Gallai-Ramsey numbers for 3-uniform rainbow Berge triangles and monochromatic linear paths or cycles*

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Abstract

Given two non-empty t-uniform hypergraphs $G^{(t)}, H^{(t)}$ and a positive integer k, the hypergraph Gallai-Ramsey number $\mathrm{gr}_k\left(G^{(t)} : H^{(t)}\right)$ is defined as the minimum positive integer N such that for all $n \geq N$, every k-hyperedge-coloring of t-uniform complete hypergraph $K_n^{(t)}$ contains either a rainbow subhypergraph $G^{(t)}$ or a monochromatic subhypergraph $H^{(t)}$. A k-hyperedge-coloring of a hypergraph is exact if all colors are used at least once. In this paper, we get exact values of hypergraph Gallai-Ramsey numbers for rainbow 3-uniform Berge triangles and monochromatic 3-uniform linear paths or cycles under exact k-hyperedge-coloring.

Keywords: t-uniform hypergraphs; Gallai-Ramsey number; k-hyperedge-coloring; Berge triangle

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1 Introduction

Let V be a finite set and E be a family of subsets of V. A hypergraph H is a pair (V, E), denoted as $H = (V, E)$. Given a hypergraph $H = (V, E)$, V is called the vertex set of H, and can be denoted as $V(H)$, E is called the hyperedge set of H, and can be denoted as $E(H)$. A hypergraph is *simple* if there is no hyperedge containing another hyperedge. A hypergraph is *linear* if it is simple and any two hyperedges intersect at most one vertex. A hypergraph is *t-uniform* if all hyperedges have the same cardinality t. If a hypergraph H is t-uniform, then we denote it as $H^{(t)}$. Obviously, a 2-uniform hypergraph is a graph.

A *t-uniform complete hypergraph* on n vertices is a hypergraph which the hyperedge set contains all t-subset of the vertex set, and can be denoted as $K_n^{(t)}$. A *t-uniform complete bipartite hypergraph* is a hypergraph which the vertex set has a bipartition (X, Y) such

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that any t vertices chosen from X and Y can form a hyperedge, but all t vertices chosen from X (or Y) cannot form a hyperedge. The definition of a t-uniform complete k-partite hypergraph is also similar. In this paper, we always use $K_{n,m}^{(t)}$ to denote a t-uniform complete bipartite hypergraph with bipartition (X, Y) with $|X| = n$ and $|Y| = m$.

Let $H = (V, E)$ be a hypergraph without isolated vertex, a path of length n from x to y is denoted as P_n, is a vertex-hyperedge alternative sequence:

$$x = v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_{n+1} = y,$$

such that $v_1, v_2, \ldots, v_{n+1}$ are distinct vertices, $e_1, e_2, \ldots, e_n$ are distinct hyperedges and $v_i, v_{i+1} \in e_i$ for each $i \in [n]$, where $[n]$ is denoted as $\{1, 2, \ldots, n\}$. A cycle of n length is defined similarly, and can be denoted as C_n. If $e_1, e_2, \ldots, e_n$ are n consecutive hyperedges of a linear path P_n (i.e., $|e_i \cap e_{i+1}| = 1$ for each $i \in [n-1]$), then we denote P_n as $(e_1, e_2, \ldots, e_n)$ for the convenience.

A hypergraph H is called a *Berge graph* of a simple graph G, if there is a bijection $f : E(H) \to E(G)$ such that for each $e \in E(G)$ we have $e \subseteq f(e)$. A hypergraph with at least four vertices and three distinct edges e_1, e_2, e_3 is called a *Berge triangle*, denoted as BC_3, if there exist three distinct vertices, say u, v, w, with $u, v \in e_1$, $v, w \in e_2$, and $u, w \in e_3$. In fact, there are four non-isomorphic 3-uniform Berge triangles, denoted as $B_1^{(3)}$, $B_2^{(3)}$, $B_3^{(3)}$, and $B_4^{(3)}$. Among them,

$V\left(B_1^{(3)}\right) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E\left(B_1^{(3)}\right) = \{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_1\}\}$;

$V\left(B_2^{(3)}\right) = \{v_1, v_2, v_3, v_4\}$ and $E\left(B_2^{(3)}\right) = \{\{v_1, v_2, v_3\}, \{v_3, v_4, v_1\}, \{v_4, v_1, v_2\}\}$;

$V\left(B_3^{(3)}\right) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E\left(B_3^{(3)}\right) = \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_4, v_5, v_1\}\}$;

$V\left(B_4^{(3)}\right) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E\left(B_4^{(3)}\right) = \{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_2, v_3, v_4\}\}$.

It is easy to see that a linear path must be a Berge path, but conversely, there are many non-isomorphic Berg paths with given number of vertices (or length). If $P_n^{(3)}$ is a 3-uniform Berge path with length n, then by definition $n + 1 \leq |V(P_n^{(3)})| \leq 2n + 1$. Since there is a bijection between a 3-uniform Berge path $P_n^{(3)}$ and a n-length simple path P_n, we call the corresponding simple path P_n the *core* of the 3-uniform Berge path $P_n^{(3)}$.

A *k-hyperedge-coloring* of hypergraph H is a function $c : E(H) \to [k]$, where $[k]$ is a set of colors. When the integer k is small, we usually use red, blue, green and other specific colors to represent these colors. A hyperedge-colored hypergraph is called *rainbow* if all its hyperedges have distinct colors and *monochromatic* if all its hyperedges have the same color. If a monochromatic hypergraph has color i, then we call it an i-color hypergraph. A k-hyperedge-coloring of a hypergraph is *exact* if all colors are used at least once. For more notation and terminology not defined here, we refer to [1].

Since Ramsey's original paper [10] was published in 1930, the development of Ramsey theory has been very rapid. Determining the Ramsey number of graphs has always been one of the hottest topics in graph theory. However, after decades of research by mathematicians, some results have been proven, and the remaining problems are of considerable difficulty. The theory of hypergraphs originated in the 1960s, and it was a natural idea to extend Ramsey theory to hypergraphs.

Given k non-empty hypergraphs $H_1^{(t)}, H_2^{(t)}, \ldots, H_k^{(t)}$, the hypergraph Ramsey number $R\left(H_1^{(t)}, H_2^{(t)}, \ldots, H_k^{(t)}\right)$ is defined as the minimum positive integer n such that every k-hyperedge-coloring of complete hypergraph $K_n^{(t)}$ contains a monochromatic subhypergraph $H_i^{(t)}$ with color i, where $i \in [k]$.

Fortunately, the research of Ramsey numbers in hypergraphs has achieved considerable results in the past decade, especially in the research of 3-uniform linear paths and cycles, which has completely solved this problem. In 2014, Omidi and Shahsiah in [8] gave the value of hypergraph Ramsey numbers involving 3-uniform linear paths and cycles.

Theorem 1.1. [8]

(1). *For positive integers $n \geq m$, we have*

$$R\left(P_n^{(3)}, P_m^{(3)}\right) = R\left(P_n^{(3)}, C_m^{(3)}\right) = R\left(C_n^{(3)}, C_m^{(3)}\right) + 1 = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

(2). *For positive integers $n > m$, we have*

$$R\left(P_m^{(3)}, C_n^{(3)}\right) = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor - 1.$$

An edge-colored complete graph without rainbow triangles has very special structure. In 1967, Gallai in [4] first examined this structure under the guise of transitive orientations of graphs and it can also be traced back to the paper [2]. For this reason, an edge-coloring of a complete graph containing no rainbow triangles is called a *Gallai coloring*. Gallai's result was restated in [5] by the terminology of graphs. For the following statement, a nontrivial partition is a partition with at least two parts.

Theorem 1.2. [2, 4, 5] *In any edge-coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In 2010, Faudree, Gould, Jacobson and Magnant in [3] defined *Gallai-Ramsey number* $\mathrm{gr}_k(G : H)$. For more recent results about Gallai-Ramsey numbers, we refer to [7].

Definition 1.3. [3] *Given two non-empty graphs G, H and a positive integer k, define the Gallai-Ramsey number $\mathrm{gr}_k(G : H)$ to be the minimum integer N such that for all $n \geq N$, every k-edge-coloring of K_n contains either a rainbow subgraph G or a monochromatic subgraph H.*

Note that rainbow triangles play a very important role in Gallai-Ramsey number, and to extend Gallai-Ramsey number to hypergraphs, it is natural to think of extending triangles to hypergraphs.

Let $BC_3^{(3)} = \left\{B_1^{(3)}, B_2^{(3)}, B_3^{(3)}, B_4^{(3)}\right\}$. A hyperedge-colored hypergraph $G^{(3)}$ does not have a rainbow $BC_3^{(3)}$ if $G^{(3)}$ contains no rainbow $B_i^{(3)}$ for all $i = 1, 2, 3, 4$. We say that $G^{(3)}$ has a rainbow $BC_3^{(3)}$ if $G^{(3)}$ contains a rainbow $B_i^{(3)}$ for some $i = 1, 2, 3, 4$. In 2019, Magnant extended the definition of Gallai-Ramsey number to hypergraphs.

Definition 1.4. [6] *Given two non-empty hypergraphs $G^{(t)}, H^{(t)}$ and a positive integer k, define the hypergraph Gallai-Ramsey number $\mathrm{gr}_k\left(G^{(t)} : H^{(t)}\right)$ to be the minimum integer N such that for all $n \geq N$, every k-hyperedge-coloring of complete hypergraph $K_n^{(t)}$ contains either a rainbow subhypergraph $G^{(t)}$ or a monochromatic subhypergraph $H^{(t)}$.*

We say that the edge (hyperedge) coloring of complete graph (hypergraph) without rainbow subgraph (subhypergraph) G is *rainbow G-free coloring*. In 2019, Magnant in [6] obtained the structure of rainbow $BC_3^{(3)}$-free coloring of 3-uniform complete hypergraphs.

Theorem 1.5. [6] *For positive integers $k \geq 3$ and $n \geq 4$, if $K_n^{(3)}$ is a rainbow $BC_3^{(3)}$-free colored complete hypergraph with k colors, then after renumbering the colors, there exists a partition $(V_2, \ldots, V_k)$ of $V\left(K_n^{(3)}\right)$ such that for each i, all the hyperedges with three vertices in V_i are colored by either 1 or i, and all the hyperedges with three vertices in different parts are colored by 1.*

Also, Magnant in [6] determined the sharp bound of k such that any k-hyperedge-coloring of $K_n^{(3)}$ always has a rainbow $BC_3^{(3)}$.

Proposition 1.6. [6] *For integers $n \geq 4$ and $\lfloor \frac{n}{3} \rfloor + 2 \leq k \leq \binom{n}{3}$, there is always a rainbow $BC_3^{(3)}$ under any k-hyperedge-coloring of $K_n^{(3)}$.*

The following proposition implies that studying the Gallai-Ramsey numbers for rainbow $BC_3^{(3)}$ and monochromatic general Berge paths is meaningless because there are many non-isomorphic Berge paths with given number of vertices (or length). This result is also the same for monochromatic general Berge cycles.

Proposition 1.7. *Let $k \geq 3$ be an integer. Each rainbow $BC_3^{(3)}$-free colored $K_n^{(3)}$ must contain a monochromatic 3-uniform Berge path with n vertices.*

Proof. Since $k \geq 3$ and $K_n^{(3)}$ does not have rainbow $BC_3^{(3)}$, it follows that Theorem 1.5 holds. The exact k-hyperedge-coloring means that $|V_i| \geq 3$ for each $i \in \{2, 3, \ldots, k\}$. Let $V(K_n^{(3)}) = \{v_1, v_2, \ldots, v_n\}$ and $\ell \in \{2, 3, \ldots, k-1\}$. Assume that $\bigcup_{i=2}^{\ell} V_i = \{v_1, v_2, \ldots, v_s\}$ and $\bigcup_{i=\ell+1}^{k} V_i = \{v_{s+1}, \ldots, v_n\}$. Choosing an edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ as the edge set of simple path, and this simple path be the core of a 3-uniform Berge path. Next, choosing $e_i = \{v_i, v_{i+1}, v_n\}$ for each $i \in [s]$ and $e_{s+i} = \{v_{s+i}, v_{s+i+1}, v_1\}$ for each $i \in [n-s-1]$. Therefore, $\{e_1, e_2, \ldots, e_{n-1}\}$ is the hyperedge set of a 3-uniform Berge path with n vertices. $\square$

Remark 1. The proof of Proposition 1.7 only provides a method for constructing a Berge path. In fact, there are many methods for constructing non-isomorphic Berge paths, but this is not the main content of the results in this paper.

In the next section of this paper, the method of proving the lower bound is usually to construct a k-hyperedge-colored complete hypergraph. To more accurately and concisely describe the coloring structure of hypergraphs, we give a family of k-hyperedge-colored complete hypergraphs as follows. According to Theorem 1.5, we know that each element of this family contains no rainbow $BC_3^{(3)}$.

Definition 1.8. *Let $\left[K_{t_1}^{(3)}, K_{t_2}^{(3)}, \dots, K_{t_{k-1}}^{(3)}\right]$ be a k-hyperedge-colored complete hypergraph obtained from $k-1$ vertex-disjoint complete hypergraphs $K_{t_1}^{(3)}, K_{t_2}^{(3)}, \dots, K_{t_{k-1}}^{(3)}$ such that all the hyperedges of $K_{t_i}^{(3)}$ are colored by $i+1$ for each $1 \leq i \leq k-1$ and all the hyperedges between $K_{t_i}^{(3)}$ and $K_{t_j}^{(3)}$ are colored by 1 for any two integers $1 \leq i < j \leq k-1$.*

2 Main Results

The following theorem states that when the number of colors k is sufficiently large, the Gallai-Ramsey number for rainbow $BC_3^{(3)}$ only depends on k.

Theorem 2.1. *Let $k \geq 3$ be an integer and $H^{(3)}$ be a subhypergraph of balanced complete $(k-1)$-partite hypergraph $K_{3,3,\dots,3}^{(3)}$. Then*

$$\mathrm{gr}_k\left(BC_3^{(3)} : H^{(3)}\right) = \lceil N_k \rceil,$$

where N_k is the unique real number such that $k = \binom{N_k}{3}$.

Proof. For the lower bound, since the number of colors less then the number of hyperedges in a k-hyperedge-colored $K_N^{(3)}$, we have $k \leq \binom{N}{3}$. It follows that $\mathrm{gr}_k\left(BC_3^{(3)} : H^{(3)}\right) \geq \lceil N_k \rceil$.

For the upper bound, we consider any k-hyperedge-coloring of $K_N^{(3)}$ with $N \geq \lceil N_k \rceil$. It follows from $k = \binom{N_k}{3}$ and $k \geq 3$ that $\lceil N_k \rceil < 3k-3$. If $\lceil N_k \rceil \leq N \leq 3k-4$, then it follows from Proposition 1.6 that there is always a rainbow $BC_3^{(3)}$ in $K_N^{(3)}$, the result thus follows. Next we assume $N \geq 3k-3$. Suppose to the contrary that $K_N^{(3)}$ contains neither a rainbow $BC_3^{(3)}$ nor a monochromatic $H^{(3)}$. It follows from $k \geq 3$ that Theorem 1.5 holds. Since each color is used at least once, then $|V_i| \geq 3$ for each $i \in \{2, 3, \dots, k\}$. Hence, there is a monochromatic $K_{3,3,\dots,3}^{(3)}$, which implies that there is a monochromatic $H^{(3)}$, a contradiction. The result thus follows. $\square$

2.1 Results involving monochromatic 3-uniform linear paths

Lemma 2.2. *Let $K_{n,m}^{(3)}$ be a 3-uniform complete bipartite hypergraph with bipartition (X, Y) and $n \geq m \geq 3$. Then*

(1). If $n \geq 3m+1$, then $K_{n,m}^{(3)}$ contains a linear path $P_{2m}^{(3)}$.

(2). If $n \leq 3m$ and $n+m$ is even, then $K_{n,m}^{(3)}$ contains a linear path $P_{\frac{n+m-2}{2}}^{(3)}$.

(3). If $n \leq 3m$ and $n+m$ is odd, then $K_{n,m}^{(3)}$ contains a linear path $P_{\frac{n+m-1}{2}}^{(3)}$.

Proof. Let $X=\{x_1,x_2,\ldots,x_n\}$ and $Y=\{y_1,y_2,\ldots,y_m\}$.

Since $n\geq 3m+1$, set $e_i=\left\{x_{\frac{3i-1}{2}},x_{\frac{3i+1}{2}},y_{\frac{i+1}{2}}\right\}$ when $i\in[2m]$ and i is odd, and $e_i=\left\{x_{\frac{3i}{2}},x_{\frac{3i}{2}+1},y_{\frac{i}{2}}\right\}$ when $i\in[2m]$ and i is even. Note that $(e_1,e_2,\ldots,e_{2m})$ is a linear path with length $2m$. The statement (1) holds.

Assume that $n\leq 3m$. Let $Y_1=\{y_1,y_2,\ldots,y_\ell\}$, $X_1=\{x_1,x_2,\ldots,x_{3\ell}\}$, $Y_2=\{y_{\ell+1},\ldots,y_m\}$ and $X_2=\{x_{3\ell+1},\ldots,x_n\}$ such that $0\leq |X_2|-|Y_2|\leq 2$. Obviously, when $|X|=|Y|$, $|X|=|Y|+1$ or $|X|=|Y|+2$, it can be seen that X_1 and Y_1 are empty sets. Next, we choose $e_i=\left\{x_{\frac{3i-1}{2}},x_{\frac{3i+1}{2}},y_{\frac{i+1}{2}}\right\}$ when $i\in[2\ell]$ and i is odd, and $e_i=\left\{x_{\frac{3i}{2}},x_{\frac{3i}{2}+1},y_{\frac{i}{2}}\right\}$ when $i\in[2\ell]$ and i is even.

If $n+m$ is even, then $|X_2|=|Y_2|$ or $|X_2|=|Y_2|+2$. We choose $e_{2\ell+i}=\{x_{3\ell+i},x_{3\ell+i+1},y_{\ell+i}\}$ for each $i\in[m-\ell-1]$ when $|X_2|=|Y_2|$, and $e_{2\ell+i}=\{x_{3\ell+i},x_{3\ell+i+1},y_{\ell+i}\}$ for each $i\in[m-\ell]$ when $|X_2|=|Y_2|+2$. Note that $(e_1,e_2,\ldots,e_{m+\ell-\epsilon})$ is a linear path with $n+m-1$ vertices, where $\epsilon=1$ if $|X_2|=|Y_2|$ and $\epsilon=0$ if $|X_2|=|Y_2|+2$. The statement (2) holds.

If $n+m$ is odd, then $|X_2|=|Y_2|+1$. We choose $e_{2\ell+i}=\{x_{3\ell+i},x_{3\ell+i+1},y_{\ell+i}\}$ for each $i\in[m-\ell]$. Note that $(e_1,e_2,\ldots,e_{m+\ell})$ is a linear path with $n+m$ vertices. The statement (3) holds. $\square$

Lemma 2.3. *Let $K_{n,m}^{(3)}$ be a 3-uniform complete bipartite hypergraph with bipartition (X,Y) satisfies $n,m\geq\lceil\frac{\ell}{2}\rceil$ and $n+m\geq 2\ell+1$. Then $K_{n,m}^{(3)}$ contains a linear path $P_\ell^{(3)}$.*

Proof. By symmetry, set $n\geq m$. If $n\geq 3m+1$, then it follows from Lemma 2.2 (1) that we can construct a linear path $P_\ell^{(3)}$ in $K_{n,m}^{(3)}$. Next, we assume $m\leq n\leq 3m$. If $n+m\geq 2\ell+2$, then it follows from Lemma 2.2 (2) that we can construct a linear path $P_\ell^{(3)}$ in $K_{n,m}^{(3)}$. If $n+m=2\ell+1$, then $n+m$ is odd and it follows from Lemma 2.2 (3) that we can construct a linear path $P_\ell^{(3)}$ in $K_{n,m}^{(3)}$. The result thus follows. $\square$

Lemma 2.4. *For integers $k\geq 3$ and $\lceil\frac{n}{2}\rceil\geq 4$, if $a_1,a_2,\ldots,a_{k-1}$ are $k-1$ integers with $\lceil\frac{n}{2}\rceil-1\geq a_1\geq a_2\geq\ldots\geq a_{k-1}\geq 3$ and $\sum_{i=1}^{k-1}a_i\geq 2n$, then there exists a partition (A_1,A_2) of $[k-1]$ such that $\sum_{i\in A_1}a_i\geq\lceil\frac{n}{2}\rceil$ and $\sum_{i\in A_2}a_i\geq\lceil\frac{n}{2}\rceil$.*

Proof. We prove this result by contradiction. Choosing a partition (A_1,A_2) of $[k-1]$ such that $\sum_{i\in A_1}a_i\geq\sum_{i\in A_2}a_i$ and $\sum_{i\in A_1}a_i-\sum_{i\in A_2}a_i$ as small as possible. Thus, $A_2\leq\lceil\frac{n}{2}\rceil-1$ and $A_1\geq 2n-\lceil\frac{n}{2}\rceil+1$. Now we construct a new partition (A_1',A_2') of $[k-1]$ from (A_1,A_2) by setting $A_1'=A_1-j$ and $A_2'=A_2+j$ for any $j\in A_1$. Noticing that $a_j\leq\lceil\frac{n}{2}\rceil-1$, then $\sum_{i\in A_2'}a_i\leq 2\lceil\frac{n}{2}\rceil-2$ and $\sum_{i\in A_1'}a_i\geq 2n-2\lceil\frac{n}{2}\rceil+2>\sum_{i\in A_2'}a_i$, which contradicts the minimality of $\sum_{i\in A_1}a_i-\sum_{i\in A_2}a_i$. The lemma follows. $\square$

Theorem 2.5. *For an integer $k\geq 3$, we have*

$$\mathrm{gr}_k\left(BC_3^{(3)}:P_n^{(3)}\right)=\begin{cases}\lceil N_k\rceil, & 1\leq n\leq\lceil\frac{3k-5}{2}\rceil;\\ 2n+1, & \lceil\frac{3k-3}{2}\rceil\leq n\leq 6k-12;\\ \lceil\frac{5n}{2}\rceil, & n\geq 6k-11.\end{cases}$$

Proof. An obvious fact is that when k is an odd integer, $P^{(3)}_{\frac{3k-5}{2}}$ is a subhypergraph of balanced complete $(k-1)$-partite hypergraph $K^{(3)}_{3,3,\ldots,3}$ but $P^{(3)}_{\frac{3k-3}{2}}$ is not; when k is an even integer, $P^{(3)}_{\frac{3}{2}k-2}$ is a subhypergraph of balanced complete $(k-1)$-partite hypergraph $K^{(3)}_{3,3,\ldots,3}$ but $P^{(3)}_{\frac{3}{2}k-1}$ is not. Based on the above fact, if $1 \leq n \leq \lceil \frac{3k-5}{2} \rceil$, then it follows from Theorem 2.1 that $\mathrm{gr}_k\left(BC_3^{(3)} : P_n^{(3)}\right) = \lceil N_k \rceil$. Next, we complete the proof in two cases.

Case 1. $\lceil \frac{3k-3}{2} \rceil \leq n \leq 6k-12$.

It is clear that $\mathrm{gr}_k\left(BC_3^{(3)} : P_n^{(3)}\right) \geq \left|V\left(P_n^{(3)}\right)\right| = 2n+1$. Next, we consider the upper bound. For any k-hyperedge-coloring of $K_N^{(3)}$ with $N \geq 2n+1$, suppose to the contrary that $K_N^{(3)}$ contains no a rainbow $BC_3^{(3)}$ or a monochromatic linear path $P_n^{(3)}$. Since $k \geq 3$ and the hyperedge-coloring of $K_N^{(3)}$ is rainbow $BC_3^{(3)}$-free, thus Theorem 1.5 holds. Recall that the hyperedge-coloring of $K_N^{(3)}$ is exact, so each color appears at least once, that is $|\bigcup_{i=2}^{k} V_i| \geq 2n+1$ and $|V_i| \geq 3$ for each $i \in \{2, 3, \ldots, k\}$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 3$.

We assert that $|V_2| \leq \lceil \frac{n}{2} \rceil - 1$. Otherwise, assume $|V_2| \geq \lceil \frac{n}{2} \rceil$. It follows from $n \leq 6k-12$ that $|\bigcup_{i=3}^{k} V_i| \geq 3(k-2) \geq \lceil \frac{n}{2} \rceil$. Let $X = V_2$ and $Y = \bigcup_{i=3}^{k} V_i$ and further assume $|X| \geq |Y|$ (otherwise, let $X = \bigcup_{i=3}^{k} V_i$ and $Y = V_2$). It follows from Lemma 2.3 that we can construct a linear path $P_n^{(3)}$ between X and Y, a contradiction. Based on the above discussion, we know that $\lceil \frac{n}{2} \rceil - 1 \geq |V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 3$. Since $N \geq \lceil \frac{5n}{2} \rceil > 2n$, so according to Lemmas 2.4 and 2.3, $K_N^{(3)}$ contains a linear path $P_n^{(3)}$, a contradiction. The result thus follows.

Case 2. $n \geq 6k-11$.

Since $\lceil \frac{n}{2} \rceil - 1 \geq 3(k-2)$, it follows from $\lceil \frac{n}{2} \rceil - 1 \equiv a \pmod{k-2}$ that $\frac{\lceil \frac{n}{2} \rceil - 1 - a}{k-2}$ is an integer and $\frac{\lceil \frac{n}{2} \rceil - 1 - a}{k-2} \geq 3$. For the lower bound, let $t_1 = 2n$, $t_2 = \frac{\lceil \frac{n}{2} \rceil - 1 - a}{k-2} + a$ and $t_i = \frac{\lceil \frac{n}{2} \rceil - 1 - a}{k-2}$ for each $i \in \{3, \ldots, k-1\}$. Let $K_{\lceil \frac{5n}{2} \rceil - 1} = \left[K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}\right]$ be a k-hyperedge-colored complete hypergraph, and its construction is described in Definition 1.8. One can easily check that there is neither a rainbow $BC_3^{(3)}$ nor a monochromatic linear path $P_n^{(3)}$, and so $\mathrm{gr}_k\left(BC_3^{(3)} : P_n^{(3)}\right) \geq \lceil \frac{5n}{2} \rceil$.

For the upper bound, we consider any k-hyperedge-coloring of $K_N^{(3)}$ with $N \geq \lceil \frac{5n}{2} \rceil$. Suppose to the contrary that $K_N^{(3)}$ contains no a rainbow $BC_3^{(3)}$ or a monochromatic linear path $P_n^{(3)}$. It follows from $k \geq 3$ and the definition of exact hyperedge-coloring that Theorem 1.5 holds, then $|\bigcup_{i=2}^{k} V_i| \geq \lceil \frac{5n}{2} \rceil$ and $|V_i| \geq 3$ for each $i \in \{2, 3, \ldots, k\}$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 3$.

If $|V_3| \geq \lceil \frac{n}{2} \rceil$, then $|V_2| \geq \lceil \frac{n}{2} \rceil$ and $|\bigcup_{i=3}^{k-1} V_i| > \lceil \frac{n}{2} \rceil$. From Lemma 2.3, we can construct a linear path $P_n^{(3)}$ between $\bigcup_{i=3}^{k-1} V_i$ and V_2, a contradiction. Hence, $\lceil \frac{n}{2} \rceil - 1 \geq |V_3| \geq |V_4| \geq \ldots \geq |V_k| \geq 3$. If $|V_2| \leq \lceil \frac{n}{2} \rceil - 1$, then $\lceil \frac{n}{2} \rceil - 1 \geq |V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 3$. Recall that $N \geq \lceil \frac{5n}{2} \rceil > 2n$. Using Lemmas 2.4 and 2.3, $K_N^{(3)}$ contains a linear path $P_n^{(3)}$, a contradiction. Consequently, $|V_2| \geq \lceil \frac{n}{2} \rceil$. Using Lemma 2.3 again, we have $|\bigcup_{j=3}^{k} V_j| \leq \lceil \frac{n}{2} \rceil - 1$.

From Lemma 2.2 (1), we can construct a linear path $P^{(3)}_{2|\cup_{j=3}^{k} V_j|}$ between V_2 and $\bigcup_{j=3}^{k} V_j$. Note that $n - 2|\bigcup_{j=3}^{k} V_j| \geq 1$. Using Theorem 1.1, we have

$$R\left(P^{(3)}_{n-2|\cup_{j=3}^{k} V_j|}, P^{(3)}_{n}\right) = 2n + \left\lfloor \frac{n - 2|\bigcup_{j=3}^{k} V_j| + 1}{2} \right\rfloor = \left\lceil \frac{5n}{2} \right\rceil - \left|\bigcup_{j=3}^{k} V_j\right| \leq |V_2|.$$

Therefore, if there is no a 2-color linear path $P^{(3)}_{n}$ with all vertices in V_2, then there must be a 1-color linear path $P^{(3)}_{n-2|\cup_{j=3}^{k} V_j|}$ with all vertices in V_2. So, the 1-color linear path $P^{(3)}_{n-2|\cup_{j=3}^{k} V_j|}$ located inside V_2 and the 1-color linear path $P^{(3)}_{2|\cup_{j=3}^{k} V_j|}$ between V_2 and $\bigcup_{j=3}^{k} V_j$ can be connected to form a 1-color linear path $P^{(3)}_{n}$, a contradiction. The result thus follows. $\square$

2.2 Results involving monochromatic 3-uniform linear cycles

Lemma 2.6. *Let $K^{(3)}_{n,m}$ be a 3-uniform complete bipartite hypergraph with bipartition (X, Y) and $n \geq m \geq 3$. Then*

(1). If $n \geq 3m + 1$, then $K^{(3)}_{n,m}$ contains a linear cycle $C^{(3)}_{2m}$.

(2). If $n \leq 3m$ and $n + m$ is even, then $K^{(3)}_{n,m}$ contains a linear cycle $C^{(3)}_{\frac{n+m}{2}}$.

(3). If $n \leq 3m$ and $n + m$ is odd, then $K^{(3)}_{n,m}$ contains a linear cycle $C^{(3)}_{\frac{n+m-1}{2}}$.

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$.

Since $n \geq 3m + 1$, set $e_i = \left\{x_{\frac{3i-1}{2}}, x_{\frac{3i+1}{2}}, y_{\frac{i+1}{2}}\right\}$ when $i \in [2m]$ and i is odd, and $e_i = \left\{x_{\frac{3i}{2}}, x_{\frac{3i}{2}+1}, y_{\frac{i}{2}}\right\}$ when $i \in [2m-2]$ and i is even, and $e_{2m} = \{y_m, x_{3m}, x_1\}$. Note that $(e_1, e_2, \ldots, e_{2m})$ is a linear cycle of length $2m$. The statement (1) holds.

Assume that $n \leq 3m$. Let $Y_1 = \{y_1, y_2, \ldots, y_\ell\}$, $X_1 = \{x_1, x_2, \ldots, x_{3\ell}\}$, $Y_2 = \{y_{\ell+1}, \ldots, y_m\}$ and $X_2 = \{x_{3\ell+1}, \ldots, x_n\}$ such that $0 \leq |X_2| - |Y_2| \leq 2$. Obviously, when $|X| = |Y|$, $|X| = |Y| + 1$ or $|X| = |Y| + 2$, it can be seen that X_1 and Y_1 are empty sets. Next, we choose $e_i = \left\{x_{\frac{3i-1}{2}}, x_{\frac{3i+1}{2}}, y_{\frac{i+1}{2}}\right\}$ when $i \in [2\ell]$ and i is odd, and $e_i = \left\{x_{\frac{3i}{2}}, x_{\frac{3i}{2}+1}, y_{\frac{i}{2}}\right\}$ when $i \in [2\ell]$ and i is even.

If $n+m$ is even, then $|X_2| = |Y_2|$ or $|X_2| = |Y_2|+2$. We choose $e_{2\ell+i} = \{x_{3\ell+i}, x_{3\ell+i+1}, y_{\ell+i}\}$ for each $i \in [m - \ell - 1]$ and $e_{m+\ell} = \{x_{m+2\ell}, y_m, x_1\}$ when $|X_2| = |Y_2|$, and $e_{2\ell+i} = \{x_{3\ell+i}, x_{3\ell+i+1}, y_{\ell+i}\}$ for each $i \in [m - \ell]$ and $e_{m+\ell+1} = \{y_m, x_n, x_1\}$ when $|X_2| = |Y_2| + 2$. Note that $(e_1, e_2, \ldots, e_{m+\ell+\epsilon})$ is a linear cycle with $n+m$ vertices, where $\epsilon = 0$ if $|X_2| = |Y_2|$ and $\epsilon = 1$ if $|X_2| = |Y_2| + 2$. The statement (2) holds.

If $n + m$ is odd, then $|X_2| = |Y_2| + 1$. We choose $e_{2\ell+i} = \{x_{3\ell+i}, x_{3\ell+i+1}, y_{\ell+i}\}$ for each $i \in [m - \ell - 1]$ and $e_{m+\ell} = \{x_{m+2\ell-1}, y_m, x_1\}$. Note that $(e_1, e_2, \ldots, e_{m+\ell})$ is a linear cycle with $n + m - 1$ vertices. The statement (3) holds. $\square$

Similar to the proof of Lemma 2.3, we directly give the following lemma based on Lemma 2.6.

Lemma 2.7. *Let $K_{n,m}^{(3)}$ be a 3-uniform complete bipartite hypergraph with bipartition (X,Y) satisfies $n,m \geq \lceil \frac{\ell}{2} \rceil$ and $n+m \geq 2\ell$. Then $K_{n,m}^{(3)}$ contains a linear cycle $C_\ell^{(3)}$.*

Theorem 2.8. *For an integer $k \geq 3$, we have*

$$\mathrm{gr}_k\left(BC_3^{(3)} : C_n^{(3)}\right) = \begin{cases} \lceil N_k \rceil, & 1 \leq n \leq \lceil \frac{3}{2}k \rceil - 2; \\ 2n, & \lceil \frac{3}{2}k \rceil - 1 \leq n \leq 6k-12; \\ \lceil \frac{5n}{2} \rceil - 1, & n \geq 6k-11. \end{cases}$$

Proof. An obvious fact is that when k is an odd integer, $C_{\frac{3k-3}{2}}^{(3)}$ is a subhypergraph of balanced complete $(k-1)$-partite hypergraph $K_{3,3,\ldots,3}^{(3)}$ but $C_{\frac{3k-1}{2}}^{(3)}$ is not; when k is an even integer, $C_{\frac{3}{2}k-2}^{(3)}$ is a subhypergraph of balanced complete $(k-1)$-partite hypergraph $K_{3,3,\ldots,3}^{(3)}$ but $C_{\frac{3}{2}k-1}^{(3)}$ is not. Based on the above facts, if $1 \leq n \leq \lceil \frac{3}{2}k \rceil - 2$, then it follows from Theorem 2.1 that $\mathrm{gr}_k\left(BC_3^{(3)} : C_n^{(3)}\right) = \lceil N_k \rceil$. Next, we complete the proof in two cases.

Case 1. $\lceil \frac{3k}{2} \rceil - 1 \leq n \leq 6k-12$.

For the lower bound, $\mathrm{gr}_k\left(BC_3^{(3)} : C_n^{(3)}\right) \geq \left|V\left(C_n^{(3)}\right)\right| = 2n$. For the upper bound, we consider any k-hyperedge-coloring of $K_N^{(3)}$ with $N \geq 2n$ and suppose to the contrary that $K_N^{(3)}$ containa no a rainbow $BC_3^{(3)}$ or a monochromatic linear cycle $C_n^{(3)}$. It follows from $k \geq 3$ and the definition of exact hyperedge-coloring that Theorem 1.5 holds, then $|\bigcup_{i=2}^k V_i| \geq 2n$ and $|V_i| \geq 3$ for each $i \in \{2,3,\ldots,k\}$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 3$.

We claim that $|V_2| \leq \lceil \frac{n}{2} \rceil - 1$. Otherwise, $|V_2| \geq \lceil \frac{n}{2} \rceil$ and $|\bigcup_{i=3}^k V_i| \geq 3(k-2) \geq \lceil \frac{n}{2} \rceil$. Thus according to Lemma 2.7, we can construct a linear cycle $C_n^{(3)}$ between V_2 and $\bigcup_{i=3}^k V_i$, a contradiction. Now that $\lceil \frac{n}{2} \rceil - 1 \geq |V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 3$, so similar to the proof of Case 1 in Theorem 2.5, we can use Lemmas 2.4 and 2.7 to know that there exists a monochromatic linear cycle $C_n^{(3)}$ in $K_N^{(3)}$, a contradiction. The result thus follows.

Case 2. $n \geq 6k-11$.

Similar to the proof of Case 2 in Theorem 2.5, we know from $\lceil \frac{n}{2} \rceil - 1 \equiv a \pmod{k-2}$ that $\frac{\lceil \frac{n}{2} \rceil - 1 - a}{k-2}$ is an integer and $\frac{\lceil \frac{n}{2} \rceil - 1 - a}{k-2} \geq 3$. For the lower bound, let $t_1 = 2n-1$, $t_2 = \frac{\lceil \frac{n}{2} \rceil - 1 - a}{k-2} + a$ and $t_i = \frac{\lceil \frac{n}{2} \rceil - 1 - a}{k-2}$ for each $i \in \{3,\ldots,k-1\}$. Let $K_{\lceil \frac{5n}{2} \rceil - 2} = \left[K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}\right]$ be a k-hyperedge-colored complete hypergraph, and its construction is described in Definition 1.8. One can easily check that there is neither a rainbow $BC_3^{(3)}$ nor a monochromatic linear cycle $C_n^{(3)}$, and so $\mathrm{gr}_k\left(BC_3^{(3)} : C_n^{(3)}\right) \geq \lceil \frac{5n}{2} \rceil - 1$.

For the upper bound, we consider any k-hyperedge-coloring of $K_N^{(3)}$ with $N \geq \lceil \frac{5n}{2} \rceil - 1$. Suppose to the contrary that $K_N^{(3)}$ contains no a rainbow $BC_3^{(3)}$ or a monochromatic linear cycle $C_n^{(3)}$. It follows from $k \geq 3$ and the definition of exact hyperedge-coloring that Theorem 1.5 holds, then $|\bigcup_{i=2}^{k} V_i| \geq \lceil \frac{5n}{2} \rceil - 1$ and $|V_i| \geq 3$ for each $i \in \{2, 3, \ldots, k\}$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 3$.

We claim that $|V_3| \leq \lceil \frac{n}{2} \rceil - 1$, $|V_2| \geq \lceil \frac{n}{2} \rceil$ and $|\bigcup_{i=3}^{k-1} V_i| \leq \lceil \frac{n}{2} \rceil - 1$. To the contrary, if $|V_3| \geq \lceil \frac{n}{2} \rceil$, then $|V_2| \geq \lceil \frac{n}{2} \rceil$ and $|\bigcup_{i=3}^{k-1} V_i| > \lceil \frac{n}{2} \rceil$. From Lemma 2.7, we can construct a linear cycle $C_n^{(3)}$ between $\bigcup_{i=3}^{k-1} V_i$ and V_2, a contradiction. Hence, $\lceil \frac{n}{2} \rceil - 1 \geq |V_3| \geq |V_4| \geq \ldots \geq |V_k| \geq 3$. If $|V_2| \leq \lceil \frac{n}{2} \rceil - 1$, then $\lceil \frac{n}{2} \rceil - 1 \geq |V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 3$. Recall that $N \geq \lceil \frac{5n}{2} \rceil - 1 \geq 2n$. Using Lemmas 2.4 and 2.7, $K_N^{(3)}$ contains a linear cycle $C_n^{(3)}$, a contradiction. Consequently, $|V_2| \geq \lceil \frac{n}{2} \rceil$ and $|\bigcup_{j=3}^{k} V_j| \leq \lceil \frac{n}{2} \rceil - 1$ according to Lemma 2.7 again.

From Lemma 2.2 (1), we can construct a a linear path $P_{2|\cup_{j=3}^{k} V_j|}^{(3)}$ between V_2 and $\bigcup_{j=3}^{k} V_j$. Similar to the proof of Theorem 2.5, using Theorem 1.1 again, we have

$$R\left(P_{n-2|\cup_{j=3}^{k} V_j|}^{(3)}, C_n^{(3)}\right) = 2n + \left\lfloor \frac{n - 2|\bigcup_{j=3}^{k} V_j| + 1}{2} \right\rfloor - 1 = \left\lceil \frac{5n}{2} \right\rceil - 1 - \left|\bigcup_{j=3}^{k} V_j\right| \leq |V_2|.$$

Therefore, if there is no a 2-color linear cycle $C_n^{(3)}$ with all vertices in V_2, then there must be a 1-color linear path $P_{n-2|\cup_{j=3}^{k} V_j|}^{(3)}$ with all vertices in V_2. So, the 1-color linear path $P_{n-2|\cup_{j=3}^{k} V_j|}^{(3)}$ located inside V_2 and the 1-color linear path $P_{2|\cup_{j=3}^{k} V_j|}^{(3)}$ between V_2 and $\bigcup_{j=3}^{k} V_j$ can be connected to form a 1-color linear cycle $C_n^{(3)}$, a contradiction. The result thus follows. $\square$

3 Conclusion

When we study Gallai-Ramsey numbers, the rainbow subgraph (subhypergraph) free coloring structure of complete graph (hypergraph) is very important. In addition, there are also considerations regarding the choice of monochromatic subhypergraphs of Gallai-Ramsey numbers. Linear paths and linear cycles are important structures in hypergraph theory, so we chose to study these two types of hypergraphs. We can also see that Ramsey numbers play a crucial role in studying Gallai-Ramsey numbers, so in most cases, we need to first obtain the values of Ramsey numbers.

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