A characterization of 4-χ_S-vertex-critical graphs for packing sequences with $s_1 = 1$ with $s_2 \geq 3$

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Abstract

If $S = (s_1, s_2, \ldots)$ is a non-decreasing sequence of positive integers, then the S-packing k-coloring of a graph G is a mapping $c : V(G) \to [k]$ such that if $c(u) = c(v) = i$ for $u \neq v \in V(G)$, then $d_G(u, v) > s_i$. The S-packing chromatic number of G is the smallest integer k such that G admits an S-packing k-coloring. A graph G is χ_S-vertex-critical if $\chi_S(G - u) < \chi_S(G)$ for each $u \in V(G)$. If G is χ_S-vertex-critical and $\chi_S(G) = k$, then G is k-χ_S-vertex-critical. In this paper, 4-χ_S-vertex-critical graphs are characterized for sequences $S = (1, s_2, s_3, \ldots)$ with $s_2 \geq 3$. There are 28 sporadic examples and two infinite families of such graphs.

Keywords: graph coloring; distance in graph; S-packing coloring; S-packing chromatic number; S-packing chromatic vertex-critical graph

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1 Introduction

A *packing k-coloring* of a graph $G = (V(G), E(G))$ is a mapping $c : V(G) \to [k]$ such that if $u \neq v$ and $c(u) = c(v) = i$, then $d_G(u, v) > i$. Here and later, $d_G(u, v)$ denotes the length of a shortest u, v-path, and $[k] = \{1, \ldots, k\}$. The *packing chromatic number*, $\chi_\rho(G)$, of G is the smallest integer k such that G admits a packing k-coloring. This concept was proposed in [14]. The seminal paper was followed by [7], where the nowadays established name and notation was proposed. The development on the packing chromatic number up to 2020 has been summarized in the substantial survey [6]. Research into this concept is still flourishing, the developments after the survey include [1, 2, 5, 8, 10].

A more general concept is the S-packing coloring. Let $S = (s_1, s_2, \ldots)$ be a non-decreasing sequence of positive integers; we will refer to S as a *packing sequence*. An *S-packing k-coloring* of G is a mapping $c : V(G) \to [k]$ such that if $u \neq v$ and $c(u) = c(v) = i$, then $d_G(u, v) > s_i$. For example, a $(1, 1, 1, \ldots)$-packing coloring is the standard proper vertex coloring, and if $S = (1, 2, 3, \ldots)$, then it is just the packing coloring. The *S-packing chromatic number*, $\chi_S(G)$, of G is the smallest integer k such that G admits an S-packing k-coloring. This concept was introduced by Goddard and Xu [15]; for more results see [4, 11, 13, 16, 19, 20].

If $S_1 = (s^1_1, s^1_2, \ldots)$ and $S_2 = (s^2_1, s^2_2, \ldots)$ are (packing) sequences with $|S_1| = |S_2|$, then $S_2 \succeq S_1$ means the coordinate order, that is, $S_2 \succeq S_1$ if $s^2_i \geq s^1_i$ for every $i \in [|S_1|]$. If $S_2 \succeq S_1$ and G admits an S_2-packing k-coloring, then G also admits an S_1-packing k-coloring. In [11, Theorem 3.1], Gastineau proved the following appealing dichotomy result: If S is a packing sequence with $|S| = 4$, then the decision problem whether a given graph G admits an S-packing coloring is polynomial-time solvable if $S \succeq S'$, where $S' \in \{(2, 3, 3, 3), (2, 2, 3, 4), (1, 4, 4, 4), (1, 2, 5, 6)\}$, and NP-complete otherwise.

We have now arrived to the central concept of interest in this paper. A graph G is *packing chromatic vertex-critical* if $\chi_\rho(G - u) < \chi_\rho(G)$ holds for each $u \in V(G)$. When $\chi_\rho(G) = k$, we more precisely say that G is *k-χ_ρ-vertex-critical*. More generally, if S is a packing sequence, then G is *S-packing chromatic vertex-critical* if $\chi_S(G - u) < \chi_S(G)$ holds for each $u \in V(G)$, and if $\chi_S(G) = k$, then we say that G is *k-χ_S-vertex-critical*. We also add that a closely related concept of *packing chromatic critical graphs*, where the packing chromatic number strictly decreases on an arbitrary proper subgraph, has been studied in [3].

Packing chromatic vertex-critical graphs were introduced in [18]. Among other results, 3-χ_ρ-vertex-critical graphs were characterized and a partial characterization of 4-χ_ρ-vertex-critical graphs was provided. The latter characterization has been completed in [9]. In [17], 3-χ_S-vertex-critical graphs were characterized for all possible packing sequences, while 4-χ_S-vertex-critical graphs were characterized for packing sequences $(s_1, s_2, s_3, \ldots)$ with $s_1 \geq 2$.

In this article we supplement the latter result by characterizing 4-χ_S-vertex-critical graphs for packing sequences with $s_1 = 1$ and $s_2 \geq 3$. The result is given in Section 3, while in the next section we introduce some additional notation and list known properties of S-packing colorings needed here.

2 Preliminaries

If G is a graph, then we use $n(G)$ to denote its order, $\mathrm{diam}(G)$ to denote its diameter, and $\chi(G)$ to denote its chromatic number. For $x \in V(G)$, let $N_G^i(x)$ be the set of vertices which are at distance i from x in G. In particular, $N_G(x) = N_G^1(x)$ is the neighborhood of x. The degree of x is $d_G(x) = |N_G(x)|$. Let C_n, P_n, and K_n denote the cycle, the path, and the complete graph on n vertices, respectively. A set $A \subseteq V(G)$ is *k-independent* if A induces a subgraph that can be properly colored by k colors. Let $\alpha_k(G)$ be the cardinality of a largest k-independent set of G.

If in a packing sequence the term i repeats ℓ times, we may abbreviate the corresponding subsequence by i^ℓ. For example, if $S = (1, \ldots, 1, s_{\ell+1}, \ldots)$ (where clearly 1 appears ℓ times), then we may shortly write $S = (1^\ell, s_{\ell+1}, \ldots)$. If $\varphi : V(G) \to [k]$ is an S-packing k-coloring of G, then $\varphi^{-1}(i)$, $i \in [k]$, is the set of vertices x with $\varphi(x) = i$. We will also use the following convention. Consider the vertex set $V(G) = \{v_1, \ldots, v_n\}$ of G as an ordered set, and let φ be an S-packing coloring of G. Then we will explicitly describe φ as follows: $\varphi =$ "$\varphi(v_1)\ \cdots\ \varphi(v_n)$". Typically, the order of vertices will be alphabetic. For instance, if $V(G) = \{a, b, c, d\}$, and $\varphi(a) = 1$, $\varphi(b) = 2$, $\varphi(c) = 1$, and $\varphi(d) = 3$, then $\varphi =$ "1 2 1 3".

We next recall some known results that will be needed in the rest.

Proposition 2.1 [14] *Let $n \geq 3$. If $n = 3$ or $n = 4k$, $k \geq 1$, then $\chi_\rho(C_n) = 3$; otherwise $\chi_\rho(C_n) = 4$.*

Lemma 2.2 [15] *If S is a packing sequence and H is a subgraph of G, then $\chi_S(H) \leq \chi_S(G)$.*

Proposition 2.3 [15] *Let $S = (1^\ell, s_{\ell+1}, \ldots)$, where $\ell \geq 1$ and $s_{\ell+1} \geq 2$, and let G be a graph. Then $\chi_S(G) \leq n(G) - \alpha_\ell(G) + \min\{\ell, \chi(G)\}$ with equality if and only if* $\mathrm{diam}(G) \leq s_{\ell+1}$.

Lemma 2.4 [18] *If S is a packing sequence and G is a χ_S-vertex-critical graph, then G is connected.*

Finally, the following notation will be useful. Suppose we wish to consider all the packing sequences $S = (s_1, s_2, s_3, \ldots)$, for which $s_1 = 2$, $s_2 \geq 4$, and $s_3 = 5$ hold. We will denote the set of all such packing sequences by $\mathcal{S}_{2,\overline{4},5}$, that is,

$$\mathcal{S}_{2,\overline{4},5} = \{(s_1, s_2, s_3, \ldots) : \ s_1 = 2, s_2 \geq 4, s_3 = 5\}.$$

Note that since $\mathcal{S}_{2,\overline{4},5}$ is a set of packing sequences, we have $s_2 \in \{4,5\}$ when $S \in \mathcal{S}_{2,\overline{4},5}$. The general notation should be clear from this example. For instance, using this notation we can state that $S \succeq (s_1, s_2, s_3, \ldots)$ if and only $S \in \mathcal{S}_{\overline{s_1},\overline{s_2},\overline{s_3},\ldots}$.

3 Vertex-critical graphs for different packing sequences

As mentioned in the introduction, a characterization of 3-χ_S-vertex-critical graphs is known for all possible packing sequences, while 4-χ_S-vertex-critical graphs were by now characterized for packing sequences from $\mathcal{S}_{\overline{2}}$. In this section we supplement the latter result by characterizing 4-χ_S-vertex-critical graphs for packing sequences S from $\mathcal{S}_{1,\overline{3}}$. To this end note that

$$\mathcal{S}_{1,\overline{3}} = \mathcal{S}_{1,\overline{4}} \cup \mathcal{S}_{1,3,\overline{4}} \cup \mathcal{S}_{1,3,3}\ .$$

In view of this fact we will solve our problem by characterizing 4-χ_S-vertex-critical graphs for packing sequences from each of the sets $\mathcal{S}_{1,\overline{4}}$, $\mathcal{S}_{1,3,\overline{4}}$, and $\mathcal{S}_{1,3,3}$.

In Figs. 1 and 2, several graphs are drawn that will turn out to be 4-χ_S-vertex-critical for packing sequences from $\mathcal{S}_{1,\overline{3}}$. Fig. 1 contains two small families of graphs, the family $\mathcal{C}_5$ contains four graphs of order 5, while $\mathcal{C}_6$ contains three graphs of order 6. Fig. 2 displays the family of graphs $\mathcal{H}$ consisting of graphs H_i, $i \in [15]$.

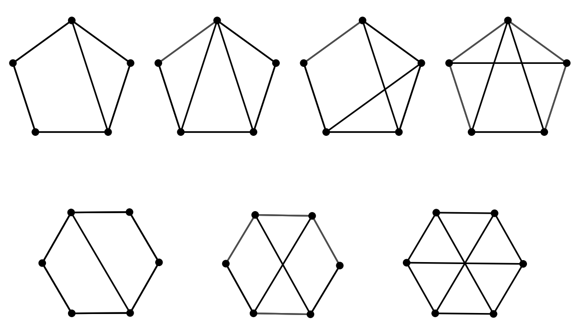

Figure 1: Family $\mathcal{C}_5$ (top row) and family $\mathcal{C}_6$ (bottom row)

In the rest we will frequently consider different subsets of $\mathcal{H}$. To shorten the presentation, we will specify subsets of $\mathcal{H}$ by (ranges of) indices. For instance, $\mathcal{H}_{1-3,7,9-11} = \{H_1, H_2, H_3, H_7, H_9, H_{10}, H_{11}\}$.

First we detect the following critical graphs.

Lemma 3.1 *Let $S \in \mathcal{S}_{1,\overline{3}}$. Then each of the graphs from $\mathcal{G} = \{K_4, C_5, C_6\} \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{H}_{1-5,7}$ is 4-χ_S-vertex-critical.*

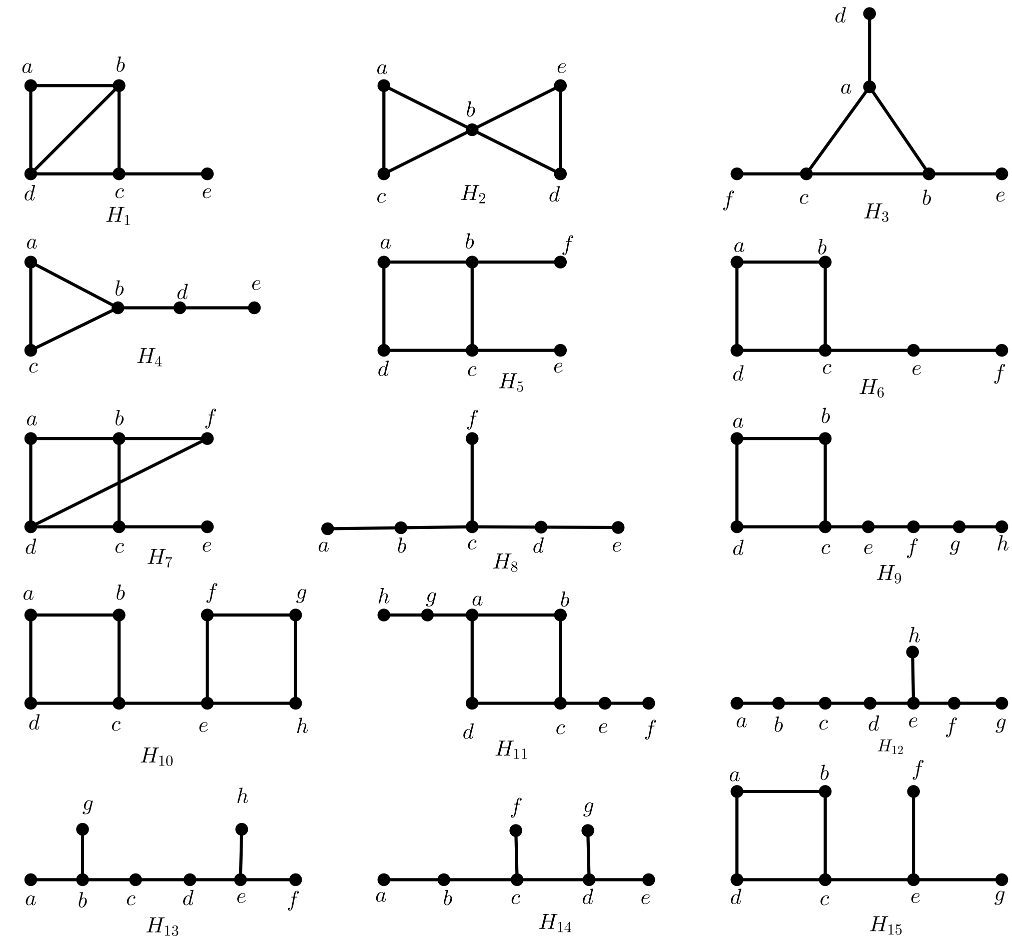

Figure 2: Family $\mathcal{H} = \{H_i : \ i \in [15]\}$

Proof. Observe that for each $G \in \mathcal{G}$, $\text{diam}(G) \leq s_2$. Using Proposition 2.3 it is then straightforward to check that $\chi_S(G) = 4$ for each $G \in \mathcal{G}$. It remains to show that each graph $G \in \mathcal{G}$ is χ_S-vertex-critical.

By Proposition 2.3, we have $\chi_S(K_3) = 3 - 1 + 1 = 3$, $\chi_S(P_k) \leq k - \alpha(P_k) + 1 \leq 3$ for $k \leq 5$, $\chi_S(G-x) = 4-2+1 = 3$ for any $G \in \mathcal{C}_5$ and $x \in V(G)$, and $\chi_S(G-x) = 5-3+1 = 3$ for any $G \in \mathcal{C}_6$ and $x \in V(G)$. Therefore, K_4, C_5, C_6, each of the graphs from $\mathcal{C}_5$, and each of the graphs from $\mathcal{C}_6$ are χ_S-vertex-critical.

Now we prove that each graph in $\mathcal{H}_{1-5,7}$ is 4-χ_S-vertex-critical where $S \in \mathcal{S}_{1,\overline{3}}$. First consider the case that $S \in \mathcal{S}_{1,\overline{4}}$. We give an S-packing 3-colorings φ for every $G - x$, where $G \in \mathcal{H}_{1-5,7}$ and $x \in V(G)$. (By symmetry, we need not to consider all the vertices.) Suppose $G = H_1$. Then we define φ as "1 2 3 1","1 1 3 2","1 2 3 2","1 2 1 3" when $x = a, b, c, e$, respectively. Suppose $G = H_2$. Then we define φ as "2 1 1 3", when $x = a$

or $x = b$. Suppose $G = H_3$. Then we define φ as "2 3 1 1 1", "1 2 3 1 1" when $x = a, d$, respectively. Suppose $G = H_4$. Then we define φ as "3 1 1 2", "2 1 1 2", "2 1 3 2", "2 3 1 1" when $x = a, b, d, e$, respectively. Suppose $G = H_5$. Then we define φ as "1 2 1 1 3", "3 2 1 1 3", "3 1 2 1 1" when $x = a, b, f$, respectively. Suppose $G = H_7$. Then we define φ as "1 2 1 1 3", "1 2 3 1 1", "2 1 1 1 3", "1 2 1 3 1", when $x = a, b, c, e$, respectively. We have thus verified that each $G \in \mathcal{H}_{1-5,7}$ is χ_S-vertex-critical for $S \in \mathcal{S}_{1,\overline{4}}$.

Finally suppose that $S \in \mathcal{S}_{1,3,\overline{4}} \cup \mathcal{S}_{1,3,3}$. Let $G \in \mathcal{H}_{1-5,7}$ and $x \in V(G)$ be an arbitrary vertex. Since $\chi_S(G) = 4$, it suffices to show that $\chi_S(G - x) = 3$. Observe that for any packing sequence $S \in \mathcal{S}_{1,\overline{3}}$ there is a packing sequence $S' \in \mathcal{S}_{1,\overline{4}}$ such that $S' \succeq S$. Thus, the above S'-packing 3-coloring of $G - x$, where $G \in \mathcal{H}_{1-5,7}$ and $x \in V(G)$, yields an S-packing 3-coloring of $G - x$. Therefore, we are finished. ∎

3.1 4-χ_S-vertex-critical graphs for $S \in \mathcal{S}_{1,\overline{4}} \cup \mathcal{S}_{1,3,\overline{4}}$

In this subsection we characterize 4-χ_S-vertex-critical graphs for $S \in \mathcal{S}_{1,\overline{4}}$ and for $S \in \mathcal{S}_{1,3,\overline{4}}$. The results are given in Theorems 3.5, 3.6, and 3.7.

Lemma 3.2 *P_6, H_6, and H_8 are 4-χ_S-vertex-critical graph for $S \in \mathcal{S}_{1,\overline{4}}$.*

Proof. Let $S \in \mathcal{S}_{1,\overline{4}}$.

First, we prove that P_6 is 4-χ_S-vertex-critical. Suppose that $P_6 = abcdef$ has an S-packing 3-coloring φ. Since $|\varphi^{-1}(1)| \leq 3$, we have $|\varphi^{-1}(2)| \geq 2$ or $|\varphi^{-1}(3)| \geq 2$. Since $s_2 \geq 4$, we must have $\varphi(a) = \varphi(f) = \alpha \in \{2, 3\}$. Then at least three vertices of $\{b, c, d, e\}$ must receive color 1, but this is impossible. The pattern "1 2 1 3 1 4" gives an S-packing 4-coloring of P_6, so $\chi_S(P_6) = 4$. By Proposition 2.3, $\chi_S(P_k) \leq k - \alpha(P_k) + 1 \leq 3$ for $k \leq 5$. Hence, P_6 is 4-χ_S-vertex-critical.

Now, we prove that both H_6 and H_8 are 4-χ_S-vertex-critical. Observe that $\alpha(H_6) = \alpha(H_8) = 3$. By Proposition 2.3, we have $\chi_S(H_6) = \chi_S(H_8) = 6 - 3 + 1 = 4$. Now we give an S-packing 3-coloring ϕ of $G - x$ for $G \in \{H_6, H_8\}$ and $x \in V(G)$. If $G = H_6$, then we define ϕ as "1 2 1 1 3", "1 1 2 3 1", "2 1 1 1 3", "3 1 2 1 3", "3 1 2 1 1" when $x = a, b, c, e, f$, respectively. If $G = H_8$, then we define ϕ as "1 2 1 3 1", "2 2 1 3 1", "2 1 1 3 1", "1 2 1 3 1" when $x = a, b, c, f$, respectively. ∎

Lemma 3.3 *Each of the graphs from $\{P_8, C_8\} \cup \mathcal{H}_{9,11-15}$ is 4-χ_S-vertex-critical for $S \in \mathcal{S}_{1,3,\overline{4}}$.*

Proof. Let $S \in \mathcal{S}_{1,3,\overline{4}}$.

Suppose that $P_8 = abcdefgh$ has an S-packing 3-coloring φ. Since $|\varphi^{-1}(1)| \leq 4$, $|\varphi^{-1}(2)| \leq 2$, and $|\varphi^{-1}(3)| \leq 2$, we have $|\varphi^{-1}(1)| = 4$, $|\varphi^{-1}(2)| = 2$ and $|\varphi^{-1}(3)| = 2$.

Without loss of generality assume $\varphi(a) = \varphi(c) = \varphi(e) = \varphi(g) = 1$. Then we have $c(b) = c(h) = 3$ because $s_3 \geq 4$. Thus d and f must receive color 2, a contradiction. The pattern "1 2 1 3 1 2 1 4" gives an S-packing 4-coloring of P_8, so $\chi_S(P_8) = 4$. Since $\chi_S(P_8) = 4$ and the pattern "1 2 1 3 1 2 1 4" gives an S-packing 4-coloring of C_8, we have $\chi_S(C_8) = 4$. The first k entries in the pattern "1 2 1 3 1 2 1" gives an S-packing 3-coloring of P_k with $k \leq 7$, so P_8 and C_8 are 4-χ_S-vertex-critical.

If $G = H_9$, then the pattern "2 1 3 1 1 2 1 4" is an S-packing 4-coloring of H_9, so $\chi_S(H_9) \leq 4$. Suppose that H_9 has an S-packing 3-coloring φ. Observe that $\{\varphi(a), \varphi(c)\} = \{2, 3\}$, which implies that $\varphi(e) = \varphi(f) = 1$ or $\varphi(g) = \varphi(h) = 1$, a contradiction. Hence $\chi_S(H_9) = 4$. Now we give an S-packing 3-coloring ϕ of $H_9 - x$ for any $x \in V(H_9)$. We define ϕ as "1 2 1 1 3 1 2", "2 1 1 1 2 1 3", "1 2 1 3 1 2 1", "2 1 3 1 2 1 3", "2 1 3 1 1 1 3", "2 1 3 1 1 2 3", "2 1 3 1 1 2 1" when $x = a, c, d, e, f, g, h$, respectively.

If $G = H_{11}$, then the pattern "2 1 3 1 1 2 1 4" is an S-packing 4-coloring, so $\chi_S(H_{11}) \leq 4$. Suppose that H_{11} admits an S-packing 3-coloring φ. Observe that $\{\varphi(a), \varphi(c)\} = \{2, 3\}$. Then $\varphi(g) = \varphi(h) = 1$ or $\varphi(e) = \varphi(f) = 1$, a contradiction. Hence $\chi_S(H_{11}) = 4$. Now we give an S-packing 3-coloring ϕ of $H_{11} - x$ for any $x \in V(H_{11})$. We define ϕ as "1 3 1 1 2 1 3", "1 3 1 2 1 2 1", "2 1 3 1 1 2 3", "2 1 3 1 1 2 1" when $x = a, d, g, h$, respectively.

If $G = H_{12}$, then the pattern "4 1 2 1 3 1 2 1" is an S-packing 4-coloring of H_{12}. Hence $\chi_S(H_{12}) \leq 4$. Suppose H_{12} admits an S-packing 3-coloring φ, then $\varphi(e) = 2$ or 3. If $\varphi(e) = 2$, then we have $\{\varphi(f), \varphi(g)\} = \{1, 3\}$ and $\varphi(d) = 1$. Thus $\varphi(c) \in \{2, 3\}$, a contradiction. If $\varphi(e) = 3$, then $\varphi(d) = 1$, $\varphi(c) = 2$, $\varphi(b) = 1$. Thus $\varphi(a) \in \{2, 3\}$, a contradiction. Therefore $\chi_S(H_{12}) = 4$. Now we give an S-packing 3-coloring ϕ of $H_{12} - x$ for any $x \in V(H_{12})$. We define ϕ as "1 2 1 3 1 2 1", "3 2 1 3 1 2 1", "3 1 1 3 1 2 1", "3 1 2 3 1 2 1", "3 1 2 1 1 2 1", "2 1 3 1 2 2 1", "2 1 3 1 2 1 1", "1 2 1 3 1 2 1" when $x = a, b, c, d, e, f, g, h$, respectively.

If $G = H_{13}$, then the pattern "1 2 1 3 1 2 1 4" is an S-packing 4-coloring. Hence $\chi_S(H_{13}) \leq 4$. Suppose that H_{13} admits an S-packing 3-coloring φ, then $\{\varphi(b), \varphi(e)\} = \{2, 3\}$. Then $\varphi(c) = \varphi(d) = 1$, a contradiction. Therefore $\chi_S(H_{13}) = 4$. Now we give an S-packing 3-coloring ϕ of $H_{13} - x$ for any $x \in V(H_{13})$. We define ϕ as "1 3 1 2 1 1 1","1 3 1 2 1 1 1", "2 1 3 1 2 1 1" when $x = b, c, g$, respectively.

If $G = H_{14}$, then the pattern "2 1 3 1 2 1 4" is an S-packing 4-coloring of H_{14}. Hence $\chi_S(H_{14}) \leq 4$. Suppose that H_{14} admits an S-packing 3-coloring φ, then $\{\varphi(c), \varphi(d)\} = \{2, 3\}$. Thus $\varphi(a) = \varphi(c) > 1$ or $\varphi(a) = \varphi(d) > 1$, a contradiction. Therefore $\chi_S(H_{14}) = 4$. Now we give an S-packing 3-coloring ϕ of $H_{14} - x$ for any $x \in V(H_{14})$. We define ϕ as "1 2 3 1 1 1","2 2 3 1 1 1", "1 2 3 1 1 1", "2 1 3 2 1 1","1 2 1 3 1 1","2 1 3 1 2 1", when $x = a, b, c, d, f, g$, respectively.

Finally, if $G = H_{15}$, then the pattern "4 1 2 1 3 1 1" is an S-packing 4-coloring of H_{15}. Hence $\chi_S(H_{15}) \leq 4$. Suppose that H_{15} admits an S-packing 3-coloring φ, then $\{\varphi(c), \varphi(e)\} = \{2, 3\}$ and $\varphi(b) = \varphi(d) = 1$. Thus $\varphi(a) \in \{2, 3\}$, a contradiction. Now we give an S-packing 3-coloring ϕ of $H_{15} - x$ for any $x \in V(H_{15})$. We define ϕ as "1 2 1 3 1 1", "1 1 2 3 1 1 ", "2 1 1 3 1 1", "1 3 1 2 1 1", "2 1 3 1 1 2", when $x = a, b, c, e, f$, respectively.■

Lemma 3.4 *If $S \in \mathcal{S}_{1,\bar{4}} \cup \mathcal{S}_{1,3,\bar{4}}$, G is a 4-χ_S-vertex-critical graph with at least one cycle, and C is a longest cycle of G, then the following hold.*

(a) If $n(C) = 3$, then $G \in \mathcal{H}_{2-4}$.

(b) If $n(C) = 4$ and C contains a chord, then $G \in \{K_4, H_1\}$.

(c) If $n(C) \in \{5, 6\}$, then $G \in \{C_{n(C)}\} \cup \mathcal{C}_{n(C)}$.

Proof. Let $S \in \mathcal{S}_{1,\bar{4}} \cup \mathcal{S}_{1,3,\bar{4}}$. Note that the graphs from Lemma 3.1 are 4-χ_S-vertex-critical. Let now G be 4-χ_S-vertex-critical and C its longest cycle.

(a) Suppose $n(C) = 3$. Let $V(C) = \{a, b, c\}$. We first assume that G contains only one triangle. If H_3 or H_4 is a subgraph of G, then we actually have $G = H_3$ or $G = H_4$, for otherwise we find another triangle in G or a cycle longer than 3. If $d_G(v) \geq 3$ holds for each vertex of C, then $G = H_3$ since H_3 is 4-χ_S-vertex-critical. If $d_G(v) = 2$ for some $v \in \{a, b, c\}$, then assume without loss of generality that $d_G(a) = 2$. If $N_G^2(b) \setminus N_G(c) = \emptyset$ and $N_G^2(c) \setminus N_G(b) = \emptyset$, then $V(G) \setminus \{b, c\}$ is an independent set in G, and so a coloring φ with $\varphi(b) = 2$, $\varphi(c) = 3$ and other vertices with color 1 is an S-packing 3-coloring of G, a contradiction. So $N_G^2(b) \setminus N_G(c) \neq \emptyset$ or $N_G^2(c) \setminus N_G(b) \neq \emptyset$. Since H_4 is 4-χ_S-vertex-critical, $G = H_4$.

Suppose secondly that there are at least two triangles in G. Since H_4 is χ_S-vertex-critical, the triangles in G have exactly one common vertex, for otherwise H_4 is a proper subgraph of G. This implies that H_2 is a spanning subgraph of G. Since $n(C) = 3$, we conclude that $G = H_2$.

(b) Suppose $n(C) = 4$. Let $C = abcda$. If $ac \in E(G)$ and $bd \in E(G)$, then $G = K_4$ by Lemma 3.1. Suppose $bd \in E(G)$. If there is a vertex $x \in N_G(b) \setminus V(C)$ such that $N_G(x) \setminus V(C) \neq \emptyset$, then $H_4 \subseteq G - a$, a contradiction. Therefore, for any vertex $x \in (N_G(b) \cup N_G(d)) \setminus V(C)$ we have $N_G(x) \setminus V(C) = \emptyset$. If $d_G(a) = d_G(c) = 2$, then $V(G) \setminus \{b, d\}$ is an independent set in G, and so a mapping φ with $\varphi(b) = 2$, $\varphi(d) = 3$ and $\varphi(N_G(b) \cup N_G(d) \setminus \{b, d\}) = 1$ is an S-packing 3-coloring of G, a contradiction. Thus $d_G(a) \geq 2$ or $d_G(c) \geq 2$. It implies that H_1 is a subgraph of G. Since $n(C) = 4$ and by Lemma 3.1 H_1 is 4-χ_S-vertex-critical, we have $G = H_1$.

(c) Suppose finally that $n(C) \in \{5, 6\}$. Since C is 4-χ_S-vertex-critical, C is a spanning subgraph of G. If $n(C) = 5$, then since K_4 and all the four graphs from $\mathcal{C}_5$ are 4-χ_S-vertex-critical, $\mathcal{C}_5$ is the family of 4-χ_S-vertex-critical graphs that contain C_5 as a proper spanning subgraph. If $n(C) = 6$, then since C_5 is 4-χ_S-vertex-critical, any two vertex at distance 2 are not adjacent in C_6. Hence $\mathcal{C}_6$ is the families of 4-χ_S-vertex-critical graphs that contain C_6 as a proper spanning subgraph by Lemma 3.1. Therefore if $n(C) \in \{5, 6\}$ and G is 4-χ_S-vertex-critical, then $G \in \{C_{n(C)}\} \cup \mathcal{C}_{n(C)}$. ■

We can now state out first characterization.

Theorem 3.5 *Let $S \in \mathcal{S}_{1,\overline{4}}$. Then a graph G is 4-χ_S-vertex-critical if and only if*

$$G \in \{K_4, C_5, C_6, P_6\} \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{H}_{1-8}\,.$$

Proof. Let $S \in \mathcal{S}_{1,\overline{4}}$ and let G be 4-χ_S-vertex-critical. First suppose that G contains a cycle, and let C be a longest cycle of G. Since P_6 is 4-χ_S-vertex-critical, Lemma 3.2 implies $n(C) \leq 6$. By Lemma 3.4 and the fact that $\chi_S(C_4) = 3$, it remains to consider the case in which $n(C) = 4$, $n(G) \geq 5$, and there is no chord in C. Let $C = abcda$. Since $\chi_S(C) \leq 3$, there is a vertex $w \in V(C)$ such that $N_G(w) \setminus V(C) \neq \emptyset$. Let $w_1 \in N_G(w) \setminus V(C)$.

First suppose $N_G(w_1) \setminus V(C) \neq \emptyset$. We may assume that $w = c$ and $w_1 = e$. Let $f \in N_G(e) \setminus V(C)$. Then H_6 is subgraph of G. By Lemma 3.2, H_6 is a spanning subgraph of G. Since G is C_k-free for $k \geq 5$, at most one of the edges $\{ae, cf\}$ can be possibly contained in G. If $ae \notin E(G)$ and $cf \notin E(G)$, then $G = H_6$ by Lemma 3.2. If $ae \in E(G)$, then $G = H_7$ by Lemma 3.1. If $cf \in E(G)$, then $H_4 \subseteq G - b$, a contradiction.

Thus we may assume that $N_G(w_1) \setminus V(C) = \emptyset$ for each $w_1 \in N_G(w) \setminus V(C)$. It implies that $N_G(u) \setminus V(C)$ is an independent set for any $u \in V(C)$. If $N_G(b) \cup N_G(d) \setminus V(C) = \emptyset$, then $V(G) \setminus \{a, c\} = N(a) \cup N(c)$ is an independent set in G, and so a mapping φ with $\varphi(a) = 2$, $\varphi(c) = 3$ and $\varphi(N(a) \cup N(c)) = 1$ is an S-packing 3-coloring of G, a contradiction. Thus $N_G(b) \cup N_G(d) \setminus V(C) \neq \emptyset$ and $N_G(a) \cup N_G(c) \setminus V(C) \neq \emptyset$, and so H_5 is a spanning subgraph of G by Lemma 3.1. If some edge from $\{af, de\}$ or from $\{ef, cf, be\}$ is contained in G, then $H_4 \subseteq G - y$ for some $y \in V(G)$ or $C_k \subseteq G$ with $k \geq 5$, a contradiction. Therefore, only one of df and ae can be contained in G, and so $G \in \mathcal{H}_{5,7}$ by Lemma 3.1.

Suppose now that G is acyclic. If P is a longest path in G, then $n(P) \leq 6$ by Lemma 3.2. If $n(P) = 6$, then $G = P_6$. If $n(P) = 5$, then let $P_5 = abcde$. If $d_G(c) = 2$, then the mapping φ with $\varphi(b) = 2$, $\varphi(d) = 3$ and $\varphi(N_G(b) \cup N_G(d)) = 1$ is an S-packing 3-coloring of G which implies that $\chi_S(G) \leq 3$, a contradiction. Therefore $d_G(c) \geq 3$. But then $G = H_8$ by Lemma 3.2. If $n(P) \leq 4$, then we have that $\chi_S(G) \leq 3$, so we get no new graph. ■

Theorem 3.6 *Let $S \in \mathcal{S}_{1,3,\overline{4}}$ and let G be a graph with a cycle. Then G is 4-χ_S-vertex-critical if and only if*

$$G \in \{K_4, C_5, C_6, C_8\} \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{H}_{1-5,7,9,11,15}\,.$$

Proof. Let $S \in \mathcal{S}_{1,3,\overline{4}}$ and let C be a longest cycle of G. Since P_8 is 4-χ_S-vertex-critical, $n \leq 8$. If $n(C) = 8$, then C is a spanning subgraph of G by Lemma 3.3. Since $\chi_S(C_5) = \chi_S(C_6) = 4$, and $\chi_S(C_7) = 7 - \alpha(C_7) + 1 > 4$ by Proposition 2.3, there is no chord in C. Therefore $G = C$ when $n(C) = 8$. Since $\chi_S(C_7) = 5$, we have $n(C) \neq 7$. By Lemma 3.4 and the fact $\chi_S(C_4) = 3$ it remains to consider the case that $n(C) = 4$, $n(G) \geq 5$, and there is no chord in C.

Let $C = abcda$. First, suppose that there is an edge in $E(C)$, say bc, such that $d_G(b) \geq 3$ and $d_G(c) \geq 3$. Then $N_G(b) \cap N_G(c) = \emptyset$, otherwise, G has a cycle of length at least 5. It implies that H_5 is a spanning subgraph of G by Lemma 3.1 because there is no chord in C. If af or de is contained in G, then $H_4 \subseteq G - y$ for some $y \in V(G)$. Hence at most one of df and ae can be added to G, therefore $G \in \mathcal{H}_{5,7}$ by Lemma 3.1.

Now consider the case in which $d_G(s) = 2$ or $d_G(t) = 2$ for each edge $st \in E(C)$. Without loss of generality, suppose that $d_G(b) = 2$ and $d_G(d) = 2$. Let P be a longest path with endpoint c, such that $a, b, d \notin V(P)$, and let P' be a longest path with endpoint a, such that $c, b, d \notin V(P')$. Without loss of generality assume that $n(P) \geq n(P')$. If $n(P) \geq 3$ and $V(P) \cap V(P') \neq \emptyset$, then $G = H_7$. Indeed, for otherwise by the definition of P and P' we have $a, c \notin V(P) \cap V(P')$, and then for some $k \geq 5$ we have $C_k \subseteq G - b$, a contradiction. In the rest of the proof we may thus assume that if $n(P) \geq 3$, then $V(P) \cap V(P') = \emptyset$. Since P_8 is 4-χ_S-critical, $n(P) + n(P') \leq 6$.

Claim. If $n(P) \leq 4$ and $n(P') \leq 2$, then $G = H_{15}$.

Proof. Since $n(P') \leq 2$, we infer that if $x \in N_G(a) \setminus N_G(c)$ and $y \in N_G(a) \cap N_G(c)$, then $d_G(x) = 1$ and $d_G(y) = 2$. Hence $N_G(a)$ is an independent set in G. If there is a vertex $x \in N_G(c) \setminus N_G(a)$ such that $d_G(x) \geq 3$, then $H_{15} \subseteq G$. Since H_{15} is 4-χ_S-critical by Lemma 3.3, H_{15} is a spanning subgraph of G. Since $n(P') \leq 2$ and $d_G(b) = d_G(d) = 2$, only edges in $\{fg, cf\}$ possibly contained in G. If an edge from fg or cf is contained in G, then there is a vertex $v \in G$ such that $H_4 \subseteq G - v$, a contradiction. Hence $G = H_{15}$. It remains to consider the case in which $d_G(x) \leq 2$ holds for each $x \in N_G(c)$. Then $N_G(c)$ is an independent set in G, for otherwise $H_4 \subseteq G - b$, a contradiction. Since $n(P) \leq 4$ and $d_G(x) \leq 2$ for every $x \in N_G(c)$, the second neighborhood $N^2_G(c)$ is an independent set and $d_G(y) = 1$ for each $y \in N^3_G(c)$. (It is possible that $N^3_G(c) = \emptyset$.) Then a mapping φ with $\varphi(c) = 3$, $\varphi(N_G(c) \cup N^3_G(c)) = 1$, and $\varphi(N^2_G(c)) = 2$, is an S-packing 3-coloring of G. This contradiction proves the claim. □

It remains to consider the following two cases: (i) $n(P) = 5$, $n(P') = 1$, and (ii) $n(P) = n(P') = 3$. If $n(P) = 5$, then H_9 is a spanning subgraph of G. (The vertices of H_9 are denoted as in Fig. 2.) If some edge from $\{cf, eg, fh\}$ or ch or cg is contained in G, then $H_4 \subseteq G - b$ or $C_5 \subseteq G - b$ or $H_5 \subseteq G - b$, respectively, a contradiction. Now we only need to check that whether eh can be added to H_9. The graph obtained from H_9 by adding the edge eh is H_{10}, cf. Fig. 2 again. Then $H_{15} \subseteq H_{10} - g$, a contradiction. Hence $G = H_9$. If $n(P) = 3$ and $n(P') = 3$, then $H_{11} \subseteq G$. By symmetry, if some edge from $\{af, ge, gf\}$ or ah or ae is contained in G, then $C_k \subseteq G - b$ with $k \geq 5$ or $H_4 \subseteq G - b$ or $H_5 \subseteq G - b$, respectively, a contradiction. Since H_{11} is 4-χ_S-vertex-critical, no additional edge can be added to H_{11}. We conclude that $G = H_{11}$. ■

It remains to consider acyclic graphs for $S \in \mathcal{S}_{1,3,\overline{4}}$.

Theorem 3.7 *Let $S \in \mathcal{S}_{1,3,\overline{4}}$ and let G be an acyclic graph. Then G is 4-χ_S-vertex-critical if and only if $G \in \{P_8\} \cup \mathcal{H}_{12-14}$.*

Proof. Let G be 4-χ_S-vertex-critical and acyclic. Denote by P a longest path in G. If $n(P) = 8$, then Lemma 3.3 implies that $G = P_8$. Since $\chi_S(G) \leq 3$ when $n(P) \leq 4$, it remains to consider the cases $5 \leq n(P) \leq 7$.

Suppose $n(P) = 5$ and let $P = abcde$. If $d_G(c) = 2$, then a coloring φ with $\varphi(\{c\} \cup N_G^2(c)) = 1$, $\varphi(b) = 2$, and $\varphi(d) = 3$ is an S-packing 3-coloring of G, contradicting the fact that $\chi_S(G) = 4$, hence $d_G(c) \geq 3$. If $d_G(x) \leq 2$ for any $x \in N_G(c)$, then the coloring φ with $\varphi(N_G(c)) = 1$, $\varphi(N_G^2(c)) = 2$, and $\varphi(c) = 3$ is an S-packing 3-coloring of G, a contradiction. So $G = H_{14}$ by Lemma 3.3.

Suppose $n(P) = 6$ and let $P = abcdef$. Then either $d_G(s) = 2$ or $d_G(t) = 2$ for $st \in E(P) \setminus \{ab, ef\}$, otherwise there is a vertex $x \in V(G)$ such that $H_{14} \subseteq G - x$. If $d_G(c) \geq 3$, then a mapping φ with $\varphi(N_G(c) \cup N_G^3(c)) = 1$, $\varphi(N_G^2(c)) = 2$, and $\varphi(c) = 3$ is an S-packing 3-coloring of G, a contradiction. Thus $d_G(c) = d_G(d) = 2$. If $d_G(b) = 2$, then a mapping φ with $\varphi(N_G(e) \cup N_G^3(e)) = 1$, $\varphi(a) = \varphi(e) = 2$, and $\varphi(c) = 3$ is an S-packing 3-coloring of G. Thus $d_G(b) \geq 3$ and $d_G(e) \geq 3$. Hence $G = H_{13}$ by Lemma 3.3.

Let finally $P = abcdefg$. If $d_G(x) = 2$ for any $x \in N_G(d)$, a mapping φ with $\varphi(N_G(d) \cup N_G^3(d)) = 1$, $\varphi(N_G^2(d)) = 2$, and $\varphi(d) = 3$ is an S-packing 3-coloring of G contradicting the fact $\chi_S(G) = 4$. Hence $G = H_{12}$ by Lemma 3.3. ■

Combining Theorem 3.7 with Theorem 3.6 we get:

Corollary 3.8 *Let $S \in \mathcal{S}_{1,3,\overline{4}}$ and let G be a graph. Then G is 4-χ_S-vertex-critical if and only if*

$$G \in \{K_4, C_5, C_6, C_8, P_8\} \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{H}_{1-5,7,9,11-15}\,.$$

3.2 4-χ_S-vertex-critical graphs for $S \in \mathcal{S}_{1,3,3}$

In this subsection we consider packing sequences $S \in \mathcal{S}_{1,3,3}$. In Lemmas 3.9, 3.10, and 3.11, we present some graphs that are 4-χ_S-vertex-critical. After that we characterize 4-χ_S-vertex-critical graphs by distinguishing the distance between vertices of degree at least 3.

Lemma 3.9 *If $S \in \mathcal{S}_{1,3,3}$, then the following hold.*

(a) If $n \geq 4$, then $\chi_S(P_n) = 3$.

(b) Let $n \geq 3$. If $n = 3$ or $n \equiv 0 \bmod 4$, then $\chi_S(C_n) = 3$. If $n \equiv 1, 2 \bmod 4$, or $n \equiv 3 \bmod 4$ and $s_4 < \lfloor n/2 \rfloor$, then $\chi_S(C_n) = 4$; otherwise, $\chi_S(C_n) = 5$. Moreover, C_n is χ_S-critical when $n \not\equiv 0 \bmod 4$ and $n \geq 5$.

Proof. (a) Note that $\chi_S(P_n) \geq 3$ for $n \geq 4$. The pattern "1 2 1 3 1 2 1 3 . . ." is an S-packing 3-coloring of P_n. Thus $\chi_S(P_n) = 3$ for $n \geq 4$.

(b) First, $\chi_S(C_n) \geq 3$ for $n \geq 3$. The pattern "1 2 3" gives an S-packing 3-coloring of C_3 and the pattern "1 2 1 3 1 2 1 3 . . . 1 2 1 3" gives an S-packing 3-coloring of C_n when $n \equiv 0 \bmod 4$. Thus $\chi_S(C_n) = 3$ when $n = 3$ or $n \equiv 0 \bmod 4$.

Next, if $n \geq 4$ and $n \not\equiv 0 \bmod 4$, then since $(1, 3, 3) \succeq (1, 2, 3)$, we have $\chi_S(C_n) \geq 4$ by Proposition 2.1. The pattern "1 2 1 3 1 2 1 3 . . . 1 2 1 3 4" gives an S-packing 4-coloring of C_n when $n \equiv 1 \bmod 4$ and the pattern "1 2 1 3 1 2 1 3 . . . 1 2 1 3 1 4" gives an S-packing 4-coloring of C_n when $n \equiv 2 \bmod 4$. Thus $\chi_S(C_n) = 4$ when $n \equiv 1, 2 \bmod 4$.

Consider now the case $n \equiv 3 \bmod 4$. When $n = 4k + 3$, $n \geq 7$, and $s_4 < \lfloor n/2 \rfloor$, we give an S-packing 4-coloring φ of $C_n = v_0 v_1 \dots v_{n-1} v_0$ as:

$$\varphi(v_i) = \begin{cases} 1; & (i \equiv 0 \bmod 4) \text{ or } (i \equiv 2 \bmod 4 \text{ and } i \neq 4k+2), \\ 2; & (i \equiv 3 \bmod 4 \text{ and } i < 2k+1) \text{ or } (i \equiv 1 \bmod 4 \text{ and } i > 2k+1), \\ 3; & (i \equiv 1 \bmod 4 \text{ and } i < 2k+1) \text{ or } (i \equiv 3 \bmod 4 \text{ and } i > 2k+1), \\ 4; & i \in \{2k+1, 4k+2\}. \end{cases}$$

Hence $\chi_S(C_n) = 4$ when $n \equiv 3 \bmod 4$, $n \geq 7$, and $s_4 < \lfloor n/2 \rfloor$.

When $n = 4k + 3$, $n \geq 7$, and $s_4 \geq \lfloor n/2 \rfloor$, the pattern "1 2 1 3 1 2 1 3. . . 1 2 1 3 1 4 5" is an S-packing 5-coloring of C_n. Hence $4 \leq \chi_S(C_n) \leq 5$. Now suppose that there is an S-packing 4-coloring φ of C_n. Since $s_4 \geq \lfloor n/2 \rfloor$, we have $|\varphi^{-1}(4)| = 1$. Without loss of generality we may assume that $\varphi(v_0) = 4$. We claim that for any edge in C_n which is not incident with v_0, one of its endpoints receives color 1. Suppose on the contrary that there is an edge $v_i v_{i+1} \in E(C_n)$ such that $\{\varphi(v_i), \varphi(v_{i+1})\} = \{2, 3\}$, where $1 \leq i \leq n-2$. Since $n \geq 7$, one of v_{i-2} and v_{i+3} (indices taken modulo n) cannot be colored under φ.

This contradiction proves the claim. Since $s_2 = s_3 = 3$, we only need to consider two cases: $\varphi(v_1) = 2$ and $\varphi(v_1) = 1$. If $\varphi(v_1) = 2$, then the colors of $v_0, v_1, \ldots, v_{4k+2}$ under φ can be described as the pattern "4 2 1 3 1 2 1 3 1…2 1 3 1 2 1". We have $\varphi(v_1) = \varphi(v_{4k+1}) = 2$ with $d_{C_n}(v_1, v_{4k+1}) = 3 \leq s_2$, a contradiction. If $\varphi(v_1) = 1$, then we may without loss of generality assume $\varphi(v_2) = 2$. Then the colors of $v_0, v_1, \ldots, v_{4k+2}$ under φ can be described as the pattern "4 1 2 1 3 1 2 1 3…1 2 1 3 1 2". However, we have $\varphi(v_2) = \varphi(v_{4k+2}) = 2$ with $d_{C_n}(v_2, v_{4k+2}) = 3 \leq s_2$, a contradiction. Therefore $\chi_S(C_n) = 5$ when $n \equiv 3 \bmod 4$, $n \geq 7$, and $s_4 \geq \lfloor n/2 \rfloor$.

If $n \not\equiv 0 \bmod 4$, then C_n is χ_S-critical because $\chi_S(P_n) \leq 3$ and $\chi_S(C_n) \geq 4$. ■

Let G_{2k}, $k \geq 3$, be the graph obtained from the path P_{2k} by attaching a pendent vertex to each of the two support vertices of P_{2k}. Equivalently, G_{2k} is obtained from P_{2k-2} by attaching two pendant vertices to each of the two leaves of P_{2k-2}.

Lemma 3.10 *If $S \in \mathcal{S}_{1,3,3}$ and $k \geq 3$, then G_{2k} is 4-χ_S-vertex-critical.*

Proof. Let $P_{2k} = v_1 v_2 \ldots v_{2k}$, and let v'_2 and v'_{2k-1} be the pendent vertices attached to v_2 and v_{2k-1}, respectively. Coloring the vertices of P_{2k} with the pattern "1 2 1 3 1 2 1 3…" and the vertices v'_2 and v'_{2k-1} with 1 and 4, respectively, we get $\chi_S(G_{2k}) \leq 4$.

Suppose now that G_{2k} admits an S-packing 3-coloring φ. Observe that $\varphi(v_2) \in \{2, 3\}$, without loss of generality assume that $\varphi(v_2) = 2$. Then we have $\varphi(v_1) = 1$ and $\varphi(v_3) = 1$, for otherwise $\varphi(v_3) = \varphi(v_4) = 1$ or $\varphi(v_4) = \varphi(v_5) = 1$. If $2 \leq i \leq 2k-2$, then at least one of v_i and v_{i+1} must be colored 1. Indeed, if we would have $\varphi(v_i) = 2$ and $\varphi(v_{i+1}) = 3$, then v_{i-2} or v_{i+3} can not be colored under φ. Thus we have $\varphi(v_{2k-2}) = 2$ and $\varphi(v_{2k-1}) = 1$, or $\varphi(v_{2k-2}) = 3$ and $\varphi(v_{2k-1}) = 1$. However, this implies that v'_{2k-1} or v_{2k} can not be colored under φ, a contradiction. Hence, $\chi_S(G_{2k}) = 4$.

If $v \in G_{2k}$, then an S-packing 3-coloring of $G_{2k} - v$ can be given by coloring a longest path of each component of $G_{2k} - v$ with either the pattern "1 2 1 3 …" or the pattern "2 1 3 1 …" and coloring the pendent vertices with 1. Therefore G_{2k} is 4-χ_S-vertex-critical.■

Lemma 3.11 *If $S \in \mathcal{S}_{1,3,3}$, then the graphs H_{14} and H_{15} are 4-χ_S-vertex-critical.*

Proof. Since H_{14} and H_{15} are 4-$\chi_{S'}$-vertex-critical, where $S' \in \mathcal{S}_{1,3,\overline{4}}$, and $(1,3,4) \succeq (1,3,3)$, it suffices to show that $\chi_S(H_{14}) = \chi_S(H_{15}) = 4$. The pattern "4 1 2 3 1 1 1" is an S-packing 4-coloring of H_{14}, so $\chi_S(H_{14}) \leq 4$. Suppose that H_{14} admits an S-packing 3-coloring φ. Then $\{\varphi(c), \varphi(d)\} = \{2, 3\}$, and so the vertex a cannot be colored under φ. It follows that $\chi_S(H_{14}) = 4$. The pattern "4 1 2 1 3 1 1" is an S-packing 4-coloring of H_{15}. Moreover, since $H_{14} \subseteq H_{15}$, we conclude that $\chi_S(H_{15}) = 4$. ■

Our next result, Theorem 3.14, follows from the following lemma and theorem.

Lemma 3.12 *Let $S \in \mathcal{S}_{1,3,3}$ and let $n \not\equiv 0 \bmod 4$, $n > 3$. If a graph G contains a cycle C_n and $V(G) - V(C_n) \neq \emptyset$, then G is not 4-χ_S-critical.*

Proof. Since $V(G) - V(C_n) \neq \emptyset$, there exists a vertex $x \in V(G)$ such that $C_n \subseteq G - x$. By Lemma 2.2, we have $\chi_S(G - x) \geq \chi_S(C_n) \geq 4$, and so G is not 4-χ_S-critical. ■

Theorem 3.13 [18, Theorem 4.3] *If G is a graph that contains a cycle of length $n \geq 5$, where $n \not\equiv 0 \bmod 4$, then G is 4-χ_ρ-vertex-critical if and only if one of the following holds.*

- *$n = 5$ and $G \in \{C_5\} \cup \mathcal{C}_5$,*
- *$n = 6$ and $G \in \{C_6\} \cup \mathcal{C}_6$,*
- *$n \geq 7$ and G is isomorphic to C_n.*

Theorem 3.14 *Let $S \in \mathcal{S}_{1,3,3}$. If G is a graph that contains a cycle of length $n \geq 5$, where $n \not\equiv 0 \bmod 4$, then G is 4-χ_S-vertex-critical if and only if one of the following holds.*

- *$n = 5$ and $G \in \{C_5\} \cup \mathcal{C}_5$,*
- *$n = 6$ and $G \in \{C_6\} \cup \mathcal{C}_6$,*
- *$n \geq 7$ and $G = C_n$ except when $n \equiv 3 \bmod 4$ and $s_4 \geq \lfloor n/2 \rfloor$.*

In order to characterize 4-χ_S-vertex-critical graphs, where $S \in \mathcal{S}_{1,3,3}$, we need to distinguish whether there are two vertices of degree at least 3 that are at odd distance. For this sake we need the following classes of cycles that depend on a positive integer s_4 (this s_4 will, of course, be the fourth component of a packing sequence S):

$$\mathcal{C}_{s_4} = \{C_n, n \geq 5 : \ (n \equiv 1, 2 \bmod 4) \text{ or } (n \equiv 3 \bmod 4 \text{ and } s_4 < \lfloor n/2 \rfloor\}\,.$$

Theorem 3.15 *Let $S \in \mathcal{S}_{1,3,3}$ and let G be a 4-χ_S-vertex-critical graph. If all the vertices of G of degree at least 3 are pairwise at even distances in G, then $G \in \{H_2, H_4\} \cup \mathcal{C}_{s_4}$.*

Proof. By Lemmas 3.9 and 3.1, every graph from $\{H_2, H_4\} \cup \mathcal{C}_{s_4}$ is 4-χ_S-vertex-critical. If $\Delta(G) \leq 2$, then $G \in \{P_n, C_n\}$, hence $G \in \mathcal{C}_{s_4}$. Suppose now that $\Delta(G) \geq 3$ and that all the vertices of degree at least 3 are pairwise at even distances in G. Let $u \in V(G)$ be an arbitrary vertex of degree at least 3. Then define $\varphi : V(G) \to [3]$ by:

$$\varphi(v) = \begin{cases} 1; & d_G(u,v) \equiv 1, 3 \bmod 4, \\ 2; & d_G(u,v) \equiv 0 \bmod 4, \\ 3; & d_G(u,v) \equiv 2 \bmod 4. \end{cases}$$

By Lemma 2.2, G is a connected graph, and so φ is well-defined. Since G is 4-χ_S-vertex-critical, there are two vertices $x, y \in V(G) \setminus \{u\}$ such that $\varphi(x) = \varphi(y) = i$ and $d_G(x, y) \leq s_i$ for some $i \in [3]$. Let P and P' be arbitrary shortest u, x-path and u, y-path in G, respectively. Let $w \in V(P) \cap V(P')$ such that $d_G(u, w)$ is as large as possible. Then we have $w \neq x, y$ and $d_G(u, x) = d_G(u, y)$. If $w = x$ or $d_G(u, x) < d_G(u, y)$, then $d_G(u, y) \leq d_G(u, x) + s_i < d_G(u, x) + 4$, and so $\varphi(x) \neq \varphi(y)$ by the definition of φ, which leads to a contradiction. Thus we have $d_G(w) \geq 3$, and so $\varphi(w) \in \{2, 3\}$.

Claim. G contains a cycle consisting of wPx, $wP'y$, and a shortest x, y-path.

Proof. Let P'' be a shortest x, y-path. By the choice of w, it suffices to show that $(V(P'') \cap V(P)) \setminus \{x\} = (V(P'') \cap V(P')) \setminus \{y\} = \emptyset$.

Suppose that $(V(P'') \cap V(P)) \setminus \{x\} \neq \emptyset$. Note that this can only happen when $\varphi(x) = \varphi(y) \in \{2, 3\}$ and $d_G(x, y) \in \{2, 3\}$. Without loss of generality, let $\varphi(x) = \varphi(y) = 3$. If $d_G(x, y) = 2$, let $P'' = xzy$. Then $d_G(z) \geq 3$ and $d_G(u, z) = d_G(u, x) - 1 \equiv 1 \bmod 4$, contradicting the fact that the vertices of degree at least 3 are at even distance. For $d_G(x, y) = 3$, let $P'' = xz_1z_2y$. Then $xz_2 \notin E(G)$. If $z_2 \in V(P'') \cap V(P)$, then $d_G(u, y) \leq d_G(u, z_2) + d_G(z_2, y) = d_G(u, x) - 2 + 1 < d_G(u, x)$, contradicting the fact that $d_G(u, x) = d_G(u, y)$. Therefore $z_2 \notin V(P'') \cap V(P)$. However, if $z_1 \in V(P'') \cap V(P)$, then $d_G(z_1) \geq 3$ and $d_G(u, z_1) = d_G(u, x) - 1 \equiv 1 \bmod 4$, a contradiction. Therefore G contains a cycle, say C_n, consisting of wPx, $wP'y$ and a shortest x, y-path, and so $n = 2d_G(u, x) - 2d_G(u, w) + d_G(x, y)$. □

Suppose $\varphi(x) = \varphi(y) = 1$ and $d_G(x, y) = 1$. Then we have $n \equiv 3 \bmod 4$. If $n \equiv 3 \bmod 4$ and $n > 3$, then $G = C_n$ with $s_4 < \lfloor n/2 \rfloor$ by Theorem 3.14. If $n = 3$, then wxy is a triangle with $d_G(w) \geq 3$ and $d_G(x) = d_G(y) = 2$. If $N_G^2(w) \neq \emptyset$, then $G = H_4$. If $N_G^2(w) = \emptyset$ and $N_G(w) \setminus \{x, y\}$ is an independent set, then a mapping φ with $\varphi(N_G(w) \setminus \{x, y\}) = \varphi(x) = 1$, $\varphi(w) = 2$ and $\varphi(y) = 3$ is an S-packing 3-coloring of G. Moreover, it is easy to see that no edge can be added to H_2 and to H_4. Hence $G \in \{H_2, H_4\}$ when $n = 3$. Suppose $\varphi(x) = \varphi(y) = 2$ or $\varphi(x) = \varphi(y) = 3$. If $d_G(x, y) = i$, $i \in [3]$, then $n \equiv i \bmod 4 \geq 5$ because $d_G(u, x)$ and $d_G(u, w)$ are even. Therefore $G \in \mathcal{C}_{s_4}$ by Theorem 3.14. ■

Theorem 3.16 *Let $S \in \mathcal{S}_{1,3,3}$ and let G be a 4-χ_S-vertex-critical graph in which there exist two vertices with degree at least 3 that are at odd distance. Then*

$$G \in \{K_4\} \cup \mathcal{H}_{1,3,5,7,14,15} \cup \{G_{2k} : \ k \geq 3\} \cup \mathcal{C}_5 \cup \mathcal{C}_6 \,.$$

Proof. Let $u, v \in V(G)$ with $d_G(u), d_G(v) \geq 3$ such that $d_G(u, v) = \ell$ is odd and as small as possible. We consider the following two cases.

Case 1: $\ell \geq 3$.

By the choice of u and v, $N_G(u) \cap N_G(v) = \emptyset$ and each vertex on a shortest u,v-path has degree 2 in G. Therefore $G_{\ell+3} = G_{2k}$ is a spanning subgraph of G, where $k = \frac{\ell+3}{2}$ by Lemma 3.10. Let the vertices of G_{2k} be denoted as in Lemma 3.10 with $u = v_2$ and $v = v_{2k-1}$. By symmetry, only some of the edges v_1v_2', v_1v_{2k}, and v_1v_{2k-1} can possibly be added to G_{2k}. If $v_1v_2' \in E(G)$, then $H_4 \subseteq G - v_{2k}$, a contradiction. If $v_1v_{2k} \in E(G)$, then $C_{2k} \subseteq G$. Further, we have $2k \equiv 0 \bmod 4 \geq 8$, for otherwise $\chi_S(C_{2k}) = 4$ by Theorem 3.14 and $C_{2k} \subseteq G - v_2'$. Then we can find a copy of G_6 consisting of $v_3v_2v_1v_{2k}v_{2k-1}v_{2k-2}$ and the pendent vertices v_2' and v_{2k-1}' contained in $G - v_4$, which also leads to a contradiction. If $v_1v_{2k-1} \in E(G)$, then there is a cycle $C_{2k-1} \subseteq G - v_2'$ with $2k-1 \not\equiv 0 \pmod 4 > 3$, again a contradiction. We conclude that in Case 1, $G = G_{2k}$.

Case 2: $\ell = 1$.

We claim that in this case, $G \in \{K_4\} \cup \mathcal{H}_{1,3,5,7,14,15} \cup \mathcal{C}_5 \cup \mathcal{C}_6$.

If G contains a cycle from $\mathcal{C}_{s_4}$, then $G \in \mathcal{C}_5 \cup \mathcal{C}_6$ by Theorem 3.14 since there are two vertices of degree degree at least 3 which are of distance 1 in G. Thus we may assume G does not contain a cycle from $\mathcal{C}_{s_4}$ as a subgraph. Suppose G contains H_4 as a subgraph. Then H_4 is a spanning subgraph of G. Since G has two vertex of degree at least 3, $G \in \mathcal{C}_5 \cup H_1$. By the same argument, if G contains H_2 as a spanning subgraph, then $G \in \mathcal{C}_5$. Therefore we may assume G does not contain a graph from $\mathcal{C}_{s_4} \cup H_2 \cup H_4$ as a subgraph. Let $a = u$ and $c = v$.

Suppose that $|N_G(a) \cap N_G(c)| \geq 2$. If $d_G(x) = 2$ for any $x \in N_G(a) \cap N_G(c)$, then $V(G) \setminus \{a, c\}$ is an independent set in G since G does not contain H_2 or H_4 as a subgraph, and so a coloring φ of G with $\varphi(V(G) \setminus \{a, c\}) = 1$, $\varphi(a) = 2$ and $\varphi(c) = 3$ is an S-packing 3-coloring, a contradiction. Then either $bd \in E(G)$ for some $b, d \in N_G(a) \cap N_G(c)$ and so $G = K_4$, or there is a vertex $x \in N_G(a) \cap N_G(c)$ such that $N_G(x) \setminus (\{a, c\} \cup V(N_G(a) \cap N_G(c))) \neq \emptyset$ and so $H_1 \subseteq G$. Since G contains no cycle from $\mathcal{C}_{s_4}$, we infer that no more edges can be added to H_1. Hence $G = H_1$.

Suppose that $|N_G(a) \cap N_G(c)| = 1$, and let $b \in N_G(a) \cap N_G(c)$. If $d_G(b) = 2$, then since G does not contain H_2 and H_4 as a subgraph, a coloring φ of G with $\varphi(V(G) \setminus \{a, c\}) = 1$, $\varphi(a) = 2$, and $\varphi(c) = 3$ is an S-packing 3-coloring, a contradiction. Therefore, $d_G(b) \geq 3$. If $|N_G(b) \cap N_G(a)| \geq 2$ or $|N_G(b) \cap N_G(c)| \geq 2$, then $G \in \{K_4 \cup H_1\}$ because $d_G(a) \geq 3$, $d_G(c) \geq 3$, and $ac \in E(G)$. If $N_G(b) \cap N_G(a) = c$ and $N_G(b) \cap N_G(c) = a$, then $H_3 \subseteq G$ since $d_G(z) \geq 3$ for $z \in \{a, b, c\}$. Let d, e, f be the three vertices of H_3 as shown in Fig. 2. If $af \in E(G)$, then $H_1 \subseteq G - d$, a contradiction. If $df \in E(G)$, then $C_5 \subseteq G - e$, again a contradiction. Therefore $G = H_3$.

Lastly, consider the case when $N_G(a) \cap N_G(c) = \emptyset$. If $d_G(w) = 1$ for each $w \in (N_G(a) \cup N_G(c)) \setminus \{a, c\}$, then a mapping φ with $\varphi(V(G) \setminus \{a, c\}) = 1$, $\varphi(a) = 2$, and $\varphi(c) = 3$ is an

S-packing 3-coloring of G, contradicting the fact that $\chi_S(G) = 4$. Let $x_1 \neq x_2 \in N_G(a) \setminus \{c\}$ and $y_1 \neq y_2 \in N_G(c) \setminus \{a\}$. Without loss of generality assume that $N_G(x_1) \setminus \{a\} \neq \emptyset$. If $x_1x_2 \in E(G)$, then $H_4 \subseteq G - y_2$, a contradiction. Hence $x_1x_2 \notin E(G)$ and $y_1y_2 \notin E(G)$ by symmetry. If $x_1y_1 \in E(G)$, then $H_5 \subseteq G$. Moreover, H_5 is a spanning subgraph of G by Lemma 3.1. If $x_1y_1 \in E(G)$ and $x_2y_2 \in E(G)$, then C_6 is a proper subgraph of G, again a contradiction. If $x_1y_1 \in E(G)$ and only one of x_1y_2 and x_2y_1 is contained in G, then $H_7 \subseteq G$. Since there is no more edge which can be added to H_7, we get $G \in \mathcal{H}_{5,7}$ when $x_1y_1 \in E(G)$. If there is a vertex $x'_1 \in N(x_1) \setminus \{a, x_2, y_1, y_2\}$, then H_{14} is a spanning subgraph of G by Lemma 3.11. If one of the edges x'_1a, x_1x_2, and y_1y_2 is contained in G, there is a vertex $y \in V(G)$ such that $H_4 \subseteq G - y$, a contradiction. If one of the edges x'_1c, x_1y_1, x_1y_2, y_1x_2, and x_2y_2 is contained in G, there is a vertex $y \in V(G)$ such that $H_5 \subseteq G - y$, a contradiction. If $x'_1y_i \in E(G)$, then $C_5 \subseteq G - x_2$, a contradiction. Since $ay_i, cx_i \notin E(G)$ for $i \in [2]$, only x'_1x_2 can be possibly contained in G, and $H_{15} \subseteq G$ when $x'_1x_2 \in E(G)$. Moreover, since H_{15} is 4-χ_S-vertex-critical, and there is no more edge can be added to G, we conclude that $G \in \mathcal{H}_{14,15}$ when $N(x_1) \setminus \{a, x_2, y_1, y_2\} \neq \emptyset$. ■

Theorems 3.16 and 3.15 are combined into the following final result of this paper.

Corollary 3.17 *Let $S \in \mathcal{S}_{1,3,3}$. Then a graph G is 4-χ_S-vertex-critical if and only if*

$$G \in \{K_4\} \cup \mathcal{H}_{1-5,7,14,15} \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_{s_4} \cup \{G_{2k} :\ k \geq 3\}\,.$$

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