2 \cup P_3$ is an even cycle in H.

Proof. Suppose $H \in \mathcal{G}_4 \cup \mathcal{G}_5$. Let $D_3 = P_2 \cup P_3$. If D_3 is an odd cycle, then $\mathcal{G}_4 = \mathcal{G}_2$ and $\mathcal{G}_5 = \mathcal{G}_1$. By **Claim 1**, we know G contains no graph from $\mathcal{G}_1 \cup \mathcal{G}_2$ as a subgraph. Therefore if $H \in \mathcal{G}_4 \cup \mathcal{G}_5$, then $D_3 = P_2 \cup P_3$ is an even cycle. □

The proof is thus complete. ■

3 Proof of Theorem 1.3

Let $\{H_i \mid i \in [k]\}$ $(k \geq 3)$ be a family of graphs satisfying the following three conditions: (1) H_i is an even cycle or a path for any $i \in [k]$; (2) there are at least two even cycles and at least one path; (3) for $i \in [k]$, if H_i is a path, then H_{i-1} and H_{i+1} are not paths, where the subscripts are taken cyclically modulo k. We define the family $\mathcal{E}'$, which is obtained from the disjoint union of k graphs $H_1, H_2, \ldots, H_k$ as follows. For each $i \in [k]$, let x_i and y_i be any two distinct vertices of H_i if H_i is an even cycle, and be the two endpoints of H_i if H_i is a path, where we only require that $\sum_{i=1}^{k} d_{H_i}(x_i, y_i)$ is odd. A graph $H \in \mathcal{E}'$ is obtained by identifying y_i and x_{i+1} for $i \in [k-1]$, and identifying y_k and x_1. Similarly, denote the even cycle C_{2n_i} in a graph $F \in \mathcal{E}$ by H_i for $i \in [k]$. We have the following lemma.

Lemma 3.1 *If we add an open ear P with endpoints u and v to $F \in \mathcal{E}' \cup \mathcal{E}$, then $F + P$ contains a graph from $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_6 \cup \mathcal{G}_7$ as a subgraph except when u and v belong to the same H_i which is a path and the new cycle generated by $H_i \cup P$ is an even cycle.*

Proof. Let $F \in \mathcal{E}' \cup \mathcal{E}$ and P be an open ear of F with endpoints u and v.

Case 1. $u, v \in V(H_i)$ and H_i is an even cycle.

Denote by P_1 and P_2 the two internally disjoint (x_i, y_i)-paths in H_i. By the construction of F, there exists an (x_i, y_i)-path P_3 in F which is internally disjoint with P_1 and P_2 such that $D_1 = P_1 \cup P_3$ and $D_2 = P_2 \cup P_3$ are odd cycles. If $u, v \in V(P_1)$, then $D_1 \cup D_2 \cup P \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. If $u \in V(P_1)$ and $v \in V(P_2)$, then $D_1 \cup D_2 \cup P \in \mathcal{G}_6$. Therefore, $D_1 \cup D_2 \cup P \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_6$.

Case 2. $u \in V(H_i)$ and $v \in V(H_j)$ $(i < j)$, where H_i and H_j are even cycles.

If u or $v \in V(H_i) \cap V(H_j)$, then it can be reduced to **Case 1**. So we assume $u, v \notin V(H_i) \cap V(H_j)$. Denote by P_1 and P_2 the two internally disjoint (x_i, y_i)-paths in H_i, P_3 and P_4 the two internally disjoint (x_j, y_j)-paths in H_j. By the construction of F, there exist (y_i, x_j)-path P_5 and (y_j, x_i)-path P_6 such that (i) P_s and P_t are internally disjoint for $s \neq t$ and $s, t \in [6]$, (ii) $D_1 = P_1 \cup P_5 \cup P_3 \cup P_6$ and $D_2 = P_2 \cup P_5 \cup P_3 \cup P_6$ are odd cycles. Since H_i and H_j are even cycles, we have $P_1 \cup P_5 \cup P_4 \cup P_6$ is also an odd cycle. Without loss of generality, let $u \in V(P_1)$ and $v \in V(P_3)$. Suppose $u \notin \{x_i, y_i\}$. Since $D_1 = P_1 \cup P_5 \cup P_3 \cup P_6$ and $D_2 = P_2 \cup P_5 \cup P_3 \cup P_6$ are odd cycles, $D_1 \cup D_2 \cup P \in \mathcal{G}_7$. Suppose $v \notin \{x_j, y_j\}$. Since $D_1 = P_1 \cup P_5 \cup P_3 \cup P_6$ and $D_2 = P_1 \cup P_4 \cup P_5 \cup P_6$ are odd cycles, $D_1 \cup D_2 \cup P \in \mathcal{G}_7$. Suppose $u \in \{x_i, y_i\}$ and $v \in \{x_j, y_j\}$. Since $k \geq 3$, at least one of P_5 and P_6 is not an isolated vertex. Without loss of generality, we assume P_5 is not an isolate vertex. It suffices to consider the following two subcases.

Subcase 2.1. $u = x_i$ and $v = x_j$.

Suppose that $y_j \neq x_i$, otherwise it can be reduced to **Case 1**. If $P \cup P_5 \cup P_1$ is an odd cycle, then let $D_1 = P_1 \cup P_5 \cup P_3 \cup P_6$ and $D_2 = P \cup P_5 \cup P_1$. Thus $D_1 \cup D_2 \cup P_4 \in \mathcal{G}_2$. If $P \cup P_5 \cup P_1$ is an even cycle, then $P \cup P_3 \cup P_6$ is an odd cycle since $P_1 \cup P_5 \cup P_3 \cup P_6$ is an odd cycle. Let $D_1 = P_1 \cup P_5 \cup P_3 \cup P_6$ and $D_2 = P \cup P_3 \cup P_6$. Thus, $D_1 \cup D_2 \cup P_2 \in \mathcal{G}_2$.

Subcase 2.2. $u = y_i$ and $v = x_j$.

For the case that P_5 is some H_i of F and H_i is a path, we will consider it in **Case 5**. So we assume there is an even cycle H_s ($s \in [k]$ and $s \neq i, j$) of F such that $H_s \cap P_5$ is a nontrivial path. Let P' be the (u', v')-path with $u', v' \in V(P_5)$, $P' \subseteq H_s$ and $P' \nsubseteq P_5$. If $P \cup P_5$ is an odd cycle, then let $D_1 = P_1 \cup P_5 \cup P_3 \cup P_6$ and $D_2 = P \cup P_5$. Thus $D_1 \cup D_2 \cup P_4 \in \mathcal{G}_2$. If $P \cup P_5$ is an even cycle, then let $D_1 = P_1 \cup P_5 \cup P_3 \cup P_6$ and $D_2 = P_1 \cup P \cup P_3 \cup P_6$. If $\{u', v'\} \cap \{y_i, x_j\} = \emptyset$, then $D_1 \cup D_2 \cup P' \in \mathcal{G}_1$. If $|\{u', v'\} \cap \{y_i, x_j\}| = 1$, then $D_1 \cup D_2 \cup P' \in \mathcal{G}_2$. Otherwise, $D_1 \cup D_2 \cup P' \in \mathcal{G}_3$. Thus, $D_1 \cup D_2 \cup P' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Case 3. $u \in V(H_i)$ and $v \in V(H_j)$ $(i < j)$, where H_i is an even cycle and H_j is a path.

If $v \in V(H_i) \cap V(H_j)$, then it can be reduced to **Case 1**. If $u \in V(H_i) \cap V(H_j)$, then we will consider it in **Case 5**. So we assume $u, v \notin V(H_i) \cap V(H_j)$. Denote by P_1 and P_2 the two internally disjoint (x_i, y_i)-paths in H_i. By the construction of F, there are (y_i, x_j)-path P_3 and (y_j, x_i)-path P_4 such that P_1, P_2, P_3 and P_4 are four internally disjoint paths in F, $P_1 \cup P_3 \cup H_j \cup P_4$ and $P_2 \cup P_3 \cup H_j \cup P_4$ are odd cycles. Suppose $u \notin \{x_i, y_i\}$. Since $D_1 = P_1 \cup P_3 \cup H_j \cup P_4$ and $D_2 = P_2 \cup P_3 \cup H_j \cup P_4$ are odd cycles, $D_1 \cup D_2 \cup P \in \mathcal{G}_7$. Suppose $u \in \{x_i, y_i\}$. Since $k \geq 3$, we have $x_i \neq y_j$ or $x_j \neq y_i$. Without loss of generality, we assume $x_i \neq y_j$. Since $u, v \notin V(H_i) \cap V(H_j)$, we may let $u = x_i$ in the following. Since H_j is a path, by the definition of $\mathcal{E}'$, H_{j-1} and H_{j+1} are not paths, thus there is an even cycle H_s ($s \in [k]$ and $s \neq i, j$) of F such that $H_s \cap P_4$ is a path. Let P' be the (u', v')-path with $u', v' \in V(P_4)$, $P' \subseteq H_s$ and $P' \nsubseteq P_4$. Let $D_1 = P_1 \cup P_3 \cup H_j \cup P_4$. The two internally disjoint (u, v)-paths of D_1 are denoted by Q_1 and Q_2, where $P_1 \subseteq Q_1$. Obviously, $Q_1 \cup P$ or $Q_2 \cup P$ is an odd cycle. If $D_2 = Q_1 \cup P$ is an odd cycle, take the position of $v \in V(H_j)$ and the intersection of $\{u', v'\}$ and $\{x_i, y_j\}$ into consideration, then $D_1 \cup D_2 \cup P' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. If $D_2 = Q_2 \cup P$ is an odd cycle, then $D_1 \cup D_2 \cup P_2 \in \mathcal{G}_2 \cup \mathcal{G}_3$.

Case 4. $u \in V(H_i)$ and $v \in V(H_j)$ $(i < j)$, where H_i and H_j are two paths.

By the construction of F, there are (y_i, x_j)-path P_1 and (y_j, x_i)-path P_2 such that P_1, P_2, H_i and H_j are four internally disjoint paths of F and $H_i \cup P_1 \cup H_j \cup P_2$ is an odd cycle. By the definition of $\mathcal{E}'$, we have $x_i \neq y_j$, $y_i \neq x_j$, $u, v \notin V(H_i) \cap V(H_j)$ and there are two different even cycles H_s and H_t ($s, t \in [k]$ and $s, t \neq i, j$) of F such that $H_s \cap P_1$ is a path and $H_t \cap P_2$ is a path. Let P_1' be the (u_1', v_1')-path with $u_1', v_1' \in V(P_1)$, $P_1' \subseteq H_s$

and $P_1' \nsubseteq P_1$. Let P_2' be the (u_2', v_2')-path with $u_2', v_2' \in V(P_2)$, $P_2' \subseteq H_t$ and $P_2' \nsubseteq P_2$. Let $D_1 = H_i \cup P_1 \cup H_j \cup P_2$. The two internally disjoint (u, v)-paths of D_1 are denoted by Q_1 and Q_2, where $P_1 \subseteq Q_1$. Obviously, $Q_1 \cup P$ or $Q_2 \cup P$ is an odd cycle. If $D_2 = Q_1 \cup P$ is an odd cycle, then $D_1 \cup D_2 \cup P_2' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. If $D_2 = Q_2 \cup P$ is an odd cycle, then $D_1 \cup D_2 \cup P_1' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Case 5. $u, v \in V(H_i)$, where H_i is a path and the new cycle generated by $H_i \cup P$ is an odd cycle.

There are two internally disjoint (u, v)-paths in F, say P_1 and P_2 such that $P_1 \cup P_2$ is an odd cycle and $P \cup P_1$ is the new odd cycle generated by $H_i \cup P$. By the definition of $\mathcal{E}'$, there is an even cycle H_s ($s \in [k]$ and $s \neq i$) of F such that $H_s \cap P_2$ is a path. Let P' be the (u', v')-path with $u', v' \in V(P_2)$, $P' \subseteq H_s$ and $P' \nsubseteq P_2$. Let $D_1 = P_1 \cup P_2$ and $D_2 = P_1 \cup P$. Since H_i is a path, we have $|\{u', v'\} \cap \{x_i, y_i\}| \leq 1$ by the construction of F. Therefore, $D_1 \cup D_2 \cup P' \in \mathcal{G}_1 \cup \mathcal{G}_2$.

The proof of Lemma 3.1 is thus complete. ■

Proof of Theorem 1.3. Let G be a $(3, 2)$-critical graph without isolated vertices with at least five odd cycles. By Theorem 2.6, G contains a graph H from $\mathcal{G}_4 \cup \mathcal{G}_5$ (see Figure 1) as a subgraph and $P_2 \cup P_3$ is an even cycle in H. Obviously, $H \in \mathcal{E}'$. By Lemma 2.4, G is nonseparable. Then by Proposition 2.3, G has a decomposition as $D_0, Q_1, \ldots, Q_\ell$ and $G = D_0 \cup Q_1 \cup \ldots \cup Q_\ell$, where $D_0 = H \in \mathcal{G}_4 \cup \mathcal{G}_5$, Q_1 is an open ear of D_0, and Q_i is an open ear of $D_{i-1} = D_{i-2} \cup Q_{i-1}$ for $2 \leq i \leq \ell$. By combining Lemma 3.1 with Theorem 2.6, $D_i = D_{i-1} \cup Q_i \in \mathcal{E}' \cup \mathcal{E}$ for $i \in [\ell - 1]$. Note that if $F \in \mathcal{E}'$, then there exists an edge $e \in E(F)$ such that $\chi(F - e) = 2$. Since G is a $(3, 2)$-critical graph, we have $G = D_\ell = D_{\ell-1} \cup Q_\ell \in \mathcal{E}$.

Thus, we complete the proof. ■

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