Positivity of the determinants of the partition function and the overpartition function

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Abstract. In this paper, we give an iterated approach to concern with the positivity of

$$\det\,(p(n-i+j))_{1\leq i,j\leq k}>0,$$

where $p(n)$ is the partition function. We first apply a general method to prove that for given k_1, k_2, m_1, m_2, one can find a threshold $N(k_1, k_2, m_1, m_2)$ such that for $n > N(k_1, k_2, m_1, m_2)$,

$$\begin{vmatrix} p(n-k_1+m_1) & p(n+m_1) & p(n+m_1+m_2) \\ p(n-k_1) & p(n) & p(n+m_2) \\ p(n-k_1-k_2) & p(n-k_2) & p(n-k_2+m_2) \end{vmatrix} > 0.$$

Based on this result, we will prove that for $n \geq 656$, $\det\,(p(n-i+j))_{1\leq i,j\leq 4} > 0$. Employing the same technique, we will show that determinants $(\bar{p}(n-i+j))_{1\leq i,j\leq k} > 0$ are positive for $k = 3$ and 4 for overpartition $\bar{p}(n)$. Furthermore, we will give an outline of how to prove the positivity of $\det\,(p(n-i+j))_{1\leq i,j\leq k}$ for general k.

Keywords: partition function; overpartition function; log-concavity; determinant inequality; Radmacher-type series; total-positivity

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1 Introduction

The objective of this paper is to prove the positivity of certain determinants involving the partition function and the overpartition function. This problem arises from the total positivity of Toeplitz matrix and properties of the Laguerre-Pólya class.

A real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{1.1}$$

is said to be in the Laguerre-Pólya class, denoted by $\psi(x) \in \mathcal{LP}$, if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k)\, e^{-x/x_k},$$

where c, β, x_k are real numbers, $\alpha \geq 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. Pólya [25] proved that the Riemann hypothesis is equivalent to the Riemann Xi-function

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right) \pi^{\frac{iz}{2} - \frac{1}{4}} \Gamma\left(-\frac{iz}{2} + \frac{1}{4}\right) \zeta\left(-iz + \frac{1}{2}\right)$$

belonging to the Laguerre-Pólya class. Jensen [14] proved that a real entire function $\psi(x)$ belongs to the Laguerre-Pólya class if and only if for any positive integer n, the n-th associated Jensen polynomial

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \tag{1.2}$$

has only real zeros. Thus, the Riemann hypothesis is equivalent to that Jensen polynomials associated with the Riemann Xi-function have only real zeros, see also [25].

Recently, the integer partition function has been connected to coefficients of the Riemann Xi-function. Recall that a partition of a positive integer n is a nonincreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n. Nicolas [22], DeSalvo and Pak [9] proved the quadratic polynomials $p(n) + 2p(n+1)x + p(n+2)x^2$ have only real zeros for $n \geq 25$, respectively. Chen, Jia and Wang [2] proved that the cubic polynomials $p(n) + 3p(n+1)x + 3p(n+2)x^2 + p(n+3)$ have only real zeros for $n \geq 94$ and conjectured that for any k, there exists $N(k)$ such that $\sum_{i=0}^{k} p(n+i)\binom{k}{i}x^i$ have only real zeros for $n > N(k)$. Griffin, Ono, Rolen and Zagier [11] proved this conjecture and showed that the Jensen polynomials associated with some sequences satisfying certain conditions have only real zeros as n is sufficiently large. It affirms the conjecture proposed by Chen, Jia and Wang. Griffin, Ono, Rolen, Thorner, Tripp, and Wagner [12] made this approach effective for the Riemann Xi-function. For more backgrounds for Jensen polynomials associated with the partition function and the Riemann Xi-function, see [17], [24] and [27].

In this paper, we will consider certain determinantal inequalities associated with the Riemann hypothesis. Following [23], the Riemann Xi-function can be rewritten as follow:

$$\Xi(z) = \sum_{n=0}^{\infty} \beta_n z^n,$$

where

$$\beta_n := \frac{1}{\Gamma(2n+1)} \int_0^{\infty} \mathrm{d}t \Phi(t) t^{2n},$$

and $\Phi(t)$ is defined by

$$\Phi(t) := \sum_{m=1}^{\infty} [2m^4\pi^2 e^{9t} - 3m^2\pi e^{5t}] \exp(-m^2\pi e^{4t}).$$

Karlin [16] gave a way to ensure that $\Xi(z)$ has only real and negative zeros, which is to check that $\{\beta_n\}$ satisfies (1.3) below. Denote a semi-infinite matrix

$$B := \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\ & \beta_0 & \beta_1 & \beta_2 & \cdots \\ & & \beta_0 & \beta_1 & \cdots \\ & & & \beta_0 & \cdots \\ & & & & \ddots \end{bmatrix}$$

and its minors

$$D(n, r) := \det(B_{i,j+n})_{i,j=1,\ldots,r}.$$

The Riemann hypothesis is equivalent to

$$D(n, r) > 0, \quad n = 0, 1, \ldots;\ r = 1, 2, \ldots, \tag{1.3}$$

or equivalently, the total positivity of B. Csordas, Norfolk and Varga [4] proved that $D(n, 2) > 0$ for all n (see also [8] and [20]). Nuttall [23] proved $D(n, 3) > 0$ for all n. The other cases are still open.

The positivity of $D(n, 3)$ have been proved to be relative to 2-log-concavity in combinatorics. Recall that the operator $\mathcal{L}$ is defined by

$$\mathcal{L}a_n = a_{n+1}^2 - a_n a_{n+2} \quad \textit{and} \quad \mathcal{L}^r a_n = \mathcal{L}(\mathcal{L}^{r-1} a_n). \tag{1.4}$$

A sequence $\{a_n\}_{n\geq 0}$ is said to be r-log-concave if for $0 \leq i \leq r$,

$$\{\mathcal{L}^i a_n\}_{n\geq 0} \tag{1.5}$$

are all nonnegative sequences, see [1]. For $\mathcal{LP}$ class, Craven and Csordas [5] proved the following theorem.

Theorem 1.1 (Craven and Csordas, [5]). *For $\gamma_k > 0$, if $\sum_{k=0}^{\infty} \gamma_k x^k$ belongs to $\mathcal{LP}$ class, then for $k \geq 2$,*

$$\begin{vmatrix} \gamma_k & \gamma_{k+1} & \gamma_{k+2} \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} \\ \gamma_{k-2} & \gamma_{k-1} & \gamma_k \end{vmatrix} \geq 0. \tag{1.6}$$

They pointed out that the left-hand side of (1.6) is equal to

$$\frac{1}{\gamma_k}\left((\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})^2 - (\gamma_{k-1}^2 - \gamma_{k-2}\gamma_k)(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2})\right).$$

Thus Theorem 1.1 gives

$$(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})^2 - (\gamma_{k-1}^2 - \gamma_{k-2}\gamma_k)(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) \geq 0,$$

which is 2-log-concavity of γ_k.

Motivated by the results on Jensen polynomials associatd with partition function, Jia and Wang [15] considered determinants involving the partition function and proved that for $n \geq 222$, det $(p(n-i+j))_{1\leq i,j\leq 3} > 0$. Hou and Zhang [13] proved that $p(n)$ satisfies the higher order log-concavity for sufficiently large n and independently gave the threshold for the 2-log-concavity of the partition function. However, there are few results concerning with determinants ordered 4 or higher.

The remaining of this paper is organized as follow. In Section 2, we shall give effective bounds for $p(n)$ which will be used in our proofs. In Section 3 and 4, we shall give an iterated approach to show the positivity of det $(p(n-i+j))_{1\leq i,j\leq k}$. Applying this method, we will show

$$\begin{vmatrix} p(n-k_1+m_1) & p(n+m_1) & p(n+m_1+m_2) \\ p(n-k_1) & p(n) & p(n+m_2) \\ p(n-k_1-k_2) & p(n-k_2) & p(n-k_2+m_2) \end{vmatrix} > 0.$$

Moreover, we will prove that for $n \geq 656$, det $(p(n-i+j))_{1\leq i,j\leq 4} > 0$. In Section 5, we shall prove similar results for the overpartition function, which is a generalizaion of the partition function. At last, we will give a sketch of how to prove det $(p(n-i+j))_{1\leq i,j\leq k} > 0$.

2 Preparation

In the following three sections, we shall prove determinantal inequalities for $p(n)$, which need bounds of $p(n)$ that are elementary functions. The Hardy-

Ramanujan-Rademacher formula for $p(n)$ states that for $n \geq 1$,

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] + R_2(n, N), \tag{2.1}$$

where $A_k(n)$ is an arithmetic function, $R_2(n, N)$ is the remainder term and

$$\mu(n) := \frac{\pi}{6}\sqrt{24n-1}, \tag{2.2}$$

see [19] and [26]. Lehmer [19] gave an error bound

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)}\right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2 \right], \tag{2.3}$$

which is valid for all positive integers n and N. Using this estimation, we have the following lemma.

Lemma 2.1. *For any given integer t, there exists $N(t)$ such that for all $n \geq N(t)$,*

$$\frac{\sqrt{12}\pi^2 e^{\mu}}{36\mu^2} \left(1 - \frac{1}{\mu} - \frac{1}{\mu^t}\right) < p(n) < \frac{\sqrt{12}\pi^2 e^{\mu}}{36\mu^2} \left(1 - \frac{1}{\mu} + \frac{1}{\mu^t}\right), \tag{2.4}$$

where μ is the abbreviation for $\mu(n)$.

Proof. Setting $N = 2$ in (2.1) and (2.3), we get

$$p(n) = \frac{\sqrt{12}}{24n-1} \left(\left(1 + \frac{1}{\mu}\right) e^{-\mu} + \left(1 - \frac{1}{\mu}\right) e^{\mu} + \frac{(-1)^n}{\sqrt{2}} \left(1 + \frac{2}{\mu}\right) e^{-\frac{\mu}{2}} + \frac{(-1)^n}{\sqrt{2}} \left(1 - \frac{2}{\mu}\right) e^{\frac{\mu}{2}} \right) + R_2(n, 2) \tag{2.5}$$

and

$$|R_2(n, 2)| \leq \frac{\pi^2}{3^{\frac{1}{2}} 2^{\frac{2}{3}}} \left(\frac{8}{\mu^3} \sinh\left(\frac{\mu}{2}\right) - \frac{4}{\mu^2} + \frac{1}{6} \right). \tag{2.6}$$

Restate (2.5) as

$$p(n) = \frac{\sqrt{12} e^{\mu}}{24n-1} \left(\left(1 - \frac{1}{\mu}\right) + R(n) \right), \tag{2.7}$$

where

$$
\begin{aligned}
|R(n)|=\frac{1}{\sqrt{2}}&\left(\left(1-\frac{2}{\mu}\right)e^{-\frac{\mu}{2}}+\left(1+\frac{2}{\mu}\right)e^{-\frac{3\mu}{2}}\right)\\
&+\left(1+\frac{1}{\mu}\right)e^{-2\mu}+\frac{36\mu^2|R(n,2)|}{\sqrt{12}\pi^2e^{\mu}}.
\end{aligned} \tag{2.8}
$$

It can be checked that for $n>350$,

$$
|R(n)|<\frac{e^{-\frac{\mu}{2}}}{\sqrt{2}}\left(1+e^{-\mu}\right)+\left(1+\frac{1}{\mu}\right)e^{-2\mu}+\frac{\mu^2e^{-\mu}}{\sqrt[3]{4}}+\frac{12\sqrt[3]{2}e^{-\frac{\mu}{2}}}{\mu}<6e^{-\frac{\mu}{2}}. \tag{2.9}
$$

It is easily seen that for any given t, there exists $N(t)$ such that for all $n\geq N(t)$,

$$
6e^{-\frac{\mu}{2}}<\frac{1}{\mu^t}, \tag{2.10}
$$

which implies that

$$
\frac{\sqrt{12}e^{\mu}}{24n-1}\left(1-\frac{1}{\mu}-\frac{1}{\mu^t}\right)<p(n)<\frac{\sqrt{12}e^{\mu}}{24n-1}\left(1-\frac{1}{\mu}+\frac{1}{\mu^t}\right). \tag{2.11}
$$

Since $\mu=\pi\sqrt{24n-1}/6$, the desired inequality is immediate by substituting n with $(36\mu^2+\pi^2)/(24\pi^2)$ in the above inequality. ∎

In particular, for $t=10$ and $t=22$, we get the following bounds that will be used in the rest of this paper.

$$
\begin{aligned}
&\frac{\sqrt{12}\pi^2e^{\mu}}{36\mu^2}\left(1-\frac{1}{\mu}-\frac{1}{\mu^{10}}\right)<p(n)<\frac{\sqrt{12}\pi^2e^{\mu}}{36\mu^2}\left(1-\frac{1}{\mu}+\frac{1}{\mu^{10}}\right), \quad n\geq 1520,\\
&\frac{\sqrt{12}\pi^2e^{\mu}}{36\mu^2}\left(1-\frac{1}{\mu}-\frac{1}{\mu^{22}}\right)<p(n)<\frac{\sqrt{12}\pi^2e^{\mu}}{36\mu^2}\left(1-\frac{1}{\mu}+\frac{1}{\mu^{22}}\right), \quad n\geq 6084.
\end{aligned} \tag{2.12}
$$

Denote

$$
f(y):=\frac{1}{y^2}\left(1-\frac{1}{y}-\frac{1}{y^{10}}\right), \quad g(y):=\frac{1}{y^2}\left(1-\frac{1}{y}+\frac{1}{y^{10}}\right), \tag{2.13}
$$

then f and g are both polynomials in y^{-1}, and from Lemma 2.1 we have

$$
\frac{\sqrt{12}\pi^2e^{\mu}}{36}f(\mu)<p(n)<\frac{\sqrt{12}\pi^2e^{\mu}}{36}g(\mu), \quad n\geq 1520. \tag{2.14}
$$

We also use the first several terms of the Taylor expansion to approximate the exponential function, specifically, for $y < 0$, $e_1(y) < e^y < e_2(y)$ where

$$e_1(y) := 1 + y + \frac{y^2}{2} + \frac{y^3}{6}, \quad e_2(y) := 1 + y + \frac{y^2}{2}. \tag{2.15}$$

We still need the bounds for $\mu(n+k)$ that are polynomials in μ and μ^{-1}. For example, if $x_1 = \sqrt{y^2 + a}$, then we have that for $y^2 \geq a$, $x_{11} < x_1 < x_{12}$ where

$$\begin{aligned} x_{11} &:= y + \frac{a}{2y} - \frac{a^2}{8y^3} + \frac{a^3}{16y^5} - \frac{5a^4}{128y^7} + \frac{7a^5}{256y^9} - \frac{21a^6}{512y^{11}}, \\ x_{12} &:= y + \frac{a}{2y} - \frac{a^2}{8y^3} + \frac{a^3}{16y^5} - \frac{5a^4}{128y^7} + \frac{7a^5}{256y^9}. \end{aligned} \tag{2.16}$$

Note that $x_1 = \mu(n+k)$ if y represents $\mu(n)$ and $a = 2k\pi^2/3$. Now we are in a position to establish relationships among the notions above.

Lemma 2.2. *For any given positives a, b and all $y \geq 12(a+b+1)$,*

$$\begin{vmatrix} e^{x_1} f(x_1) & e^{x_3} g(x_3) \\ e^{y} g(y) & e^{x_2} f(x_2) \end{vmatrix} > 0, \tag{2.17}$$

where

$$x_1 = \sqrt{y^2 + a}, \quad x_2 = \sqrt{y^2 + b}, \quad x_3 = \sqrt{y^2 + a + b}. \tag{2.18}$$

Proof. We denote x_{11}, x_{12} as in (2.16), and $x_{21}, x_{22}, x_{31}, x_{32}$ in similar ways. Then, for $y^2 \geq (a+b)$, we have

$$x_{i1} < x_i < x_{i2}, \quad i = 1, 2, 3. \tag{2.19}$$

Moreover, if we let

$$z := \sqrt{y^2 - \frac{a+b}{2}} \tag{2.20}$$

and

$$z_2 := y + \frac{a+b}{4y} - \frac{(a+b)^2}{32y^3} + \frac{(a+b)^3}{128y^5} - \frac{5(a+b)^4}{2048y^7} + \frac{7(a+b)^5}{8192y^9}, \tag{2.21}$$

then for $y^2 \geq (a+b)$, $z < z_2$.

Before calculating the determinant, we give some inequalities for $x_{11} + x_{21} - 2z_2$ and $y + x_{32} - 2z_2$. A direct computation gives

$$\begin{aligned}
x_{11} + x_{21} - 2z_2 =& -\frac{(a-b)^2}{16y^3} + \frac{3(a-b)^2(a+b)}{64y^5} - \frac{5(a-b)^2(7a^2+10ab+7b^2))}{1024y^7} \\
&+ \frac{7(a-b)^2(a+b)(15a^2+10ab+15b^2)}{4096y^9} - \frac{21(a^6+b^6)}{512y^{11}}, \\
y + x_{32} - 2z_2 =& -\frac{(a+b)^2}{16y^3} + \frac{3(a+b)^3}{64y^5} - \frac{35(a+b)^4}{1024y^7} + \frac{105(a+b)^5}{4096y^9}.
\end{aligned} \tag{2.22}$$

For $y \geq \sqrt{(a+b)}$, we have

$$\begin{aligned}
\frac{(a-b)^2}{16y^3} >& \frac{3(a-b)^2(a+b)}{64y^5} > \frac{5(a-b)^2(7a^2+10ab+7b^2))}{1024y^7} \\
>& \frac{7(a-b)^2(a+b)(15a^2+10ab+15b^2)}{4096y^9}
\end{aligned}$$

and

$$\frac{(a+b)^2}{16y^3} > \frac{3(a+b)^3}{64y^5} > \frac{35(a+b)^4}{1024y^7} > \frac{105(a+b)^5}{4096y^9}.$$

It follows that

$$x_{11} + x_{21} - 2z_2 < 0 \tag{2.23}$$

and

$$y + x_{32} - 2z_2 < 0. \tag{2.24}$$

Now we proceed to consider the determinant $\begin{vmatrix} e^{x_1}f(x_1) & e^{x_3}g(x_3) \\ e^{y}g(y) & e^{x_2}f(x_2) \end{vmatrix}$. By (2.23) and (2.24), we obtain

$$\begin{aligned}
& e^{x_1+x_2}f(x_1)f(x_2) - e^{y+x_3}g(y)g(x_3) \\
>& e^{2z_2}\Bigg(e^{x_{11}+x_{21}-2z_2}\frac{x_1^{10} - x_1^8x_{12} - 1}{x_1^{12}}\frac{x_2^{10} - x_2^8x_{12} - 1}{x_2^{12}} \\
& - e^{y+x_{32}-2z_2}\frac{y^{10} - y^9 + 1}{y^{12}}\frac{x_3^{10} - x_3^8x_{31} + 1}{x_3^{12}}\Bigg) \\
>& e^{2z_2}\Bigg(e_1(x_{11}+x_{21}-2z_2)\frac{x_1^{10} - x_1^8x_{12} - 1}{x_1^{12}}\frac{x_2^{10} - x_2^8x_{12} - 1}{x_2^{12}} \\
& - e_2(y+x_{32}-2z_2)\frac{y^{10} - y^9 + 1}{y^{12}}\frac{x_3^{10} - x_3^8x_{31} + 1}{x_3^{12}}\Bigg).
\end{aligned} \tag{2.25}$$

In order to prove this lemma, it suffices to show that

$$\begin{aligned} & x_1^{12}x_2^{12}x_3^{12}y^{39}\Bigg(e_1(x_{11}+x_{21}-2z_2)\frac{x_1^{10}-x_1^8x_{12}-1}{x_1^{12}}\frac{x_2^{10}-x_2^8x_{12}-1}{x_2^{12}} \\ & -e_2(y+x_{32}-2z_2)\frac{y^{10}-y^9+1}{y^{12}}\frac{x_3^{10}-x_3^8x_{31}+1}{x_3^{12}}\Bigg)>0. \end{aligned} \tag{2.26}$$

The left-hand side of the above inequality is

$$A(y)=\sum_{i=0}^{80}c_iy^i, \tag{2.27}$$

which is a polynomial of y. The first few coefficients c_i's are

$$c_{80}=\frac{1}{4}ab,\quad c_{79}=-\frac{3}{2}ab,\quad c_{78}=\frac{41}{16}ab(a+b)+3ab,\quad \ldots. \tag{2.28}$$

It can be checked that for $y\geq 12(a+b+1)$,

$$c_iy^i>-\frac{1}{2^{80-i}}c_{80}y^{80},\qquad 0\leq i\leq 79. \tag{2.29}$$

It follows that for $y>4$,

$$A(y)>\frac{1}{2^{80}}c_{80}y^{80}>0. \tag{2.30}$$

We conclude that for $y\geq\max\{12(a+b+1),\sqrt{a+b}\}=12(a+b+1)$,

$$\begin{vmatrix} e^{x_1}f(x_1) & e^{x_3}g(x_3) \\ e^{y}g(y) & e^{x_2}f(x_2) \end{vmatrix}>0. \tag{2.31}$$

∎

By a similar argument, we can prove the following result concerning with the 3 order determinant.

Lemma 2.3. *For any given positives a_1,a_2,b_1,b_2, there exists $N(a_1,a_2,b_1,b_2)$ s.t. for all $y>N(a_1,a_2,b_1,b_2)$,*

$$\begin{vmatrix} e^{x_1}f(x_1) & e^{x_2}g(x_2) & e^{x_3}f(x_3) \\ e^{x_4}g(x_4) & e^{x_5}f(x_5) & e^{x_6}g(x_6) \\ e^{x_7}f(x_7) & e^{x_8}g(x_8) & e^{x_9}f(x_9) \end{vmatrix}>0, \tag{2.32}$$

where

$$\begin{aligned} & x_1=\sqrt{y^2-a_1+b_1}, && x_2=\sqrt{y^2+b_1}, && x_3=\sqrt{y^2+b_1+b_2}, \\ & x_4=\sqrt{y^2-a_1}, && x_5=y, && x_6=\sqrt{y^2+b_2}, \\ & x_7=\sqrt{y^2-a_1-a_2}, && x_8=\sqrt{y^2-a_2}, && x_9=\sqrt{y^2-a_2+b_2}. \end{aligned} \tag{2.33}$$

Proof. As in (2.16), let

$$x_{11}\colon = y - \frac{a_1 - b_1}{2y} - \frac{(a_1 - b_1)^2}{8y^3} - \frac{(a_1 - b_1)^3}{16y^5} - \frac{5(a_1 - b_1)^4}{128y^7} - \frac{7(a_1 - b_1)^5\pi^{10}}{256y^9} - \frac{21(a_1 - b_1)^6}{512y^{11}},$$

$$x_{12}\colon = y - \frac{a_1 - b_1}{2y} - \frac{(a_1 - b_1)^2}{8y^3} - \frac{(a_1 - b_1)^3}{16y^5} - \frac{5(a_1 - b_1)^4}{128y^7} - \frac{7(a_1 - b_1)^5\pi^{10}}{256y^9} \tag{2.34}$$

and so on. Moreover, denote

$$z_2\colon = y + \frac{a}{2y} - \frac{a^2}{8y^3} + \frac{a^3}{16y^5} - \frac{5a^4}{128y^7} + \frac{7a^5}{256y^9}, \tag{2.35}$$

where $a = (-a_1 - a_2 + b_1 + b_2)/3$. Notice that $x_{11} + x_{91} + y - 3z_2, x_{32} + x_{72} + y - 3z_2, x_{12} + x_{62} + x_{82} - 3z_2, x_{31} + x_{41} + x_{81} - 3z_2, x_{21} + x_{61} + x_{71} - 3z_2, x_{22} + x_{42} + x_{92} - 3z_2$ are all polynomials in $1/y$ whose constant terms vanish and $1/y$-terms are negative. Using the same technique givin in (2.22), we can find $N_1(a_1, a_2, b_1, b_2)$ such that for $y > N_1(a_1, a_2, b_1, b_2)$, these polynomials are all negative. It follows that

$$\begin{aligned}
e^{x_{11}+x_{91}+y-3z_2} &> e_1(x_{11} + x_{91} + y - 3z_2),\\
e^{x_{32}+x_{72}+y-3z_2} &< e_2(x_{32} + x_{72} + y - 3z_2),\\
e^{x_{12}+x_{62}+x_{82}-3z_2} &< e_2(x_{12} + x_{62} + x_{82} - 3z_2),\\
e^{x_{31}+x_{41}+x_{81}-3z_2} &> e_1(x_{31} + x_{41} + x_{81} - 3z_2),\\
e^{x_{21}+x_{61}+x_{71}-3z_2} &> e_1(x_{21} + x_{61} + x_{71} - 3z_2),\\
e^{x_{22}+x_{42}+x_{92}-3z_2} &< e_2(x_{22} + x_{42} + x_{92} - 3z_2).
\end{aligned} \tag{2.36}$$

Thus we have that

$$
\begin{aligned}
&\begin{vmatrix} e^{x_1}f(x_1) & e^{x_2}g(x_2) & e^{x_3}f(x_3) \\ e^{x_4}g(x_4) & e^{x_5}f(x_5) & e^{x_6}g(x_6) \\ e^{x_7}f(x_7) & e^{x_8}g(x_8) & e^{x_9}f(x_9) \end{vmatrix} = \begin{vmatrix} e^{x_1}f(x_1) & e^{x_2}g(x_2) & e^{x_3}f(x_3) \\ e^{x_4}g(x_4) & e^{y}f(y) & e^{x_6}g(x_6) \\ e^{x_7}f(x_7) & e^{x_8}g(x_8) & e^{x_9}f(x_9) \end{vmatrix} \\
=&\Big(f(x_1)f(y)f(x_9)e^{x_1+x_9+y} - f(x_3)f(y)f(x_7)e^{x_3+x_7+y} \\
&- f(x_1)g(x_6)g(x_8)e^{x_1+x_6+x_8} + f(x_3)g(x_4)g(x_8)e^{x_3+x_4+x_8} \\
&+ f(x_2)g(x_6)g(x_7)e^{x_2+x_6+x_7} - f(x_2)g(x_4)g(x_9)e^{x_2+x_4+x_9}\Big) \\
>&e^{3z_2}\Big(f(x_1)f(y)f(x_9)e_1(x_{11}+x_{91}+y-3z_2) \\
&- f(x_3)f(y)f(x_7)e_2(x_{32}+x_{72}+y-3z_2) \\
&- f(x_1)g(x_6)g(x_8)e_2(x_{12}+x_{62}+x_{82}-3z_2) \\
&+ f(x_3)g(x_4)g(x_8)e_1(x_{31}+x_{41}+x_{81}-3z_2) \\
&+ f(x_2)g(x_6)g(x_7)e_1(x_{21}+x_{61}+x_{71}-3z_2) \\
&- f(x_2)g(x_4)g(x_9)e_2(x_{22}+x_{42}+x_{92}-3z_2)\Big).
\end{aligned} \tag{2.37}
$$

Since for $1 \le i \le 9$

$$
\begin{aligned}
f(x_i) &> \frac{1}{x_i^{12}}\left(x_i^{10} - x_i^8 x_{i2} - 1\right), \\
g(x_i) &< \frac{1}{x_i^{12}}\left(x_i^{10} - x_i^8 x_{i1} + 1\right),
\end{aligned} \tag{2.38}
$$

(5.7) can be rewritten as

$$(x_1x_2x_3x_4x_6x_7x_8x_9)^{-12}y^{-74}A(y), \tag{2.39}$$

where $A(y)$ is a certain polynomial with the coefficient of the first term

$$\frac{1}{128}a_1a_2b_1b_2(a_1+b_2)(a_2+b_1) > 0. \tag{2.40}$$

Hence we can find $N_2(a_1,a_2,b_1,b_2)$ such that for $y > N_2(a_1,a_2,b_1,b_2)$, $A(y) > 0$. Let $N(a_1,a_2,b_1,b_2) = \max\{N_1(a_1,a_2,b_1,b_2), N_2(a_1,a_2,b_1,b_2)\}$, we obtain that for $y > N(a_1,a_2,b_1,b_2)$, (2.32) holds. This completes the proof. ∎

3 Positivity of the 3 order determinant

In this section, we will give an iterated approach to prove the following theorem.

Theorem 3.1. *For given positive integers k_1, k_2, m_1, m_2, there exists $N(k_1, k_2, m_1, m_2)$ such that for $n \geq N(k_1, k_2, m_1, m_2)$,*

$$\begin{vmatrix} p(n-k_1+m_1) & p(n+m_1) & p(n+m_1+m_2) \\ p(n-k_1) & p(n) & p(n+m_2) \\ p(n-k_1-k_2) & p(n-k_2) & p(n-k_2+m_2) \end{vmatrix} > 0. \tag{3.1}$$

To prove it, we first give the following important lemma which also serves for the proofs in Section 4 and 5.

Lemma 3.2. *For all $h_i \in [e^{x_i} f(x_i), e^{x_i} g(x_i)], 1 \leq i \leq 9$, if*

$$x_7 \geq 12(a_1 + a_2 + b_1 + b_2 + 1), \tag{3.2}$$

then

$$\begin{vmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & e^{x_3} f(x_3) \\ e^{x_4} g(x_4) & e^{x_5} f(x_5) & e^{x_6} g(x_6) \\ e^{x_7} f(x_7) & e^{x_8} g(x_8) & e^{x_9} f(x_9) \end{vmatrix}, \tag{3.3}$$

where x_i's are defined as in Lemma 2.3.

Proof. Denote

$$D(x) := \begin{vmatrix} x & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix}. \tag{3.4}$$

Note that

$$y = \sqrt{x_8^2 + a_2}, \quad x_9 = \sqrt{x_8^2 + b_2}, \quad x_6 = \sqrt{x_8^2 + a_2 + b_2}, \tag{3.5}$$

and $x_8 > x_7 \geq 12(a_2 + b_2 + 1)$, by Lemma 2.2 we deduce that

$$\begin{vmatrix} e^{y} f(y) & e^{x_6} g(x_6) \\ e^{x_8} g(x_8) & e^{x_9} f(x_9) \end{vmatrix} > 0. \tag{3.6}$$

Thus

$$\begin{vmatrix} h_5 & h_6 \\ h_8 & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_5} f(x_5) & e^{x_6} g(x_6) \\ e^{x_8} g(x_8) & e^{x_9} f(x_9) \end{vmatrix} > 0. \tag{3.7}$$

It implies that $D(x)$ is strictly increasing. This leads to

$$\begin{vmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_1} f(x_1) & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix}. \tag{3.8}$$

Now we consider h_4. Since

$$x_2 = \sqrt{x_8^2 + (a_1 + a_2)}, \quad x_9 = \sqrt{x_8^2 + b_2}, \quad x_3 = \sqrt{x_8^2 + (a_1 + a_2) + b_2}, \tag{3.9}$$

and $x_8 > x_7 \geq 12(a_1 + a_2 + b_2 + 1)$, we have that

$$-\begin{vmatrix} h_2 & h_3 \\ h_8 & h_9 \end{vmatrix} < -\begin{vmatrix} e^{x_2} f(x_2) & e^{x_3} g(x_3) \\ e^{x_8} g(x_8) & e^{x_9} f(x_9) \end{vmatrix} < 0. \tag{3.10}$$

Therefore,

$$\begin{vmatrix} e^{x_1} f(x_1) & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_1} f(x_1) & h_2 & h_3 \\ e^{x_4} g(x_4) & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix}. \tag{3.11}$$

We proceed to deal with h_2. By Lemma 2.2, we have

$$-\begin{vmatrix} e^{x_4} g(x_4) & h_6 \\ h_7 & h_9 \end{vmatrix} < -\begin{vmatrix} e^{x_4} f(x_4) & e^{x_6} g(x_6) \\ e^{x_7} g(x_7) & e^{x_9} f(x_9) \end{vmatrix} < 0, \tag{3.12}$$

and hence

$$\begin{vmatrix} e^{x_1} f(x_1) & h_2 & h_3 \\ e^{x_4} g(x_4) & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & h_3 \\ e^{x_4} g(x_4) & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix}. \tag{3.13}$$

Repeat the above steps, we get

$$\begin{aligned}
&\begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & h_3 \\ e^{x_4} g(x_4) & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & e^{x_3} f(x_3) \\ e^{x_4} g(x_4) & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & e^{x_3} f(x_3) \\ e^{x_4} g(x_4) & e^{x_5} f(x_5) & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix} \\
&> \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & e^{x_3} f(x_3) \\ e^{x_4} g(x_4) & e^{x_5} f(x_5) & h_6 \\ e^{x_7} f(x_7) & h_8 & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & e^{x_3} f(x_3) \\ e^{x_4} g(x_4) & e^{x_5} f(x_5) & e^{x_6} g(x_6) \\ e^{x_7} f(x_7) & h_8 & h_9 \end{vmatrix} \\
&> \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & e^{x_3} f(x_3) \\ e^{x_4} g(x_4) & e^{x_5} f(x_5) & e^{x_6} g(x_6) \\ e^{x_7} f(x_7) & e^{x_8} g(x_8) & h_9 \end{vmatrix} > \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & e^{x_3} f(x_3) \\ e^{x_4} g(x_4) & e^{x_5} f(x_5) & e^{x_6} g(x_6) \\ e^{x_7} f(x_7) & e^{x_8} g(x_8) & e^{x_9} f(x_9) \end{vmatrix},
\end{aligned}$$

as desired. ∎

Now we turn to prove Theorem 3.1.

Proof of Theorem 3.1: Let

$$y = \mu(n), \quad a_1 = \frac{2}{3} k_1 \pi^2, \quad a_2 = \frac{2}{3} k_2 \pi^2, \quad b_1 = \frac{2}{3} m_1 \pi^2, \quad b_2 = \frac{2}{3} m_2 \pi^2 \tag{3.14}$$

in (2.33), then

$$\begin{aligned} &x_1 = \mu(n-k_1+m_1), \quad &&x_2 = \mu(n+m_1), \quad &&x_3 = \mu(n+m_1+m_2), \\ &x_4 = \mu(n-k_1), &&x_5 = \mu(n), &&x_6 = \mu(n+m_2), \\ &x_7 = \mu(n-k_1-k_2), &&x_8 = \mu(n-k_2), &&x_9 = \mu(n-k_2+m_2). \end{aligned} \tag{3.15}$$

Recall that (2.14) gives the upper and lower bound for $p(n)$, thus by Lemma 3.2 we have that for $n \geq k_1 + k_2 + 1520$,

$$\begin{aligned} &\begin{vmatrix} p(n-k_1+m_1) & p(n+m_1) & p(n+m_1+m_2) \\ p(n-k_1) & p(n) & p(n+m_2) \\ p(n-k_1-k_2) & p(n-k_2) & p(n-k_2+m_2) \end{vmatrix} \\ &> \frac{\sqrt{12}\pi^6}{3888} \begin{vmatrix} e^{x_1}f(x_1) & e^{x_2}g(x_2) & e^{x_3}f(x_3) \\ e^{x_4}g(x_4) & e^{y}f(y) & e^{x_6}g(x_6) \\ e^{x_7}f(x_7) & e^{x_8}g(x_8) & e^{x_9}f(x_9) \end{vmatrix}. \end{aligned} \tag{3.16}$$

It follows from Lemma 2.3 that we can find a $N(k_1, k_2, m_1, m_2)$ such that for $n > N(k_1, k_2, m_1, m_2)$, (3.1) holds, as desired. ∎

Now we aim to find the specific value of N for given (k_1, k_2, m_1, m_2). For example, when $(k_1, k_2, m_1, m_2) = (1, 1, 1, 1)$, we have that $a_1 = a_2 = b_1 = b_2 = 2\pi^2/3$, and the polynomial $A(y)$ in the end of Lemma 2.3 can be rewritten as

$$A(y) = \sum_{i=0}^{131} c_i y^i, \tag{3.17}$$

where

$$c_{131} = \frac{2\pi^{12}}{729}, \quad c_{130} = -\frac{4\pi^{12}}{81}, \quad c_{129} = \frac{43\pi^{12}}{324}, \quad \ldots. \tag{3.18}$$

It can be checked that for $y \geq 36$,

$$c_i y^i \geq -\frac{1}{2^{131-i}} c_{131} y^{131}, \qquad 0 \leq i \leq 130, \tag{3.19}$$

and the equality holds only if $i = 130$. It implies that

$$A(y) > \frac{1}{2^{131}} c_{131} y^{131} > 0, \tag{3.20}$$

Thus, for $y \geq 36 (n \geq 197)$, we get

$$\begin{vmatrix} e^{x_1}f(x_1) & e^{x_2}g(x_2) & e^{x_3}f(x_3) \\ e^{x_4}g(x_4) & e^{y}f(y) & e^{x_6}g(x_6) \\ e^{x_7}f(x_7) & e^{x_8}g(x_8) & e^{x_9}f(x_9) \end{vmatrix} > 0. \tag{3.21}$$

It means that for $n \geq 1524$,

$$\begin{vmatrix} p(n) & p(n+1) & p(n+2) \\ p(n-1) & p(n) & p(n+1) \\ p(n-2) & p(n-1) & p(n) \end{vmatrix} > 0. \tag{3.22}$$

We actually give an alternative proof for det $(p(n-i+j))_{1\leq i,j\leq 3} > 0$ given by Jia and Wang [15]. Employing the same method, we can get the following result which will be used in the proof of the positivity of $\det(p_(n-i+j))_{1\leq i,j\leq 4}$ for $n \geq 656$.

Theorem 3.3. *For $k_1+m_2 \leq 3$ and $k_2+m_1 \leq 3$, we have that for $n \geq 1524$,*

$$\begin{aligned} &\begin{vmatrix} p(n-k_1+m_1) & p(n+m_1) & p(n+m_1+m_2) \\ p(n-k_1) & p(n) & p(n+m_2) \\ p(n-k_1-k_2) & p(n-k_2) & p(n-k_2+m_2) \end{vmatrix} \\ &> \frac{\sqrt{12}\pi^6}{3888} \begin{vmatrix} e^{x_1} f(x_1) & e^{x_2} g(x_2) & e^{x_3} f(x_3) \\ e^{x_4} g(x_4) & e^{y} f(y) & e^{x_6} g(x_6) \\ e^{x_7} f(x_7) & e^{x_8} g(x_8) & e^{x_9} f(x_9) \end{vmatrix} > 0. \end{aligned} \tag{3.23}$$

This approach can be extended to prove det $(p(n-i+j))_{1\leq i,j\leq 4} > 0$ in Section 4.

4 Positivity of the 4 order determinant

In this section, we will apply the approach in Section 3 to prove the det $(p(n-i+j))_{1\leq i,j\leq 4} > 0$. Note that we may use notions which have been used before but with different meanings.

Theorem 4.1. *For $n \geq 656$, we have*

$$\begin{vmatrix} p(n) & p(n+1) & p(n+2) & p(n+3) \\ p(n-1) & p(n) & p(n+1) & p(n+2) \\ p(n-2) & p(n-1) & p(n) & p(n+1) \\ p(n-3) & p(n-2) & p(n-1) & p(n) \end{vmatrix} > 0. \tag{4.1}$$

Proof. As in Lemma 3.2, we claim that for $y \geq 6087$

$$\begin{vmatrix} p(n) & p(n+1) & p(n+2) & p(n+3) \\ p(n-1) & p(n) & p(n+1) & p(n+2) \\ p(n-2) & p(n-1) & p(n) & p(n+1) \\ p(n-3) & p(n-2) & p(n-1) & p(n) \end{vmatrix}$$
$$> \frac{\pi^8}{11664} \begin{vmatrix} e^y f_1(y) & e^{x_5} g_1(x_5) & e^{x_6} f_1(x_6) & e^{x_7} g_1(x_7) \\ e^{x_3} g_1(x_3) & e^y f_1(y) & e^{x_5} g_1(x_5) & e^{x_6} f_1(x_6) \\ e^{x_2} f_1(x_2) & e^{x_3} g_1(x_3) & e^y f_1(y) & e^{x_5} g_1(x_5) \\ e^{x_1} g_1(x_1) & e^{x_2} f_1(x_2) & e^{x_3} g_1(x_3) & e^y f_1(y) \end{vmatrix}, \tag{4.2}$$

where $y = \mu(n)$,

$$\begin{aligned} x_1 &= \mu(n-3), \quad x_2 = \mu(n-2), \quad x_3 = \mu(n-1), \\ x_5 &= \mu(n+1), \quad x_6 = \mu(n+2), \quad x_7 = \mu(n+3), \end{aligned} \tag{4.3}$$

and

$$f_1(y) := \frac{1}{y^2}\left(1 - \frac{1}{y} - \frac{1}{y^{22}}\right), \quad g_1(y) := \frac{1}{y^2}\left(1 - \frac{1}{y} + \frac{1}{y^{22}}\right). \tag{4.4}$$

To prove (4.2), we denote

$$D(x) := \begin{vmatrix} x & p(n+1) & p(n+2) & p(n+3) \\ p(n-1) & p(n) & p(n+1) & p(n+2) \\ p(n-2) & p(n-1) & p(n) & p(n+1) \\ p(n-3) & p(n-2) & p(n-1) & p(n) \end{vmatrix}. \tag{4.5}$$

By (2.12), we have that for $n \geq 6084$,

$$\frac{\sqrt{12}\pi^2 e^y}{36} f(y) < \frac{\sqrt{12}\pi^2 e^y}{36} f_1(y) < p(n) < \frac{\sqrt{12}\pi^2 e^y}{36} g_1(y) < \frac{\sqrt{12}\pi^2 e^y}{36} g(y). \tag{4.6}$$

Thus, from (2.1) and Lemma 3.2 we deduce that for $n \geq 6087$,

$$\begin{vmatrix} p(n) & p(n+1) & p(n+2) \\ p(n-1) & p(n) & p(n+1) \\ p(n-2) & p(n-1) & p(n) \end{vmatrix} > \frac{\sqrt{12}\pi^6}{3888} \begin{vmatrix} e^y f(y) & e^{x_5} g(x_5) & e^{x_6} f(x_6) \\ e^{x_3} g(x_3) & e^y f(y) & e^{x_5} g(x_5) \\ e^{x_2} f(x_2) & e^{x_3} g(x_3) & e^y f(y) \end{vmatrix} > 0. \tag{4.7}$$

It implies that $D(x)$ is strictly increasing, i.e.,

$$\begin{aligned}&\begin{vmatrix} p(n) & p(n+1) & p(n+2) & p(n+3) \\ p(n-1) & p(n) & p(n+1) & p(n+2) \\ p(n-2) & p(n-1) & p(n) & p(n+1) \\ p(n-3) & p(n-2) & p(n-1) & p(n) \end{vmatrix} \\ &> \begin{vmatrix} \frac{\sqrt{12}\pi^2}{36} e^y f_1(y) & p(n+1) & p(n+2) & p(n+3) \\ p(n-1) & p(n) & p(n+1) & p(n+2) \\ p(n-2) & p(n-1) & p(n) & p(n+1) \\ p(n-3) & p(n-2) & p(n-1) & p(n) \end{vmatrix}.\end{aligned} \tag{4.8}$$

Repeat the above steps, we get (4.2) by Lemma 2.2 and Lemma 3.2.

Now we proceed to prove (4.1). Denote

$$\begin{aligned}e_3(y) &:= 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \frac{y^5}{120} + \frac{y^6}{720} + \frac{y^7}{5040}, \\ e_4(y) &:= 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \frac{y^5}{120} + \frac{y^6}{720} + \frac{y^7}{5040} + \frac{y^8}{40320},\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}x_{i1} &:= y - \frac{(i-4)\pi^2}{2y} - \frac{(i-4)^2\pi^4}{8y^3} - \cdots - \frac{4199(i-4)^{11}\pi^{22}}{2^{19}y^{21}} - \frac{29393(i-4)^{13}\pi^{24}}{2^{21}y^{23}}, \\ x_{i2} &:= y - \frac{(i-4)\pi^2}{2y} - \frac{(i-4)^2\pi^4}{8y^3} - \cdots - \frac{4199(i-4)^{11}\pi^{22}}{2^{19}y^{21}}.\end{aligned} \tag{4.10}$$

One can see that for $x < 0$,

$$e_3(x) < e^x < e_4(x) \tag{4.11}$$

and for $i = 1, 2, 3, 5, 6, 7$,

$$x_{i1} < x_i < x_{i2}. \tag{4.12}$$

By a similar argument as in the proof of Lemma 2.3, we get

$$\begin{vmatrix} e^y f_1(y) & e^{x_5} g_1(x_5) & e^{x_6} f_1(x_6) & e^{x_7} g_1(x_7) \\ e^{x_3} g_1(x_3) & e^y f_1(y) & e^{x_5} g_1(x_5) & e^{x_6} f_1(x_6) \\ e^{x_2} f_1(x_2) & e^{x_3} g_1(x_3) & e^y f_1(y) & e^{x_5} g_1(x_5) \\ e^{x_1} g_1(x_1) & e^{x_2} f_1(x_2) & e^{x_3} g_1(x_3) & e^y f_1(y) \end{vmatrix} > x_1^{-24} x_2^{-48} x_3^{-72} x_5^{-72} x_6^{-48} x_7^{-24} y^{-258} A(y), \tag{4.13}$$

where

$$A(y) = \sum_{i=0}^{520} c_i y^i, \quad c_{520} = \frac{4\pi^{24}}{177147}, \quad c_{519} = -\frac{160\pi^{24}}{177147}, \quad \ldots \tag{4.14}$$

It is easily seen that for $y > 80 (n \geq 973)$,

$$c_i y^i > -\frac{1}{2^{520-i}} c_{520} y^{520}, \qquad 0 \leq i \leq 519, \tag{4.15}$$

on which occasion

$$A(y) > \frac{1}{2^{520}} c_{520} y^{520} > 0. \tag{4.16}$$

It implies that (4.1) is true for $n \geq 6087$. Moreover, numerical evidence shows that it also holds for $656 \leq n \leq 6086$. The proof is completed. ∎

5 Overpartition

In this section, we confine our attention to a generalization of partition, namely, the overpartition $\bar{p}(n)$. An overpartition of n is an ordinary partition of n with the added condition that the first occurrence of any part may be overlined or not. For the background and analytical properties of $\bar{p}(n)$, see [3], [10], [18], [28] and [29]. In this paper, we shall give the same determinantal inequalities as in Section 3 and 4 for the overpartition function $\bar{p}(n)$.

Note that the whole approach presented in Section 3 and 4 also works for overpartition. Thus, in this section we will omit some tedious formulae and statements. First, we use the Rademacher-type estimation to get a bound for $\bar{p}(n)$. Let

$$\bar{\mu}(n) := \pi\sqrt{n}. \tag{5.1}$$

Liu and Zhang [18] mentioned that following the Rademacher type series given by Zuckermann [29] and error term given by Engel [10], one has

$$\bar{p}(n) = \frac{1}{8n}\left(\left(1+\frac{1}{\mu}\right)e^{-\mu} + \left(1-\frac{1}{\mu}\right)e^{\mu}\right) + \bar{R}_2(n,2), \tag{5.2}$$

with

$$|\bar{R}_2(n,2)| \leq \frac{2^{\frac{5}{2}}}{n\mu}\sinh\left(\frac{\mu}{2}\right). \tag{5.3}$$

Imitating the proof of Lemma 2.1, we can prove the following lemma.

Lemma 5.1. *For any given integer t, there exists $N(t)$ such that for all $n \geq N(t)$,*

$$\frac{\pi^2 e^{\bar{\mu}}}{8\bar{\mu}^2}\left(1-\frac{1}{\bar{\mu}}-\frac{1}{\bar{\mu}^t}\right) < \bar{p}(n) < \frac{\pi^2 e^{\bar{\mu}}}{8\bar{\mu}^2}\left(1-\frac{1}{\bar{\mu}}+\frac{1}{\bar{\mu}^t}\right), \tag{5.4}$$

where $\bar{\mu}$ is the abbreviation for $\bar{\mu}(n)$.

Specifically, for $t = 10$ and 22, we have

$$\begin{aligned} \frac{\pi^2 e^{\bar{\mu}}}{8\bar{\mu}^2}\left(1-\frac{1}{\bar{\mu}}-\frac{1}{\bar{\mu}^{10}}\right) <&\bar{p}(n) < \frac{\pi^2 e^{\bar{\mu}}}{8\bar{\mu}^2}\left(1-\frac{1}{\bar{\mu}}+\frac{1}{\bar{\mu}^{10}}\right), \quad n \geq 821, \\ \frac{\pi^2 e^{\bar{\mu}}}{8\bar{\mu}^2}\left(1-\frac{1}{\bar{\mu}}-\frac{1}{\bar{\mu}^{22}}\right) <&\bar{p}(n) < \frac{\pi^2 e^{\bar{\mu}}}{8\bar{\mu}^2}\left(1-\frac{1}{\bar{\mu}}+\frac{1}{\bar{\mu}^{22}}\right), \quad n \geq 5644. \end{aligned} \tag{5.5}$$

Let $y = \bar{\mu}(n)$ and $(a_1, a_2, b_1, b_2) = (k_1\pi^2, k_2\pi^2, m_1\pi^2, m_2\pi^2)$, then in (2.33) we have

$$\begin{aligned} &x_1 = \sqrt{y^2 + (m_1 - k_1)\pi^2}, \quad x_2 = \sqrt{y^2 + m_1\pi^2}, \quad x_3 = \sqrt{y^2 + (m_1 + m_2)\pi^2}, \\ &x_4 = \sqrt{y^2 + (-k_1)\pi^2}, \qquad x_5 = y, \qquad x_6 = \sqrt{y^2 + m_2\pi^2}. \\ &x_7 = \sqrt{y^2 + (-k_1 - k_2)\pi^2}, \quad x_8 = \sqrt{y^2 + (-k_2)\pi^2}, \quad x_9 = \sqrt{y^2 + (m_2 - k_2)\pi^2}. \end{aligned} \tag{5.6}$$

By Lemma 2.3 and Lemma 3.2, we get that

$$\begin{aligned} &\begin{vmatrix} \bar{p}(n-k_1+m_1) & \bar{p}(n+m_1) & \bar{p}(n+m_1+m_2) \\ \bar{p}(n-k_1) & \bar{p}(n) & \bar{p}(n+m_2) \\ \bar{p}(n-k_1-k_2) & \bar{p}(n-k_2) & \bar{p}(n-k_2+m_2) \end{vmatrix} \\ >&\frac{\pi^6}{512}\begin{vmatrix} e^{x_1}f(x_1) & e^{x_2}g(x_2) & e^{x_3}f(x_3) \\ e^{x_4}g(x_4) & e^{x_5}f(x_5) & e^{x_6}g(x_6) \\ e^{x_7}f(x_7) & e^{x_8}g(x_8) & e^{x_9}f(x_9) \end{vmatrix} > 0. \end{aligned} \tag{5.7}$$

Hence we deduce the following result.

Theorem 5.2. *For given positive integers k_1, k_2, m_1, m_2, there exists $N(k_1, k_2, m_1, m_2)$ such that for $n \geq N(k_1, k_2, m_1, m_2)$,*

$$\begin{vmatrix} \bar{p}(n-k_1+m_1) & \bar{p}(n+m_1) & \bar{p}(n+m_1+m_2) \\ \bar{p}(n-k_1) & \bar{p}(n) & \bar{p}(n+m_2) \\ \bar{p}(n-k_1-k_2) & \bar{p}(n-k_2) & \bar{p}(n-k_2+m_2) \end{vmatrix} > 0. \tag{5.8}$$

In particular, for $k_1 = k_2 = m_1 = m_2 = 1$ and $n \geq 823$, we can get

$$\begin{vmatrix} \bar{p}(n) & \bar{p}(n+1) & \bar{p}(n+2) \\ \bar{p}(n-1) & \bar{p}(n) & \bar{p}(n+1) \\ \bar{p}(n-2) & \bar{p}(n-1) & \bar{p}(n) \end{vmatrix} > \frac{\pi^6}{512} x_1^{-12} x_2^{-24} x_3^{-24} x_4^{-12} y^{-74} A(y), \tag{5.9}$$

where

$$A(y) = \sum_{i=0}^{131} c_i y^i, \quad c_{131} = \frac{\pi^{12}}{32}, \quad c_{130} = -\frac{9\pi^{12}}{16}, \quad \ldots. \tag{5.10}$$

It can be checked that for $y > 36 (n \geq 197)$,

$$c_i y^i > -\frac{1}{2}|c_{i+1}|y^{i+1}, \qquad 0 \leq i \leq 130. \tag{5.11}$$

Thus

$$A(y) > 2c_{130}y^{130} + c_{131}y^{131} > 0. \tag{5.12}$$

Numerical evidence shows that the above determinantal inequality also holds for $42 \leq n \leq 822$. Hence, we obtain the following theorem.

Theorem 5.3. *For $n \geq 42$,*

$$\begin{vmatrix} \bar{p}(n) & \bar{p}(n+1) & \bar{p}(n+2) \\ \bar{p}(n-1) & \bar{p}(n) & \bar{p}(n+1) \\ \bar{p}(n-2) & \bar{p}(n-1) & \bar{p}(n) \end{vmatrix} > 0. \tag{5.13}$$

By a similar argument as in Section 3, we get the following result.

Theorem 5.4. *For $k_1 + m_2 \leq 3$ and $k_2 + m_1 \leq 3$, we have that for $n \geq 825$,*

$$\begin{aligned} &\begin{vmatrix} \bar{p}(n-k_1+m_1) & \bar{p}(n+m_1) & \bar{p}(n+m_1+m_2) \\ \bar{p}(n-k_1) & \bar{p}(n) & \bar{p}(n+m_2) \\ \bar{p}(n-k_1-k_2) & \bar{p}(n-k_2) & \bar{p}(n-k_2+m_2) \end{vmatrix} \\ &> \frac{\pi^6}{512} \begin{vmatrix} e^{x_1}f(x_1) & e^{x_2}g(x_2) & e^{x_3}f(x_3) \\ e^{x_4}g(x_4) & e^{x_5}f(x_5) & e^{x_6}g(x_6) \\ e^{x_7}f(x_7) & e^{x_8}g(x_8) & e^{x_9}f(x_9) \end{vmatrix} > 0. \end{aligned} \tag{5.14}$$

Now we go forward to 4 order determinants for the overpartition function. Following the approach given in Section 4, we have that for $n \geq 5647$,

$$\begin{aligned} &\begin{vmatrix} \bar{p}(n) & \bar{p}(n+1) & \bar{p}(n+2) & \bar{p}(n+3) \\ \bar{p}(n-1) & \bar{p}(n) & \bar{p}(n+1) & \bar{p}(n+2) \\ \bar{p}(n-2) & \bar{p}(n-1) & \bar{p}(n) & \bar{p}(n+1) \\ \bar{p}(n-3) & \bar{p}(n-2) & \bar{p}(n-1) & \bar{p}(n) \end{vmatrix} \\ &> \frac{\pi^8}{4096} \begin{vmatrix} e^{y}f_1(y) & e^{x_5}g_1(x_5) & e^{x_6}f_1(x_6) & e^{x_7}g_1(x_7) \\ e^{x_3}g_1(x_3) & e^{y}f_1(y) & e^{x_5}g_1(x_5) & e^{x_6}f_1(x_6) \\ e^{x_2}f_1(x_2) & e^{x_3}g_1(x_3) & e^{y}f_1(y) & e^{x_5}g_1(x_5) \\ e^{x_1}g_1(x_1) & e^{x_2}f_1(x_2) & e^{x_3}g_1(x_3) & e^{y}f_1(y) \end{vmatrix}, \end{aligned} \tag{5.15}$$

where $y = \bar{\mu}(n)$, and

$$\begin{aligned} x_1 &= \bar{\mu}(n-3), \quad x_2 = \bar{\mu}(n-2), \quad x_3 = \bar{\mu}(n-1), \\ x_5 &= \bar{\mu}(n+1), \quad x_6 = \bar{\mu}(n+2), \quad x_7 = \bar{\mu}(n+3), \end{aligned} \tag{5.16}$$

A direct calculation leads to

$$\begin{vmatrix} e^y f_1(y) & e^{x_5} g_1(x_5) & e^{x_6} f_1(x_6) & e^{x_7} g_1(x_7) \\ e^{x_3} g_1(x_3) & e^y f_1(y) & e^{x_5} g_1(x_5) & e^{x_6} f_1(x_6) \\ e^{x_2} f_1(x_2) & e^{x_3} g_1(x_3) & e^y f_1(y) & e^{x_5} g_1(x_5) \\ e^{x_1} g_1(x_1) & e^{x_2} f_1(x_2) & e^{x_3} g_1(x_3) & e^y f_1(y) \end{vmatrix} \tag{5.17}$$

$$> \frac{\pi^8}{2048} x_1^{-24} x_2^{-48} x_3^{-72} x_5^{-72} x_6^{-48} x_7^{-24} y^{-258} A(y),$$

where

$$A(y) = \sum_{i=0}^{520} c_i y^i, \quad c_{520} = \frac{3\pi^{24}}{1024}, \quad c_{519} = -\frac{15\pi^{24}}{128}, \quad \ldots \tag{5.18}$$

It can be checked that for $y > 80 (n \geq 973)$,

$$a_i y^i > -\frac{1}{2^{520-i}} c_{520} y^{520}, \qquad 0 \leq i \leq 519. \tag{5.19}$$

It implies that

$$A(y) > \frac{1}{2^{520}} c_{520} y^{520} > 0. \tag{5.20}$$

Thus for $n \geq 5647$, $\det(\bar{p}(n-i+j))_{1\leq i,j\leq 4} > 0$. Numerical evidence shows that for $141 \leq n \leq 5646$, this inequality also holds. Hence we conclude with the following theorem.

Theorem 5.5. *For $n \geq 141$,*

$$\begin{vmatrix} \bar{p}(n) & \bar{p}(n+1) & \bar{p}(n+2) & \bar{p}(n+3) \\ \bar{p}(n-1) & \bar{p}(n) & \bar{p}(n+1) & \bar{p}(n+2) \\ \bar{p}(n-2) & \bar{p}(n-1) & \bar{p}(n) & \bar{p}(n+1) \\ \bar{p}(n-3) & \bar{p}(n-2) & \bar{p}(n-1) & \bar{p}(n) \end{vmatrix} > 0. \tag{5.21}$$

Remark. Recently, using the technique given in [15], Mukherjee [21] independently proved that $\bar{p}(n)$ satisfies (5.13) and conjectured that $\det(\bar{p}(n-i+j))_{1\leq i,j\leq k} > 0$ for all k and sufficiently large n. We actually give an affirmative answer to Mukherjee's conjecture for $k = 4$ in Theorem 5.5.

6 For general cases

As a conclusion, we give an outline of how to extend our method to prove $\det(p(n-i+j))_{1\leq i,j\leq k} > 0$ step by step. Notice that in the proofs of the positivity of $\det(p(n-i+j))_{1\leq i,j\leq 3}$ and $\det(p(n-i+j))_{1\leq i,j\leq 4}$, we actually depend on the following general lemma.

Lemma 6.1. *Suppose sequences $x(n)$, $f(n)$ and $g(n)$ satisfies*

$$f(n) \leq x(n) \leq g(n). \tag{6.1}$$

If there exists N' such that for $n \geq N'$, $l \leq k$ and integers $a_1, b_1, a_2, b_2, \ldots, a_l, b_l \in [-M, M]$, the l-order determinants

$$\begin{vmatrix} f(n) & g(n+b_1) & f(n+b_1+b_2) & \cdots & \\ g(n-a_1) & f(n-a_1+b_1) & g(n-a_1+b_1+b_2) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \\ & & & \cdots & f(n-\sum a_i + \sum b_i) \end{vmatrix}_{l\times l} \tag{6.2}$$

are always positive, then there exists N such that for $n \geq N, l = k$ the determinant $\det(x(n-k_i+m_j))_{1\leq i,j\leq l}$ is positive for any integers k_i, m_j whose absolute values are no greater than M.

The sketch of the proof. We use induction on k. Assume that the result holds for $k-1$. Denote by $D(x)$ the determinant where the element in the position (i,j) is substituted with x in $\det(x(n-k_i+m_j))_{1\leq i,j\leq k}$. Then the coefficient of x is of the form $\det(x'(n-k'_i+m'_j))_{1\leq i,j\leq k-1}$, where x',k' and m' satisfy

$$f(n) \leq x'(n) \leq g(n) \tag{6.3}$$

and

$$|k'_1|, |m'_1|, |k'_2|, |m'_2|, \ldots \leq M \tag{6.4}$$

as well. By the induction hypothesis, the coefficient of x is positive. It follows that $D(x)$ is strictly increasing (decreasing, respectively) if and only if $i+j$ is even (odd, respectively). Hence we deduce that $\det(x(n-k_i+m_j))_{1\leq i,j\leq l}$ is greater than the determinant in (6.2), which is positive by the induction hypothesis. ∎

In Section 3 and 4, we actually use this lemma for the case $k = 3$ and 4. And in the upper and lower bound for $p(n)$

$$\frac{\sqrt{12}\pi^2 e^{\mu}}{36\mu^2}\left(1 - \frac{1}{\mu} - \frac{1}{\mu^t}\right) < p(n) < \frac{\sqrt{12}\pi^2 e^{\mu}}{36\mu^2}\left(1 - \frac{1}{\mu} + \frac{1}{\mu^t}\right), \tag{6.5}$$

we find $t = 10$ and 22 are enough. For large k, one only need to find a suitable t such that the process in Section 3 and 4 works.

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