GENERALIZATIONS OF MOCK THETA FUNCTIONS AND APPELL–LERCH SUMS

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Abstract. Ramanujan named and first studied mock theta functions which can be represented by Eulerian forms, Appell–Lerch sums, Hecke-type double sums, and Fourier coefficients of meromorphic Jacobi forms. In this paper, we investigate some generalizations of mock theta functions and express them in terms of Appell–Lerch sums. For instance, one result proved in the present paper is that for any positive integer r, $|q|<1$ and x so that no denominators vanish,

$$\begin{aligned}&\left(1+x^{-1}\right)\sum_{n=0}^{\infty}\frac{(-q;q)_{2n+2r-2}q^{n+1}}{(xq^{2r-1},x^{-1}q^{2r-1};q^2)_{n+1}}\\&\quad=\frac{1}{(q,q,q^2;q^2)_\infty}\sum_{j=0}^{2r-2}q^{1-j}\frac{(q^2;q^2)_{2r-2}}{(q^2;q^2)_j(q^2;q^2)_{2r-2-j}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(n+2)}}{1-xq^{2n+2r-2j-1}}.\end{aligned}$$

In addition, we generalize not only two of Ramanujan's universal mock theta functions $g_2(x,q)$ and $g_3(x,q)$, but also two identities recorded by Ramanujan in his lost notebook.

1. Introduction

Throughout, we always assume that q is a complex number such that $|q|<1$ and adopt the following standard q-series notation [23]:

$$\begin{aligned}(a;q)_n&:=\prod_{k=0}^{n-1}(1-aq^k),\\(a;q)_\infty&:=\prod_{k=0}^{\infty}(1-aq^k),\\(a_1,a_2,\ldots,a_m;q)_n&:=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,\\(a_1,a_2,\ldots,a_m;q)_\infty&:=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.\end{aligned}$$

For notational convenience, we denote

$$j(x;q):=(x,q/x,q;q)_\infty.$$

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The Jacobi triple product identity [8, Theorem 1.3.3] is stated as

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = (-xq, -q/x, q^2; q^2)_\infty, \qquad \text{for } \ x \neq 0. \tag{1.1}$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, x\right) := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} x^n.$$

Mock theta functions have been a continuing source of inspiration and have motivated a tremendous amount of research over a century. In 1920, Ramanujan [38] introduced seventeen classical mock theta functions in his last letter to Hardy. It is well-known that mock theta functions can be represented by Eulerian forms, Appell–Lerch sums, Hecke-type double sums, and Fourier coefficients of meromorphic Jacobi forms. Translating from one representation form to another has been a historically difficult problem. Andrews [2] established Hecke-type double sums for the fifth and seventh order mock theta functions which were applied later to prove mock theta conjectures related to the fifth and seventh order mock theta functions in [26,27]. For example, Hickerson [27] proved the following well-known mock theta conjectures due to Ramanujan:

$$f_0(q) = \frac{j(q^5; q^{10}) j(q^2; q^5)}{j(q; q^3)} - 2q^2 g_3(q^2, q^{10}),$$
$$f_1(q) = \frac{j(q^5; q^{10}) j(q; q^5)}{j(q; q^3)} - 2q^3 g_3(q^4, q^{10}),$$

where the fifth order mock theta functions $f_0(q)$ and $f_1(q)$ are defined by

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n}, \qquad\qquad f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q)_n},$$

and

$$g_3(x, q) := x^{-1}\left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x; q)_{n+1}(x^{-1}q; q)_n}\right) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x, x^{-1}q; q)_{n+1}}.$$

Andrews and Hickerson [6] proved eleven identities concerning the sixth order mock theta functions which were recorded in Ramanujan's Lost Notebook. Berndt and Chan [9] defined two new sixth order mock theta functions and then provided four transformation formulas relating these two mock theta functions with Ramanujan's sixth order mock theta functions. Lovejoy [32] not only proved four identities involving the sixth order mock theta functions from Ramanujan's Lost Notebook, but gave a quick proof of four sixth order mock theta function identities due to Berndt and Chan. In 2000, Gordon and McIntosh [24] constructed eight eighth order mock theta functions from the Rogers–Ramanujan type identities. McIntosh [34] established the relationship between two families of mock theta functions. One was introduced by Ramanujan, the other was defined by Gordon and McIntosh. Choi [14–17] subsequently proved eight of Ramanujan's tenth order mock theta function identities. In 2020, Chen and Wang [13] provided a unified approach to find Appell–Lerch series and Hecke-type series representations of mock theta functions. For more works on mock theta functions; see, for example, [3, 11, 19, 22, 25, 28, 36, 39–41].

Now, we are in the position of universal mock theta functions. In 2012, Gordon and McIntosh [25] observed that the odd order mock theta functions can be expressed by the function $g_3(x,q)$ and the even order mock theta functions are related to $g_2(x,q)$, defined by

$$g_2(x,q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(x, x^{-1}q;q)_{n+1}}.$$

They named these two functions as universal mock theta functions because all the classical mock theta functions, including those found by Ramanujan, can be expressed in terms of them. Some other universal mock theta functions are given in [22]. For more properties of universal mock theta functions, we refer the interested reader to [7, 10, 12, 18, 21, 29–31, 33, 35].

To further study Hecke-type double sum representations of mock theta functions, Hickerson and Mortenson [28] considered the following Appell–Lerch sums.

Definition 1.1. Let x, $z \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ with neither z nor xz an integral power of q. Then

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r(r-1)/2} z^r}{1 - q^{r-1}xz}.$$

Replacing r by $r+1$ in the above series yields another representation of $m(x,q,z)$:

$$m(x,q,z) = \frac{-z}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r(r+1)/2} z^r}{1 - q^r xz}. \tag{1.2}$$

Mortenson and Hickerson [28, p. 398, Propositions 4.2 and 4.4] established universal mock theta functions $g_2(x,q)$ and $g_3(x,q)$ in terms of Appell–Lerch sums, that is,

$$g_2(x,q) = \frac{1}{j(q;q^2)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1 - xq^n} = -x^{-1} m\big(x^{-2}q, q^2, x\big), \tag{1.3}$$

$$\begin{aligned} g_3(x,q) &= \frac{1}{j(q;q^3)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 - xq^n} \\ &= -x^{-1} m\big(x^{-3}q^2, q^3, x^2\big) - x^{-2} m\big(x^{-3}q, q^3, x^2\big). \end{aligned} \tag{1.4}$$

Meanwhile, they [28] derived that

$$m(xq,q,z) = 1 - x m(x,q,z), \tag{1.5}$$

$$m(x,q,z)=m(x,q,qz). \tag{1.6}$$

Mortenson [37, Proposition 2.6] obtained the following identities, some of which were first proved in [1].

$$\big(1+x^{-1}\big)\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{(n+1)^2}}{(-xq,-x^{-1}q;q^2)_{n+1}}=m(x,q,-1)-\frac{j(q;q^2)^2}{2j(-x;q)}, \tag{1.7}$$

$$\sum_{n=0}^{\infty}{}^{*}\frac{(-1)^n(q;q^2)_n}{(-x;q)_{n+1}(-x^{-1}q;q)_n}=m(x,q,-1), \tag{1.8}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2}}{(-x;q^2)_{n+1}(-x^{-1}q^2;q^2)_n}&=m(x,q,-1)+\frac{j(q;q^2)^2}{2j(-x;q)}\\&=2m(x,q,-1)-m(x,q,\sqrt{-x^{-1}q})\\&=m\big(-x^2q,q^4,-q^{-1}\big)\\&\quad-xq^{-1}m\big(-x^2q^{-1},q^4,-q\big),\end{aligned} \tag{1.9}$$

$$\big(1+x^{-1}\big)\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(xq,x^{-1}q;q^2)_{n+1}}=-m\big(x,q^2,q\big), \tag{1.10}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q^2;q^4)_nq^{2n^2}}{(-x;q^4)_{n+1}(-x^{-1}q^4;q^4)_n}=m\big(x,q^2,q\big)+\frac{j(-q;q^4)^2j(-xq^2;q^4)}{j(-x;q^4)j(xq;q^2)}, \tag{1.11}$$

where [4, Entry 12.3.3]

$$\sum_{n=0}^{\infty}{}^{*}\frac{(-1)^n(q;q^2)_n}{(-x;q)_{n+1}(-x^{-1}q;q)_n}:=\frac{1}{j(-1;q)}\sum_{n=-\infty}^{\infty}\frac{(1+x^{-1})q^{n(n+1)/2}}{(1+xq^n)(1+x^{-1}q^n)}.$$

The identities (1.3), (1.4) and (1.7)–(1.11) reveal that there is an inseparable relationship between some generalizations of mock theta functions and Appell–Lerch sums. In the present paper, we further investigate some generalizations of mock theta functions by introducing an additional integer parameter. More precisely, we consider the following two-variable generalizations of mock theta functions, then express them in terms of Appell–Lerch sums and obtain some comparable identities. For any positive integer r,

$$\begin{aligned}A_r(x,q)&:=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_{n+r}q^{n^2}}{(xq^r,x^{-1}q^r;q^2)_{n+1}},\\B_r(x,q)&:=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_{n+r}q^{(n+1)^2}}{(xq^{r+1},x^{-1}q^{r+1};q^2)_{n+1}},\\C_r(x,q)&:=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+r-1}q^{n(n+1)}}{(xq^r,x^{-1}q^r;q^2)_{n+1}},\end{aligned}$$

$$D_r(x,q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+r)}}{(xq^r, x^{-1}q^r; q^2)_{n+1}},$$
$$E_r(x,q) := \sum_{n=0}^{\infty}{}^{*} \frac{(-1)^n (q;q^2)_{n+r}}{(xq^r, x^{-1}q^r; q)_{n+1}},$$
$$F_r(x,q) := (1+x^{-1}) \sum_{n=0}^{\infty} \frac{(-q;q)_{2n+2r-2}\, q^{n+1}}{(xq^{2r-1}, x^{-1}q^{2r-1}; q^2)_{n+1}}.$$

Of course, the left-hand side of (1.8) and the definition of $E_r(x,q)$ have an obvious problem; namely, the involved series are divergent series. However, Andrews noted that [1, p. 37]

$$\lim_{\alpha\to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha q;q)_n (q;q^2)_n}{(q;q)_n (-\alpha a q;q)_n (-\alpha q/a;q)_n} \alpha^n = \frac{1}{j(-1;q)} \sum_{n=-\infty}^{\infty} \frac{(1+1/a)(1+a) q^{n(n+1)/2}}{(1+aq^n)(1+q^n/a)}, \tag{1.12}$$

which follows from Watson's q-analogue of Whipple's theorem [23, Appendix (III.18)]:

$$ {}_8\phi_7\left(\begin{matrix} a, & \sqrt{a}q, & -\sqrt{a}q, & b, & c, & d, & e, & q^{-N} \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{N+1} \end{matrix}; q, \frac{a^2q^{N+2}}{bcde}\right) = \frac{(aq, aq/de;q)_N}{(aq/d, aq/e;q)_N}\, {}_4\phi_3\left(\begin{matrix} aq/bc, & d, & e, & q^{-N} \\ & aq/b, & aq/c, & deq^{-N}/a \end{matrix}; q, q\right).$$

The details of the derivation of (1.12) can be found in [4, p. 266]. Therefore, the right-hand side of (1.12) is a suitable representation of the following divergent series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-x;q)_n (-x^{-1}q;q)_n}.$$

Similarly, in the Watson–Whipple transformation formula (see (2.1) below), replacing a by αq^{2r}, then setting $b = xq^r$, $c = x^{-1}q^r$, $d = -e = q^{r+1/2}$, and finally letting $f \to \infty$ and $\alpha \to 1^-$, after simplification, we obtain that

$$\sum_{n=0}^{\infty}{}^{*} \frac{(-1)^n (q;q^2)_{n+r}}{(xq^r, x^{-1}q^r;q)_{n+1}} = \frac{1}{j(-1;q)} \sum_{n=-\infty}^{\infty} \frac{(q^{n+1};q)_{2r-1}\, q^{n(n+1)/2}}{1-xq^{n+r}}.$$

Before stating the main results, we recall that the q-binomial coefficients are defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}, & \text{if } 0 \le n \le m, \\ 0, & \text{otherwise.} \end{cases}$$

We prove the following theorems.

Theorem 1.2. *For any integer $r \geq 1$,*

$$A_r(x,q) = \sum_{j=0}^{2r-1} (-1)^j \begin{bmatrix} 2r-1 \\ j \end{bmatrix}_q \Big\{ m\big(-x^2q^{2r-2j+1}, q^4, -q^{2+(-1)^j}\big) + xq^{r-j-1} m\big(-x^2q^{2r-2j-1}, q^4, -q^{2-(-1)^j}\big) \Big\}.$$

Theorem 1.3. *For any integer $r \geq 1$,*

$$B_r(x,q) = \sum_{j=0}^{2r-1} (-1)^j \begin{bmatrix} 2r-1 \\ j \end{bmatrix}_q \Big\{ m\big(-x^2q^{2r-2j+1}, q^4, -q^{2-(-1)^j}\big) + xq^{r-j-1} m\big(-x^2q^{2r-2j-1}, q^4, -q^{2+(-1)^j}\big) \Big\}.$$

Theorem 1.4. *For any integer $r \geq 1$,*

$$C_r(x,q) = -\sum_{j=0}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^4} x^{-1}q^{2j-r} m\big(x^{-2}q^{4j-2r+2}, q^4, xq^{r-2j}\big).$$

Theorem 1.5. *For any integer $r \geq 1$,*

$$\begin{aligned} D_r(x,q) = & \sum_{\substack{j=0 \\ r+j\equiv 0 \pmod 3}}^{r-1} (-1)^{(r+4j)/3} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} q^{j^2+j-(r+j)(r+j+1)/3} \\ & + \sum_{k=0}^{2} \sum_{\substack{j=0 \\ r+j\equiv k \pmod 3}}^{r-1} (-1)^{j+1+(r+j-k)/3} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} x^{-k} q^{-k(r-2j+2k)/3} \\ & \times q^{j^2+j-(r+j-k)(r+j+k+1)/3} \Big\{ m\big(x^{-3}q^{-r+2j-2k+4}, q^6, x^2q^{(2r-4j+4k)/3}\big) \\ & + x^{-1}q^{-(r-2j+2k)/3} m\big(x^{-3}q^{-r+2j-2k+2}, q^6, x^2q^{(2r-4j+4k)/3}\big) \Big\}. \end{aligned}$$

Theorem 1.6. *For any integer $r \geq 1$,*

$$E_r(x,q) = \sum_{j=0}^{2r-1} (-1)^j \begin{bmatrix} 2r-1 \\ j \end{bmatrix}_q m\big(-xq^{r-j}, q, -1\big).$$

Theorem 1.7. *For any integer $r \geq 1$,*

$$F_r(x,q) = -\sum_{j=0}^{2r-2} \begin{bmatrix} 2r-2 \\ j \end{bmatrix}_{q^2} q^{-j} m\big(xq^{2r-2j-2}, q^2, q\big).$$

It is worth mentioning that the identity appeared in the abstract is equivalent to Theorem 1.7. Taking $(r,q,x) \mapsto \big(1, q^{1/2}, xq^{-1/2}\big)$ in Theorem 1.4 yields the rightmost side of (1.3). The rightmost side of (1.4) can be derived by setting $(r,q,x) \mapsto$

$(1, q^{1/2}, xq^{-1/2})$ in Theorem 1.5. Moreover, putting $(r, x) \mapsto (1, -x)$ in Theorem 1.6, we obtain (1.8). We derive (1.10) by setting $r = 1$ in Theorem 1.7. Finally, we obtain the following corollary by putting $r = 1$ in Theorem 1.2 and utilizing (1.5).

Corollary 1.8. *We have*

$$\frac{1}{x-1}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_{n+1}q^{n^2}}{(xq,x^{-1}q;q^2)_{n+1}} - \frac{1}{x-1} \\ = m\big(-x^2q, q^4, -q\big) + xq^{-1}m\big(-x^2q^{-1}, q^4, -q^3\big).$$

Remark 1.9. Some specializations of the main theorems encounter the Appell–Lerch sums expressions of classical mock theta functions.

(i) Putting $(r, q, x) \mapsto (1, -q, \sqrt{-1})$ in Theorem 1.3, we obtain the corresponding Appell–Lerch sums expression for the eighth order mock theta function $U_0(q)$, introduced by Gordon and McIntosh [24].

(ii) The case $(r, x) = (1, 1)$ in Theorem 1.4 is equivalent to the Appell–Lerch sums expressions of McIntosh's [34] second order mock theta function $B(q)$ due to Hickerson and Mortenson (see [28, p. 399, Eq. (5.2)]).

(iii) Taking $(r, q, x) \mapsto (1, q^{1/2}, \sqrt{-1})$ in Theorem 1.5, we obtain an Appell–Lerch sums expression of Watson's [40] third order mock theta function $\nu(q)$. Actually, this identity is equivalent to an identity due to Hickerson and Mortenson (see [28, p. 400, Eq. (5.9)]). Moreover, the Appell–Lerch sums expressions of two Watson's third order mock theta functions $\omega(q)$ and $\rho(q)$ (see [28, p. 400, Eqs. (5.8) and (5.10)])) can be established by utilizing Theorem 1.5 and some properties of Appell–Lerch sums.

2. Proofs of Theorems 1.2–1.7

The following identities are frequently used in the proofs of the main theorems.

$$j(x;q) = j(q/x;q),$$
$$j(qx;q) = -x^{-1}j(x;q).$$

The Watson–Whipple transformation formula plays an important role in our paper.

Lemma 2.1. [23, Appendix (III.17)] *The Watson–Whipple transformation formula is stated as*

$$ {}_8\phi_7\left(\begin{matrix} a, & \sqrt{a}q, & -\sqrt{a}q, & b, & c, & d, & e, & f \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right) \\ = \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty}\, {}_4\phi_3\left(\begin{matrix} aq/bc, & d, & e, & f \\ & aq/b, & aq/c, & def/a \end{matrix}; q, q\right). \quad (2.1)$$

Moreover, the following identity which was proved in [20] is the main ingredient in our proofs.

Lemma 2.2. [20, Lemma 2.2] *For any integer $r \geq 1$,*

$$\sum_{n=0}^{\infty} \frac{(d,e;q^2)_n}{(xq^r, x^{-1}q^r;q^2)_{n+1}} \left(\frac{q^{2r+2}}{de}\right)^n = \frac{(q^{2r+2}/d, q^{2r+2}/e;q^2)_\infty}{(q^2, q^{2r+2}/de;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (q^{2n+2};q^2)_{r-1}(d,e;q^2)_n q^{n^2+2rn+3n}}{(q^{2r+2}/d, q^{2r+2}/e;q^2)_n (1-xq^{2n+r})} \left(\frac{1}{de}\right)^n.$$

Now we are ready to prove Theorems 1.2–1.7.

Proof of Theorem 1.2. Setting $d \to \infty$ and $e = q^{2r+1}$ in Lemma 2.2, we deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} &\frac{(-1)^n (q^{2r+1};q^2)_n q^{n^2}}{(xq^r, x^{-1}q^r;q^2)_{n+1}} \\
&= \frac{1}{(q;q^2)_r j(-q;q^4)} \sum_{n=-\infty}^{\infty} \frac{(q^{2n+1};q)_{2r-1} q^{2n^2+n}}{1-xq^{2n+r}} \\
&= \frac{1}{(q;q^2)_r j(-q;q^4)} \sum_{j=0}^{2r-1} (-1)^j \begin{bmatrix} 2r-1 \\ j \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n+2nj+j(j+1)/2}}{1-xq^{2n+r}} \\
&= \frac{1}{(q;q^2)_r j(-q;q^4)} \sum_{j=0}^{r-1} \begin{bmatrix} 2r-1 \\ 2j \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n+4nj+2j^2+j}}{1-xq^{2n+r}} \\
&\quad - \frac{1}{(q;q^2)_r j(-q;q^4)} \sum_{j=0}^{r-1} \begin{bmatrix} 2r-1 \\ 2j+1 \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+3n+4nj+2j^2+3j+1}}{1-xq^{2n+r}},
\end{aligned} \tag{2.2}$$

where we obtain the penultimate step by utilizing the following identity [5, p. 11, Lemma 1.3.1]:

$$(a;q)_n = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q q^{(j^2-j)/2} a^j. \tag{2.3}$$

Then replacing n by $n-j$ on the right-hand side of (2.2), and then multiplying $(q;q^2)_r$ on both sides, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} &\frac{(-1)^n (q;q^2)_{n+r} q^{n^2}}{(xq^r, x^{-1}q^r;q^2)_{n+1}} \\
&= \frac{1}{j(-q;q^4)} \sum_{j=0}^{r-1} \begin{bmatrix} 2r-1 \\ 2j \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{\left(1+xq^{2n-2j+r}\right) q^{2n^2+n}}{1-x^2q^{4n-4j+2r}} \\
&\quad - \frac{1}{j(-q;q^4)} \sum_{j=0}^{r-1} \begin{bmatrix} 2r-1 \\ 2j+1 \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{\left(1+xq^{2n-2j+r}\right) q^{2n^2+3n+1}}{1-x^2q^{4n-4j+2r}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{r-1} \begin{bmatrix} 2r-1 \\ 2j \end{bmatrix}_q \\
&\qquad \times \left\{ m\left(-x^2q^{2r-4j+1}, q^4, -q^3\right) + xq^{r-2j-1} m\left(-x^2q^{2r-4j-1}, q^4, -q\right) \right\} \\
&\quad - \sum_{j=0}^{r-1} \begin{bmatrix} 2r-1 \\ 2j+1 \end{bmatrix}_q \\
&\qquad \times \left\{ m\left(-x^2q^{2r-4j-1}, q^4, -q\right) + xq^{r-2j-2} m\left(-x^2q^{2r-4j-3}, q^4, -q^3\right) \right\} \\
&= \sum_{j=0}^{2r-1} (-1)^j \begin{bmatrix} 2r-1 \\ j \end{bmatrix}_q \\
&\quad \times \left\{ m\left(-x^2q^{2r-2j+1}, q^4, -q^{2+(-1)^j}\right) + xq^{r-j-1} m\left(-x^2q^{2r-2j-1}, q^4, -q^{2-(-1)^j}\right) \right\},
\end{aligned}
$$

where the penultimate step follows from (1.2) and (1.6). Therefore, we complete the proof. □

Proof of Theorem 1.3. For $r \geq 2$, if we set $d \to \infty$ and $e = q^{2r-1}$ in Lemma 2.2 and multiply both sides by $(q;q^2)_{r-1}$, then we find that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_{n+r-1} q^{n(n+2)}}{(xq^r, x^{-1}q^r; q^2)_{n+1}} \\
&\quad = \frac{1}{j(-q;q^4)} \sum_{n=-\infty}^{\infty} \frac{(q^{2n+2};q)_{2r-3} q^{2n^2+3n}}{1-xq^{2n+r}} \\
&\quad = \frac{1}{j(-q;q^4)} \sum_{j=0}^{2r-3} (-1)^j \begin{bmatrix} 2r-3 \\ j \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+3n+2nj+j(j+3)/2}}{1-xq^{2n+r}} \\
&\quad = \frac{1}{j(-q;q^4)} \sum_{j=0}^{r-2} \begin{bmatrix} 2r-3 \\ 2j \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+3n+4nj+2j^2+3j}}{1-xq^{2n+r}} \\
&\qquad - \frac{1}{j(-q;q^4)} \sum_{j=0}^{r-2} \begin{bmatrix} 2r-3 \\ 2j+1 \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+5n+4nj+2j^2+5j+2}}{1-xq^{2n+r}},
\end{aligned} \tag{2.4}
$$

where the penultimate step follows from (2.3). Then letting $n \to n-j$ and $n \to n-j-1$ in the first and second terms on the right-hand side of (2.4), respectively, we derive that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_{n+r-1} q^{n(n+2)}}{(xq^r, x^{-1}q^r; q^2)_{n+1}} \\
&= \frac{1}{j(-q;q^4)} \sum_{j=0}^{r-2} \begin{bmatrix} 2r-3 \\ 2j \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{\left(1+xq^{2n-2j+r}\right) q^{2n^2+3n}}{1-x^2q^{4n-4j+2r}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{1}{j(-q;q^4)} \sum_{j=0}^{r-2} \begin{bmatrix} 2r-3 \\ 2j+1 \end{bmatrix}_q \sum_{n=-\infty}^{\infty} \frac{\left(1+xq^{2n-2j+r-2}\right)q^{2n^2+n-1}}{1-x^2q^{4n-4j+2r-4}} \\
&= \sum_{j=0}^{r-2} q^{-1} \begin{bmatrix} 2r-3 \\ 2j \end{bmatrix}_q \\
&\quad \times \left\{ m\left(-x^2q^{2r-4j-1}, q^4, -q\right) + xq^{r-2j-2} m\left(-x^2q^{2r-4j-3}, q^4, -q^3\right) \right\} \\
&\quad - \sum_{j=0}^{r-2} q^{-1} \begin{bmatrix} 2r-3 \\ 2j+1 \end{bmatrix}_q \\
&\quad \times \left\{ m\left(-x^2q^{2r-4j-3}, q^4, -q^3\right) + xq^{r-2j-3} m\left(-x^2q^{2r-4j-5}, q^4, -q\right) \right\} \\
&= \sum_{j=0}^{2r-3} (-1)^j q^{-1} \begin{bmatrix} 2r-3 \\ j \end{bmatrix}_q \\
&\quad \times \left\{ m\left(-x^2q^{2r-2j-1}, q^4, -q^{2-(-1)^j}\right) + xq^{r-j-2} m\left(-x^2q^{2r-2j-3}, q^4, -q^{2+(-1)^j}\right) \right\},
\end{aligned} \tag{2.5}
$$

where the penultimate step follows from (1.2) and (1.6). Finally, we complete the proof by changing r to $r+1$ in (2.5). $\square$

Proof of Theorem 1.4. Setting $d \to \infty$ and $e = -q^{2r}$ in Lemma 2.2, and then multiplying $(-q^2;q^2)_{r-1}$ on both sides, we derive that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_{n+r-1} q^{n(n+1)}}{(xq^r, x^{-1}q^r; q^2)_{n+1}} \\
&\quad = \frac{1}{j(q^2;q^4)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (q^{4n+4};q^4)_{r-1} q^{2n^2+2n}}{1-xq^{2n+r}} \\
&\quad = \frac{1}{j(q^2;q^4)} \sum_{j=0}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^4} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j} q^{2n^2+2n+4nj+2j^2+2j}}{1-xq^{2n+r}} \\
&\quad = \frac{1}{j(q^2;q^4)} \sum_{j=0}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+2n}}{1-xq^{2n-2j+r}},
\end{aligned} \tag{2.6}
$$

where the penultimate step follows from (2.3), and we obtain the last step by letting $n \to n-j$. Then in view of (1.3) and (2.6), we prove the theorem. $\square$

Proof of Theorem 1.5. We let d and e tend to ∞ in Lemma 2.2 to obtain that

$$
\sum_{n=0}^{\infty} \frac{q^{2n^2+2rn}}{(xq^r, x^{-1}q^r; q^2)_{n+1}}
$$

$$
\begin{aligned}
&= \frac{1}{j(q^2;q^6)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (q^{2n+2};q^2)_{r-1} q^{3n^2+(2r+1)n}}{1-xq^{2n+r}} \\
&= \frac{1}{j(q^2;q^6)} \sum_{j=0}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j} q^{3n^2+(2r+2j+1)n+j^2+j}}{1-xq^{2n+r}} \\
&= \frac{1}{j(q^2;q^6)} \sum_{k=0}^{2} \sum_{\substack{j=0 \\ r+j\equiv k \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j} q^{3n^2+(2r+2j+1)n+j^2+j}}{1-xq^{2n+r}} \\
&= \frac{1}{j(q^2;q^6)} \sum_{k=0}^{2} \sum_{\substack{j=0 \\ r+j\equiv k \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} \\
&\quad \times \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j+(r+j-k)/3} q^{3n^2+(2k+1)n+j^2+j-(r+j-k)(r+j+k+1)/3}}{1-xq^{2n+(r-2j+2k)/3}},
\end{aligned}
\tag{2.7}
$$

where we derive the second step by utilizing (2.3), and the last step follows from $n \to n-(r+j-k)/3$. Now we turn to simplify the right-hand side of (2.7). We find that

$$
\begin{aligned}
&\frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 0 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j+(r+j)/3} q^{3n^2+n+j^2+j-(r+j)(r+j+1)/3}}{1-xq^{2n+(r-2j)/3}} \\
&= \frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 0 \pmod 3}}^{r-1} (-1)^{j+(r+j)/3} q^{j^2+j-(r+j)(r+j+1)/3} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} \\
&\quad \times \sum_{n=-\infty}^{\infty} \frac{(-1)^n \left(1-xq^{2n+(r-2j)/3}+xq^{2n+(r-2j)/3}\right) q^{3n^2+n}}{1-xq^{2n+(r-2j)/3}} \\
&= \frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 0 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} \\
&\quad \times \sum_{n=-\infty}^{\infty} (-1)^{n+j+(r+j)/3} q^{3n^2+n+j^2+j-(r+j)(r+j+1)/3} \\
&\quad + \frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 0 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} \\
&\quad \times \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j+(r+j)/3} x q^{3n^2+3n+j^2+j-((r+j)(r+j+1)-(r-2j))/3}}{1-xq^{2n+(r-2j)/3}}
\end{aligned}
$$

$$= \sum_{\substack{j=0 \\ r+j\equiv 0 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} (-1)^{j+(r+j)/3} q^{j^2+j-(r+j)(r+j+1)/3}$$
$$+ \frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 0 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2}$$
$$\times \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j+(r+j)/3} x q^{3n^2+3n+j^2+j-((r+j)(r+j+1)-(r-2j))/3}}{1-xq^{2n+(r-2j)/3}}, \tag{2.8}$$

where we use (1.1) to obtain the last identity. Also, we observe that

$$\frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 2 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2}$$
$$\times \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j+(r+j-2)/3} q^{3n^2+5n+j^2+j-(r+j-2)(r+j+3)/3}}{1-xq^{2n+(r-2j+4)/3}}$$
$$= \frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 2 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} (-1)^{j+(r+j-2)/3} q^{j^2+j-(r+j-2)(r+j+3)/3}$$
$$\times \sum_{n=-\infty}^{\infty} \frac{(-1)^n \left(1-x^{-1}q^{-2n-(r-2j+4)/3}+x^{-1}q^{-2n-(r-2j+4)/3}\right) q^{3n^2+5n}}{1-xq^{2n+(r-2j+4)/3}}$$
$$= \frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 2 \pmod 3}}^{r-1} x^{-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2}$$
$$\times \sum_{n=-\infty}^{\infty} (-1)^{n+j+(r+j+1)/3} q^{3n^2+3n+j^2+j-((r+j-2)(r+j+3)+(r-2j+4))/3}$$
$$+ \frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 2 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2}$$
$$\times \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j+(r+j-2)/3} x^{-1} q^{3n^2+3n+j^2+j-((r+j-2)(r+j+3)+(r-2j+4))/3}}{1-xq^{2n+(r-2j+4)/3}}$$
$$= \frac{1}{j(q^2;q^6)} \sum_{\substack{j=0 \\ r+j\equiv 2 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2}$$

$$\times \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j+(r+j-2)/3} x^{-1} q^{3n^2+3n+j^2+j-((r+j-2)(r+j+3)+(r-2j+4))/3}}{1-xq^{2n+(r-2j+4)/3}}, \tag{2.9}$$

where we use (1.1) to obtain the last step.

Then substituting (2.8) and (2.9) into (2.7) yields that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{2n^2+2rn}}{(xq^r, x^{-1}q^r; q^2)_{n+1}} \\
&= \sum_{\substack{j=0 \\ r+j\equiv 0 \pmod 3}}^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} (-1)^{j+(r+j)/3} q^{j^2+j-(r+j)(r+j+1)/3} \\
&\quad + \frac{1}{j(q^2;q^6)} \sum_{k=0}^{2} \sum_{\substack{j=0 \\ r+j\equiv k \pmod 3}}^{r-1} (-1)^{j+(r+j-k)/3} x^{1-k} \begin{bmatrix} r-1 \\ j \end{bmatrix}_{q^2} \\
&\quad \times \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n+j^2+j-((r+j-k)(r+j+k+1)+(k-1)(r-2j+2k))/3}}{1-xq^{2n+(r-2j+2k)/3}}.
\end{aligned}$$

Finally, combining (1.4) and the above identity, we complete the proof. □

Proof of Theorem 1.6. Taking $(r, d, e) \mapsto (2r, q^{2r+1}, -q^{2r+1})$ in Lemma 2.2, and then multiplying $(q^2;q^4)_r$ on both sides, we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n (q^2;q^4)_{n+r}}{(xq^{2r}, x^{-1}q^{2r}; q^2)_{n+1}} &= \frac{1}{j(-1;q^2)} \sum_{n=-\infty}^{\infty} \frac{(q^{2n+2};q^2)_{2r-1} q^{n^2+n}}{1-xq^{2n+2r}} \\
&= \frac{1}{j(-1;q^2)} \sum_{j=0}^{2r-1} (-1)^j \begin{bmatrix} 2r-1 \\ j \end{bmatrix}_{q^2} \sum_{n=-\infty}^{\infty} \frac{q^{n^2+n+2nj+j^2+j}}{1-xq^{2n+2r}} \\
&= \frac{1}{j(-1;q^2)} \sum_{j=0}^{2r-1} (-1)^j \begin{bmatrix} 2r-1 \\ j \end{bmatrix}_{q^2} \sum_{n=-\infty}^{\infty} \frac{q^{n^2+n}}{1-xq^{2n-2j+2r}} \\
&= \sum_{j=0}^{2r-1} (-1)^j \begin{bmatrix} 2r-1 \\ j \end{bmatrix}_{q^2} m\left(-xq^{2r-2j}, q^2, -1\right),
\end{aligned} \tag{2.10}$$

where we obtain the second equality by using (2.3), and the last step follows from (1.2). Hence, replacing q by $q^{1/2}$ in (2.10), we complete the proof. □

Proof of Theorem 1.7. Replacing q, a, b, c, d, and e by q^2, q^{4r-2}, xq^{2r-1}, $x^{-1}q^{2r-1}$, $-q^{2r-1}$, and $-q^{2r}$ in (2.1), respectively, and then letting f tend to ∞, after simplification, we have

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n+2r-2} q^n}{(xq^{2r-1}, x^{-1}q^{2r-1}; q^2)_{n+1}}$$

$$
\begin{aligned}
&= \frac{1}{j(q;q^2)} \sum_{n=0}^{\infty} \frac{(-1)^n \big(1-q^{2n+2r-1}\big)(q^{2n+2};q^2)_{2r-2} q^{n^2+2n}}{(1-xq^{2n+2r-1})(1-x^{-1}q^{2n+2r-1})} \\
&= \frac{1}{(1+x^{-1})j(q;q^2)} \\
&\quad \times \sum_{n=0}^{\infty} (-1)^n (q^{2n+2};q^2)_{2r-2} q^{n^2+2n} \left(\frac{1}{1-xq^{2n+2r-1}} + \frac{x^{-1}}{1-x^{-1}q^{2n+2r-1}} \right) \\
&= \frac{1}{(1+x^{-1})j(q;q^2)} \sum_{n=0}^{\infty} \frac{(-1)^n (q^{2n+2};q^2)_{2r-2} q^{n^2+2n}}{1-xq^{2n+2r-1}} \\
&\quad + \frac{x^{-1}}{(1+x^{-1})j(q;q^2)} \sum_{n=0}^{\infty} \frac{(-1)^n (q^{2n+2};q^2)_{2r-2} q^{n^2+2n}}{1-x^{-1}q^{2n+2r-1}} \\
&= \frac{1}{(1+x^{-1})j(q;q^2)} \sum_{n=0}^{\infty} \frac{(-1)^n (q^{2n+2};q^2)_{2r-2} q^{n^2+2n}}{1-xq^{2n+2r-1}} \\
&\quad + \frac{1}{(1+x^{-1})j(q;q^2)} \left(\sum_{n=-\infty}^{-1} - \sum_{n=-2r+2}^{-1} \right) \frac{(-1)^n (q^{2n+2};q^2)_{2r-2} q^{n^2+2n}}{1-xq^{2n+2r-1}} \\
&= \frac{1}{(1+x^{-1})j(q;q^2)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (q^{2n+2};q^2)_{2r-2} q^{n^2+2n}}{1-xq^{2n+2r-1}}, \qquad (2.11)
\end{aligned}
$$

where the penultimate equality follows from $n \to -n-2r+1$ in the second sum, and we obtain the last step by observing that $(q^{2n+2};q^2)_{2r-2} = 0$ for $r \geq 2$ and $-2r+2 \leq n \leq -1$. Notice that $\sum_{n=-2r+2}^{-1} = 0$ when $r = 1$.

In light of (2.3) and (2.11), we further obtain that

$$
\begin{aligned}
&(1+x^{-1}) \sum_{n=0}^{\infty} \frac{(-q;q)_{2n+2r-2} q^n}{(xq^{2r-1}, x^{-1}q^{2r-1};q^2)_{n+1}} \\
&\quad = \frac{1}{j(q;q^2)} \sum_{j=0}^{2r-2} \begin{bmatrix} 2r-2 \\ j \end{bmatrix}_{q^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+j} q^{n^2+2n+2nj+j^2+j}}{1-xq^{2n+2r-1}} \\
&\quad = \frac{1}{j(q;q^2)} \sum_{j=0}^{2r-2} \begin{bmatrix} 2r-2 \\ j \end{bmatrix}_{q^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n-j}}{1-xq^{2n-2j+2r-1}} \\
&\quad = -q^{-1} \sum_{j=0}^{2r-2} \begin{bmatrix} 2r-2 \\ j \end{bmatrix}_{q^2} q^{-j} m\big(xq^{2r-2j-2}, q^2, q\big),
\end{aligned}
$$

where the penultimate step is established by changing n to $n-j$, and we obtain the last step by utilizing (1.2). Therefore, we complete the proof. $\square$

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Declarations

Conflict of interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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