ON BASIC 2-ARC-TRANSITIVE GRAPHS

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Abstract. A connected graph $\Gamma = (V, E)$ of valency at least 3 is called a basic 2-arc-transitive graph if its full automorphism group has a subgroup G with the following properties: (i) G acts transitively on the set of 2-arcs of Γ, and (ii) every minimal normal subgroup of G has at most two orbits on V. Based on Praeger's theorems on 2-arc-transitive graphs, this paper presents a further understanding on the automorphism group of a basic 2-arc-transitive graph.

Keywords. 2-arc-transitive graph, stabilizer, quasiprimitive permutation group, almost simple group.

1. Introduction

All graphs considered in this paper are assumed to be finite, simple and undirected.

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. Denote by $\mathrm{Aut}(\Gamma)$ the full automorphism group of the graph Γ. A subgroup G of $\mathrm{Aut}(\Gamma)$, written as $G \leqslant \mathrm{Aut}(\Gamma)$, is called a group of Γ. For a vertex $\alpha \in V$, let $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$ and $\Gamma(\alpha) = \{\beta \in V \mid \{\alpha, \beta\} \in E\}$, called the stabilizer of α in G and the neighborhood of α in Γ, respectively. A group G of Γ is call locally-primitive on Γ if for each $\alpha \in V$ the stabilizer G_α acts primitively on $\Gamma(\alpha)$, that is, $\Gamma(\alpha)$ has no nontrivial G_α-invariant partition. Recall that an arc of Γ is an ordered pair of adjacent vertices, and a 2-arc is a triple (α, β, γ) of vertices with $\{\alpha, \beta\}$, $\{\beta, \gamma\} \in E$ and $\alpha \neq \gamma$. A group G of Γ is said to be vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive on Γ if G acts transitively on the vertices, edges, arcs or 2-arcs of Γ, respectively. A graph is called vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive if it has a vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive group, respectively.

A connected regular graph $\Gamma = (V, E)$ of valency at least 3 is called a basic 2-arc-transitive graph if it has a 2-arc-transitive group G such that every minimal normal subgroup of G has at most two orbits on V. Praeger [17, 18] observed that a connected 2-arc-transitive graph of valency at least 3 is a normal cover of some basic 2-arc-transitive graph. Based on the O'Nan-Scott theorem for quasiprimitive permutation groups established in [17], Praeger [17, 18] characterized the group-theoretic structures for basic 2-arc-transitive graphs. She proved that, except for complete bipartite graphs and another case about bipartite graphs, basic 2-arc-transitive graphs are associated with quasiprimitive groups of type I, II, IIIb(i) or III(c) described as in [17, Section 2], which is named HA, AS, PA or TW in [19], respectively.

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Praeger's framework for 2-arc-transitive graphs stimulated a wide interest in classification or characterization of basic 2-arc-transitive graphs. For example, a construction of the graphs associated with quasiprimitive permutation groups of type TW is given in [2], the graphs associated with Suzuki simple groups, Ree simple groups and 2-dimensional projective linear groups are classified in [5, 6, 9] respectively, the graphs of order a prime power are classified in [10]. Besides, Li [11] proved that all basic 2-arc-transitive graphs of odd order can be constructed from almost simple groups, which inspires the ongoing project to classify basic 2-arc-transitive graphs of odd order, see [12] for some progress in this topic.

In this paper, we have a further understanding on the automorphism groups of basic 2-arc-transitive graphs, which may be helpful to study the Praeger's problem proposed in [18]: *Classify all finite basic* 2*-arc-transitive graphs.* Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to G. Fix an edge $\{\alpha, \beta\} \in E$, and set $G^* = \langle G_\alpha, G_\beta\rangle$. It is well-known that $|G : G^*| \leqslant 2$, G^* is edge-transitive on Γ, and Γ is bipartite if and only if $|G : G^*| = 2$, refer to [21, Exercise 3.8]. If Γ is not bipartite, then G is a quasiprimitive permutation group on V of type HA, AS, PA or TW, refer to [17, Theorem 2] or [19, Theorem 6.1]. In this case, it is easily deduced that G has a unique minimal normal subgroup, the socle $\mathrm{soc}(G)$ of G. Somewhat surprisingly, this is almost true for the bipartite case. If Γ is bipartite, that is, $|G : G^*| = 2$, then Praeger [18] proved that either Γ is a complete bipartite graph, or G^* acts faithfully on both parts of Γ and one of the following holds:

(I) G^* is quasiprimitive on both parts of Γ with a same type HA, AS, PA or TW;
(II) G has a normal subgroup N which is a direct product of two intransitive minimal normal subgroups of G^*.

For (I) and (II), we prove in Section 3 that $\mathrm{soc}(G^*)$ is the unique minimal normal subgroup of G, and so $\mathrm{soc}(G) = \mathrm{soc}(G^*)$. Thus, in general, $\mathrm{soc}(G)$ is the unique minimal normal subgroup of G, provided that Γ is not a complete bipartite graph. Based on this observation and the description of types HA, AS, PA or TW, we investigate in Section 4 the action of $\mathrm{soc}(G)$ on the graph Γ, including the structure of vertex-stabilizers and the semiregularity of simple direct factors of $\mathrm{soc}(G)$. Then we formulate the following result, which is finally proved in Section 4.

Theorem 1.1. *Assume that $\Gamma = (V, E)$ is a basic* 2*-arc-transitive graph with respect to a group G. Let $G^* = \langle G_\alpha, G_\beta\rangle$ and $N = \mathrm{soc}(G^*)$, where $\{\alpha, \beta\} \in E$. Then either Γ is a complete bipartite graph, or the following statements hold:*

(1) *N is the unique minimal normal subgroup of G, in particular, $N = \mathrm{soc}(G)$;*
(2) *either N is simple, or every simple direct factor of N is semiregular on V;*
(3) *either N is locally-primitive on Γ, or N_α is given as follows:*
   (i) *$N_\alpha = 1$; or*
   (ii) *$N_\alpha = \mathbb{Z}_p^k{:}(\mathbb{Z}_{m_1}.\mathbb{Z}_m) = (\mathbb{Z}_p^k \times \mathbb{Z}_{m_1}).\mathbb{Z}_m$ and $|\Gamma(\alpha)| = p^k$, where $m_1 \bigm| m$, $m \bigm| (p^d - 1)$ for some divisor d of k with $d < k$; or*
   (iii) *$N_\alpha = \mathbb{Z}_3^4{:}(Q.\mathrm{Q}_8) = (\mathbb{Z}_3^4 \times Q).\mathrm{Q}_8$ and $|\Gamma(\alpha)| = 3^4$, where Q_8 is the quaternion group and Q is isomorphic to a subgroup of Q_8.*

It is well-known that the order of a finite nonabelian simple group is divisible by 4 and two distinct odd primes. By (2) of Theorem 1.1, we have the following corollaries.

Corollary 1.2. *Assume that* $\Gamma = (V, E)$ *is a basic* 2*-arc-transitive graph with respect to a group* G*, and* $G^* = \langle G_\alpha, G_\beta\rangle$ *for an edge* $\{\alpha, \beta\} \in E$*. Assume that one of* G^**-orbits on* V *has length* $p^a q^b$*, where* a *and* b *are positive integers,* p *and* q *are distinct primes. If* Γ *is not a complete bipartite graph, then* G *is almost simple.*

Corollary 1.3. *Assume that* $\Gamma = (V, E)$ *is a basic* 2*-arc-transitive graph with respect to a group* G*, and* $G^* = \langle G_\alpha, G_\beta\rangle$ *for an edge* $\{\alpha, \beta\} \in E$*. Assume that one of* G^**-orbits on* V *has length* n *or* $2n$*, where* n *is either an odd integer or a power of* 2*. If* Γ *is not a complete bipartite graph then either* G *is almost simple, or* $|G^* : G_\alpha| = p^k$ *and* $\mathrm{soc}(G^*) \cong \mathbb{Z}_p^k$*, where* p *is a prime and* $k \geqslant 1$*.*

Another consequence of Theorem 1.1 is stated as follows.

Theorem 1.4. *Let* $\Gamma = (V, E)$ *be a connected graph,* $G \leqslant \mathrm{Aut}(\Gamma)$ *and* $G^* = \langle G_\alpha, G_\beta\rangle$ *for an edge* $\{\alpha, \beta\} \in E$*. Assume that* G *is* 2*-arc-transitive on* Γ*, and* G^* *acts primitively on each* G^**-orbit on* V*. Then one of the following holds:*

(1) Γ *is a complete bipartite graph;*
(2) $\mathrm{soc}(G) = \mathrm{soc}(G^*)$*, and* $\mathrm{soc}(G^*)$ *is either simple or regular on each* G^**-orbit;*
(3) Γ *is bipartite,* $\mathrm{soc}(G) = \mathrm{soc}(G^*) \times M$ *with* $|M| = 2$*, and* $\mathrm{soc}(G^*)$ *is either simple or regular on each* G^**-orbit.*

2. Some observations on 2-transitive permutation groups

This section gives some simple results about 2-transitive permutation groups, which serve to analyze the structures of vertex-stabilizers of 2-arc-transitive graphs.

Let X be a transitive permutation group on a finite set Ω. Recall that the socle $\mathrm{soc}(X)$ is generated by all minimal normal subgroups of X. It is easily shown that $\mathrm{soc}(X)$ is a characteristic subgroup of X. Assume that X is a 2-transitive permutation group on Ω. Then $\mathrm{soc}(X)$ is either elementary abelian and regular on Ω, or simple and primitive on Ω, refer to [3, Page 101, Theorem 4.3] and [4, Page 107, Theorem 4.1B]. In particular, X is either affine or almost simple. Inspecting the lists of finite 2-transitive permutation groups (refer to [3, Pages 195-197, Tables 7.3 and 7.4]), we have the following basic fact, see also [14, Corollary 2.5].

Lemma 2.1. *Let* X *be a* 2*-transitive permutation group on a finite set* Ω*, and* $\alpha \in \Omega$*. Assume that* K *is an insoluble normal subgroup of* X_α*. Then* K *has a unique insoluble composition factor say* S*, and* S *is isomorphic to a composition factor of* X *if and only if* X *is affine.*

Recall that a transitive permutation group X on Ω is a Frobenius group if X is not regular on Ω and, for $\alpha \in \Omega$, the point-stabilizer X_α, called a Frobenius complement of X, is semiregular on $\Omega \setminus \{\alpha\}$.

By Frobenius' Theorem (refer to [1, Pages 190-191, (35.23) and (35.24)]), for a Frobenius group X on Ω, the identity and the elements without fixed-point form a normal regular subgroup of X, which is called the Frobenius kernel of X.

Lemma 2.2. *Let* $X = KH$ *be an imprimitive Frobenius group on* Ω *with the Frobenius kernel* $K \cong \mathbb{Z}_p^k$ *and a Frobenius complement* H*, where* p *is a prime and* $k \geqslant 2$*.*

Then H is isomorphic to an irreducible subgroup of the general linear group $\mathrm{GL}_l(p)$, *and $|H|$ is a divisor of p^d-1, where $2l \leqslant k$ and d is a common divisor of k and l.*

Proof. Note that H acts faithfully and semiregularly on $K \setminus \{1\}$ by conjugation, see [1, Page 191, (35.25)]. Then $|H|$ is a divisor of p^k-1. Recall that X is imprimitive on Ω. Then K is not a minimal normal subgroup of X. By Maschke's Theorem (refer to [1, Page 40, (12.9)]), K is a direct product of two H-invariant proper subgroups. Thus we may choose a minimal H-invariant subgroup L of K with $|L|^2 \leqslant |K|$. It is easily shown that LH is a primitive Frobenius group (on an L-orbit), which has the Frobenius kernel L. Set $|L| = p^l$. Then $|H|$ is a divisor of p^l-1, $2l \leqslant k$, and H is isomorphic to an irreducible subgroup of $\mathrm{GL}_l(p)$.

Choose a minimal positive integer d such that $|H|$ is a divisor of p^d-1. Then $d \leqslant l$. Set $k = xd+y$ for integers $x \geqslant 1$ and $0 \leqslant y < d$. Then $p^k-1 = p^y(p^{xd}-1)+(p^y-1)$, and thus $|H|$ is a divisor of p^y-1. By the choice of d, we have $y=0$, and so d is a divisor of k. Similarly, d is a divisor of l. Then the lemma follows. □

Lemma 2.3. *Let X be a 2-transitive permutation group on a finite set Ω. Assume that $1 \neq N \trianglelefteq X$. Then* $\mathrm{soc}(N) = \mathrm{soc}(X)$, *and either N is primitive on Ω or one of the following holds:*

(1) $N = \mathbb{Z}_p^k{:}\mathbb{Z}_m$ *and $|\Omega| = p^k$, where p is a prime, $k \geqslant 2$, $m \mid (p^d-1)$ for some divisor d of k with $d<k$;*
(2) $N = \mathbb{Z}_3^4{:}\mathrm{Q}_8$ *and $|\Omega| = 3^4$.*

Proof. Since X is 2-transitive on Ω, by [4, Page 107, Theorem 4.1B], $\mathrm{soc}(X)$ is either abelian or nonabelian simple. By [4, Page 114, Theorem 4.3B], the centralizer $\mathbf{C}_X(\mathrm{soc}(X)) = \mathrm{soc}(X)$ or 1, respectively. In particular, $\mathrm{soc}(X)$ is the unique minimal normal subgroup of X. Noting that $\mathrm{soc}(N)$ is characteristic in N, it follows that $\mathrm{soc}(N)$ is a normal subgroup of X, and thus $\mathrm{soc}(X) \leqslant \mathrm{soc}(N)$. Suppose that $\mathrm{soc}(X) \neq \mathrm{soc}(N)$. Then $\mathrm{soc}(N)$ has a simple direct factor T with $T \cap \mathrm{soc}(X) = 1$. Since both T and $\mathrm{soc}(X)$ are normal in $\mathrm{soc}(N)$, we deduce that T centralizes $\mathrm{soc}(X)$, and so $T \leqslant \mathbf{C}_X(\mathrm{soc}(X)) = \mathrm{soc}(X)$ or 1, a contradiction. Therefore, $\mathrm{soc}(N) = \mathrm{soc}(X)$.

Next we assume that N is imprimitive on Ω, and show that one of (1) and (2) holds. By [4, Pages 215-217, Theorems 7.2C and 7.2E], $\mathrm{soc}(N) = \mathrm{soc}(X) \cong \mathbb{Z}_p^k$ for a prime p and integer $k \geqslant 2$ with $|\Omega| = p^k$, and either $N = \mathrm{soc}(X)$ or N is a Frobenius group with the Frobenius kernel $\mathrm{soc}(X)$. In particular, by Lemma 2.2, we write $N = KH$, where $K \cong \mathbb{Z}_p^k$ and $|H|$ is a divisor of p^d-1 for a divisor d of k with $d<k$. Note that X is an affine 2-transitive permutation group. Inspecting the finite affine 2-transitive permutation groups listed in [3, Page 197, Table 7.4], we conclude that either H is cyclic, or one of the following holds:

(i) $H \trianglelefteq X_0 \leqslant \Gamma\mathrm{L}_1(p^k)$, where X_0 is a point-stabilizer in X;
(ii) $p^k = 3^4$, yielding $d \in \{1,2\}$, and so $|H|$ is a divisor of 8.

If H is cyclic then N is described as in part (1) of this lemma. In the following, we assume further that H is not cyclic.

Suppose that (i) holds. If $k=2$ then H is isomorphic to a subgroup of $\mathrm{GL}_1(p)$ by Lemma 2.2, and so H is cyclic, which is not the case. If $p^k = 2^6$ then $|H|$ is a divisor of 2^d-1 with $d \in \{1,2,3\}$ by Lemma 2.2, which yields that H is cyclic, a contradiction. Thus $k > 2$ and $p^k \neq 2^6$. By the Zigmondy Theorem, there exists a prime r such that $p^k - 1 \equiv 0 \,(\mathrm{mod}\ r)$ but $p^l - 1 \not\equiv 0 \,(\mathrm{mod}\ r)$ for $1 < l < k$. In particular, p has order k modulo r, and so k is a divisor of $r-1$. Recall that $|H|$ is a divisor of $p^d - 1$, where $d < k$. It follows that r is not a divisor of $|H|$, and so H contains no element of order r. Since X is a 2-transitive group of degree p^k, the order of X_0 is divisible by $p^k - 1$. Pick an element $x \in X_0$ with order r. Write $\Gamma\mathrm{L}_1(p^k) = \langle a, \tau \mid a^{p^k-1} = 1, \tau^k = 1, \tau^{-1}a\tau = a^p\rangle$. Clearly, $\langle a\rangle$ is normal in $\Gamma\mathrm{L}_1(p^k)$, and so $\langle a\rangle\langle x\rangle \leqslant \Gamma\mathrm{L}_1(p^k)$. In particular, $|\langle a\rangle\langle x\rangle|$ is a divisor of $|\Gamma\mathrm{L}_1(p^k)| = (p^k-1)k$. Noting that $|\langle a\rangle\langle x\rangle| = \frac{|\langle a\rangle||\langle x\rangle|}{|\langle a\rangle\cap\langle x\rangle|} = \frac{(p^k-1)r}{|\langle a\rangle\cap\langle x\rangle|}$, it follows that $\frac{r}{|\langle a\rangle\cap\langle x\rangle|}$ is a divisor of k. Since $r > k$ and r is a prime, we have $|\langle a\rangle \cap \langle x\rangle| = r$, yielding $x \in \langle a\rangle$. Then $\tau^{-1}x\tau = x^p$. Since H is not cyclic, we take an element $a^i\tau^j \in H \setminus \langle a\rangle$, where $1 < j < k$. We have $x^{-1}a^i\tau^j x \in H$ as $H \trianglelefteq X_0$. Noting that $x^{-1}a^i\tau^j x = x^{p^{kj-j}-1}a^i\tau^j = x^{p^{k-j}-1}a^i\tau^j$, we deduce that $x^{p^{k-j}-1} \in H$. Since $1 < k-j < k$, by the choice of r, we have $p^{k-j}-1 \not\equiv 0 \,(\mathrm{mod}\ r)$. Thus H contains an element $x^{p^{k-j}-1}$ of order r, a contradiction.

Suppose that (ii) holds. By Lemma 2.2, H is isomorphic to an irreducible subgroup of $\mathrm{GL}_2(3)$. Choose a minimal H-invariant subgroup L of K with $L \cong \mathbb{Z}_3^2$. Then LH is a primitive Frobenius group of degree 9 and of order a divisor of 72. Confirmed by GAP [20], up to permutation isomorphism, there are four affine primitive groups of degree 9 which have order a divisor of 72, say $\mathbb{Z}_3^2{:}\mathbb{Z}_4$, $\mathbb{Z}_3^2{:}\mathbb{Z}_8$, $\mathbb{Z}_3^2{:}\mathrm{D}_8$ and $\mathbb{Z}_3^2{:}\mathrm{Q}_8$. In addition, the group $\mathbb{Z}_3^2{:}\mathrm{D}_8$ is not a Frobenius group. Since H is not cyclic, we have $H \cong \mathrm{Q}_8$, and thus part (2) of this lemma follows. This completes the proof. □

Lemma 2.4. *Let X be an affine 2-transitive permutation group, and $\mathrm{soc}(X) = K_1 \times \cdots \times K_l$, where $1 < K_i < \mathrm{soc}(X)$ for $1 \leqslant i \leqslant l$. Then there exist $x \in X$ and i such that $K_i^x \notin \{K_i \mid 1 \leqslant i \leqslant l\}$.*

Proof. Clearly, $\cup_i(K_i \setminus \{1\}) \neq \mathrm{soc}(X) \setminus \{1\}$. Let H be a point-stabilizer in X. Then H acts transitively on $\mathrm{soc}(X)\setminus\{1\}$ by conjugation. Thus H does not fix $\cup_i(K_i \setminus \{1\})$ set-wise by conjugation, and the lemma follows. □

3. The uniqueness of minimal normal subgroup

In this section, we assume that $\Gamma = (V, E)$ is a connected regular graph, and $G \leqslant \mathrm{Aut}(\Gamma)$. Denote by $G_\alpha^{\Gamma(\alpha)}$ the permutation group induced by G_α on $\Gamma(\alpha)$. Let $G_\alpha^{[1]}$ be the kernel of G_α acting on $\Gamma(\alpha)$. Then

$$G_\alpha^{\Gamma(\alpha)} \cong G_\alpha/G_\alpha^{[1]}.$$

Let $\beta \in \Gamma(\alpha)$, and set $G_{\alpha\beta}^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]}$. Then $G_{\alpha\beta}^{[1]}$ is the kernel of the arc-stabilizer $G_{\alpha\beta}$ acting on $\Gamma(\alpha)\cup\Gamma(\beta)$. Noting that $G_\alpha^{[1]} \trianglelefteq G_{\alpha\beta}$, we have

$$G_\alpha^{[1]}/G_{\alpha\beta}^{[1]} \cong (G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq G_{\alpha\beta}^{\Gamma(\beta)} = (G_\beta^{\Gamma(\beta)})_\alpha.$$

Assume that G is arc-transitive on Γ, and N is an arbitrary normal subgroup of G. Then

$$N_\alpha \trianglelefteq G_\alpha,\ N_\alpha^{[1]} \trianglelefteq G_\alpha^{[1]},\ N_{\alpha\beta} \trianglelefteq G_{\alpha\beta},\ N_{\alpha\beta}^{[1]} \trianglelefteq G_{\alpha\beta}^{[1]}.$$

Taking $x \in G$ with $(\alpha, \beta)^x = (\beta, \alpha)$, we have

$$N_\beta = N_\alpha^x,\ N_{\alpha\beta}^x = N_{\alpha\beta},\ \Gamma(\beta) = \Gamma(\alpha)^x.$$

It follows that $N_{\alpha\beta}^{\Gamma(\beta)} \cong N_{\alpha\beta}^{\Gamma(\alpha)}$. Since $N_{\alpha\beta} \trianglelefteq G_{\alpha\beta}$, we have $N_{\alpha\beta}^{\Gamma(\alpha)} \trianglelefteq G_{\alpha\beta}^{\Gamma(\alpha)}$, and so

$$N_\alpha^{[1]}/N_{\alpha\beta}^{[1]} \cong (N_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq N_{\alpha\beta}^{\Gamma(\beta)} \cong N_{\alpha\beta}^{\Gamma(\alpha)} = (N_\alpha^{\Gamma(\alpha)})_\beta \trianglelefteq (G_\alpha^{\Gamma(\alpha)})_\beta. \tag{3.1}$$

In particular, $(N_\alpha^{[1]})^{\Gamma(\beta)}$ is isomorphic to a normal subgroup of $(N_\alpha^{\Gamma(\alpha)})_\beta$.

Assume that G is 2-arc-transitive on Γ. Then $G_{\alpha\beta}^{[1]}$ has order a prime power, see [7, Corollary 2.3]. In particular, $G_{\alpha\beta}^{[1]}$ is soluble. Then $(G_\alpha^{[1]})^{\Gamma(\beta)}$ is soluble if and only if $G_\alpha^{[1]}$ is soluble. Noting that $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive group on $\Gamma(\alpha)$, by Lemma 2.1 and (3.1), we have the following fact.

Lemma 3.1. *Assume that G is 2-arc-transitive on $\Gamma = (V, E)$, $N \trianglelefteq G$ and N_α is insoluble, where $\alpha \in V$. Then $N_\alpha^{\Gamma(\alpha)}$ has a unique insoluble composition factor, and $N_\alpha^{[1]}$ has at most one insoluble composition factor. If $N_\alpha^{[1]}$ and $N_\alpha^{\Gamma(\alpha)}$ have isomorphic insoluble composition factors then $G_\alpha^{\Gamma(\alpha)}$ is an affine 2-transitive permutation group.*

Proof. Let $\beta \in \Gamma(\alpha)$. Then $N_{\alpha\beta}^{[1]}$ is soluble as $N_{\alpha\beta}^{[1]} \leqslant G_{\alpha\beta}^{[1]}$. By (3.1), we may write $N_\alpha = N_{\alpha\beta}^{[1]}.(N_\alpha^{[1]})^{\Gamma(\beta)}.N_\alpha^{\Gamma(\alpha)}$. In addition, $(N_\alpha^{[1]})^{\Gamma(\beta)}$ is isomorphic to a normal subgroup of $(N_\alpha^{\Gamma(\alpha)})_\beta$. If $N_\alpha^{\Gamma(\alpha)}$ is soluble, then $(N_\alpha^{[1]})^{\Gamma(\beta)}$ is soluble, and so N_α is soluble, a contradiction. Thus $N_\alpha^{\Gamma(\alpha)}$ is an insoluble normal subgroup of the 2-transitive permutation group $G_\alpha^{\Gamma(\alpha)}$. Inspecting the 2-transitive permutation groups listed in [3, Pages 195-197, Tables 7.3 and 7.4], it follows that $N_\alpha^{\Gamma(\alpha)}$ has a unique insoluble composition factor, which is the unique insoluble composition factor of $G_\alpha^{\Gamma(\alpha)}$.

Since $(N_\alpha^{\Gamma(\alpha)})_\beta \trianglelefteq (G_\alpha^{\Gamma(\alpha)})_\beta$, by Lemma 2.1, $(N_\alpha^{\Gamma(\alpha)})_\beta$ has at most one insoluble composition factor. Recall that $N_\alpha^{[1]}/N_{\alpha\beta}^{[1]} \cong (N_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (N_\beta^{\Gamma(\beta)})_\alpha \cong (N_\alpha^{\Gamma(\alpha)})_\beta$, see (3.1). It follows that $N_\alpha^{[1]}$ has at most one insoluble composition factor. Suppose that $N_\alpha^{[1]}$ and $N_\alpha^{\Gamma(\alpha)}$ have isomorphic insoluble composition factors. Then $(N_\alpha^{\Gamma(\alpha)})_\beta$ and $G_\alpha^{\Gamma(\alpha)}$ have isomorphic insoluble composition factors. By Lemma 2.1, $G_\alpha^{\Gamma(\alpha)}$ is an affine 2-transitive group. This completes the proof. □

Lemma 3.2. *Assume that G is 2-arc-transitive on $\Gamma = (V, E)$, and $N \trianglelefteq G$. Suppose that, for $\alpha \in V$, the stabilizer N_α has a normal subgroup $K \cong T^k$ for an integer $k \geq 1$ and a nonabelian simple group T. Then $k = 1$.*

Proof. Note that every normal subgroup of K is isomorphic to T^l for some $l \leqslant k$, where $T^0 = 1$. Set $K \cap G_\alpha^{[1]} \cong T^l$. Then

$$K^{\Gamma(\alpha)} \cong KG_\alpha^{[1]}/G_\alpha^{[1]} \cong K/(K \cap G_\alpha^{[1]}) \cong T^{k-l}.$$

Since $K^{\Gamma(\alpha)} \trianglelefteq N_\alpha^{\Gamma(\alpha)} \trianglelefteq G_\alpha^{\Gamma(\alpha)}$, by Lemma 3.1, we conclude that $l, k - l \in \{0, 1\}$. If $G_\alpha^{\Gamma(\alpha)}$ is of affine type, then $k - l = 0$, and so $k = l = 1$. If $G_\alpha^{\Gamma(\alpha)}$ is almost simple, then either $k = l = 1$ or $k - l = 1$ and $l = 0$, and so $k = 1$. This completes the proof. □

Theorem 3.3. *Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to G, and $G^* = \langle G_\alpha, G_\beta \rangle$ for $\{\alpha, \beta\} \in E$. Then either Γ is a complete bipartite* *graph, or* $\mathrm{soc}(G^*) = \mathrm{soc}(G)$ *is the unique minimal normal subgroup of* G *and one of the following holds:*

1. $\mathrm{soc}(G)$ *is semiregular on* V*;*
2. $\mathrm{soc}(G)$ *is a nonabelian simple group;*
3. G^* *is a quasiprimitive permutation group of type* PA *on each* G^**-orbit on* V*;*
4. Γ *is a bipartite graph,* G^* *is faithful on each part of* Γ*,* $\mathrm{soc}(G) = M_1 \times M_2$ *for minimal normal subgroups* M_1 *and* M_2 *of* G^**, and both* M_1 *and* M_2 *are semiregular and intransitive on each part of* Γ*.*

Proof. If Γ is not bipartite then $G = G^*$ and, by [17, Theorem 2], G has a unique minimal normal subgroup, and one of parts (1)-(3) follows. Thus we assume that Γ is a bipartite graph with two parts U and W. In particular, $|G : G^*| = 2$. By [18, Theorem 2.1], either Γ is a complete bipartite graph, or G^* is faithful on each of U and W. In the following, we assume that the latter case occurs.

Let K be an arbitrary minimal normal subgroup of G. Suppose that $K \not\leqslant G^*$. Then $K \cap G^* = 1$ and $G = G^*K$, yielding $|K| = 2$. Since K has at most two orbits on V, we have $|V| \leqslant 4$, which is impossible as Γ is bipartite and of valency at least 3. Therefore, $K \leqslant G^*$. Let K_1 be a minimal normal subgroup of G^* with $K_1 \leqslant K$, and let $x \in G \setminus G^*$. Then K_1^x is also a minimal normal subgroup of G^*. Noting that $x^2 \in G^*$, we have $(K_1^x)^x = K_1^{x^2} = K_1$. This implies that $K_1K_1^x$ is normal in G. Since $K_1^x \leqslant K^x = K$, we have $K = K_1K_1^x \leqslant \mathrm{soc}(G^*)$. It follows that $\mathrm{soc}(G) \leqslant \mathrm{soc}(G^*)$.

Case 1. Assume that G^* is quasiprimitive on both U and W. Then, by [18, Theorem 2.3], $\mathrm{soc}(G^*)$ is the unique minimal normal subgroup of G^*, and one of parts (1)-(3) of Theorem 3.3 occurs. Noting that $G^* \trianglelefteq G$ and $\mathrm{soc}(G^*)$ is characteristic in G^*, we have $\mathrm{soc}(G^*) \trianglelefteq G$, and hence $\mathrm{soc}(G^*)$ is a minimal normal subgroup of G. Then $\mathrm{soc}(G^*) \leqslant \mathrm{soc}(G)$. Recalling that $\mathrm{soc}(G) \leqslant \mathrm{soc}(G^*)$, we have $\mathrm{soc}(G) = \mathrm{soc}(G^*)$, and hence $\mathrm{soc}(G^*)$ is the unique minimal normal subgroup of G.

Case 2. Assume that G^* is not quasiprimitive on one of U and W, say U. Then G^* has a minimal normal subgroup M which is intransitive on U. Let $x \in G \setminus G^*$. Then M^x is a minimal normal subgroup of G^*, and M^x is intransitive on W. Note that MM^x is normal in G. Then MM^x is transitive on both U and W. It follows that $M \neq M^x$, and so $M \cap M^x = 1$. Then $MM^x = M \times M^x$. If M is transitive on W then M^x is semiregular on W by [4, Theorem 4.2A], and thus both M and M^x are regular on W, a contradiction. Therefore, M is intransitive on W. Similarly, M^x is intransitive on U. It follows from [8, Lemma 5.1] that M and M^x are semiregular on both U and W.

Set $N = MM^x$, and write $M = T_1 \times \cdots T_k$, where T_i are isomorphic simple groups. Then

$$N = T_1 \times \cdots T_k \times T_1^x \times \cdots T_k^x.$$

Let L be a minimal normal subgroup of G with $L \leqslant N$. Assume that $M \not\leqslant L$. Then $M \cap L = 1$ as M is a minimal normal subgroup of G^*, and so $M^x \cap L = (M \cap L)^x = 1$. Thus both M and M^x centralize L. Considering the action of G^* on U or W, by [4, Theorem 4.2A], L is nonabelian. This forces that every T_i is a nonabelian simple group. Since $L \trianglelefteq N$, it follows that L contains T_i or T_i^x for some i. Then $M \cap L \neq 1$

or $M^x \cap L \neq 1$, a contradiction. Then $M \leqslant L$, and $M^x \leqslant L^x = L$. We have $N = MM^x \leqslant L$, and so $N = L$. Therefore, N is a minimal normal subgroup of G. In addition, since M and M^x are minimal normal subgroups of G^*, we have $N = MM^x \leqslant \mathrm{soc}(G^*)$.

Suppose that $N \neq \mathrm{soc}(G^*)$. Then G^* has a minimal normal subgroup M_1 with $M_1 \cap N = 1$. This implies that $M_1 \leqslant \mathbf{C}_{G^*}(N) \not\leqslant N$. Noting that $\mathbf{C}_{G^*}(N)$ is a normal subgroup of G, it follows that $\mathbf{C}_{G^*}(N)$ acts transitively on both U and W. Considering the action of G^* on U, it follows from [4, Theorem 4.2A] that N is not abelian, $\mathbf{C}_{G^*}(N) \cong N$, and both $\mathbf{C}_{G^*}(N)$ and N are regular on U. Let $\alpha \in U$ and $X = \mathbf{C}_{G^*}(N)N$. Then X is normal in G, and $X = \mathbf{C}_{G^*}(N)X_\alpha$. We have

$$X_\alpha \cong \mathbf{C}_{G^*}(N)N/\mathbf{C}_{G^*}(N) \cong N = T_1 \times \cdots T_k \times T_1^x \times \cdots \times T_k^x.$$

Then $2k = 1$ by Lemma 3.2, a contradiction. Therefore, $N = \mathrm{soc}(G^*)$. Recall that $\mathrm{soc}(G) \leqslant \mathrm{soc}(G^*)$ and N is a minimal normal subgroup of G. We have $\mathrm{soc}(G) = N = \mathrm{soc}(G^*)$, and the result follows. □

4. Semiregular direct factors

Let $\Gamma = (V, E)$ be a connected graph, and $G \leqslant \mathrm{Aut}(\Gamma)$.

Assume that G is a 2-arc-transitive group of Γ. Then $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive permutation group on $\Gamma(\alpha)$, where $\alpha \in V$. Let $N \trianglelefteq G$ with $N_\alpha \neq 1$. It is easily shown that N_α acts transitively on $\Gamma(\alpha)$, see [13, Lemma 2.5] for example. Thus $N_\alpha^{\Gamma(\alpha)}$ is a transitive normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. By Lemma 2.3, $\mathrm{soc}(N_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(G_\alpha^{\Gamma(\alpha)})$ and one of the following holds:

(i) $N_\alpha^{\Gamma(\alpha)}$ is a primitive permutation group on $\Gamma(\alpha)$;
(ii) $N_\alpha^{\Gamma(\alpha)} = \mathbb{Z}_p^k{:}\mathbb{Z}_m$ and $|\Gamma(\alpha)| = p^k$, where $k \geqslant 2$, $m \mid (p^d - 1)$ for some divisor d of k with $d < k$;
(iii) $N_\alpha^{\Gamma(\alpha)} = \mathbb{Z}_3^4{:}\mathrm{Q}_8$ and $|\Gamma(\alpha)| = 3^4$.

Lemma 4.1. *Assume that G is 2-arc-transitive on Γ, and $N \trianglelefteq G$ with $N_\alpha \neq 1$ for $\alpha \in V$. Suppose that $N_\alpha^{\Gamma(\alpha)}$ is not primitive on $\Gamma(\alpha)$. Then one of the following holds:*

(1) $N_\alpha = \mathbb{Z}_p^k{:}(\mathbb{Z}_{m_1}.\mathbb{Z}_m) = (\mathbb{Z}_p^k \times \mathbb{Z}_{m_1}).\mathbb{Z}_m$, $|\Gamma(\alpha)| = p^k$ *and* $N_\alpha^{[1]} \cong \mathbb{Z}_{m_1}$, *where* $m_1 \mid m$, $m \mid (p^d - 1)$ *for some divisor d of k with $d < k$;*
(2) $N_\alpha = \mathbb{Z}_3^4{:}(Q.\mathrm{Q}_8) = (\mathbb{Z}_3^4 \times Q).\mathrm{Q}_8$, $|\Gamma(\alpha)| = 3^4$ *and* $Q \cong N_\alpha^{[1]}$, *where Q is isomorphic to a subgroup of* Q_8.

Proof. By the foregoing argument, we may let $N_\alpha^{\Gamma(\alpha)} = KH$, where $K = \mathrm{soc}(N_\alpha^{\Gamma(\alpha)}) \cong \mathbb{Z}_p^k$, and either $H \cong \mathbb{Z}_m$ or $p^k = 3^4$ and $H \cong \mathrm{Q}_8$. Without loss of generality, let $H = (N_\alpha^{\Gamma(\alpha)})_\beta$ for some $\beta \in \Gamma(\alpha)$. Then $N_\alpha^{[1]}/N_{\alpha\beta}^{[1]}$ is isomorphic to a normal subgroup of H, see (3.1) given in Section 3.

Assume first that $p^k = 4$. In this case, we have $H = 1$ and $N_\alpha^{\Gamma(\alpha)} = \mathbb{Z}_2^2$, and so N_α acts faithfully on $\Gamma(\alpha)$, refer to [13, Lemma 2.3]. Then $N_\alpha = \mathbb{Z}_2^2$, desired as in part (1) of this lemma.

Now assume that $p^k \neq 4$. Then $|\Gamma(\alpha)| = p^k > 5$. By [21, Theorem 4.7], $G_{\alpha\beta}^{[1]} = 1$, and so $N_{\alpha\beta}^{[1]} = 1$, where $\beta \in \Gamma(\alpha)$. Then $N_\alpha^{[1]}$ is isomorphic to a normal subgroup of H, in particular, $(p, |N_\alpha^{[1]}|) = 1$. It is easily shown that $|\mathrm{Aut}(N_\alpha^{[1]})| < p^k$. Let P be a Sylow p-subgroup of N_α. Then $P \cong \mathbb{Z}_p^k$, and $PN_\alpha^{[1]}/N_\alpha^{[1]}$ is the unique Sylow p-subgroup of $N_\alpha/N_\alpha^{[1]}$, in particular, $PN_\alpha^{[1]} \trianglelefteq N_\alpha$. Noting that $PN_\alpha^{[1]}/\mathbf{C}_{PN_\alpha^{[1]}}(N_\alpha^{[1]})$ is isomorphic to a subgroup of $\mathrm{Aut}(N_\alpha^{[1]})$, it follows that p is a divisor of $|\mathbf{C}_{PN_\alpha^{[1]}}(N_\alpha^{[1]})|$. Let Q be a Sylow p-subgroup of $\mathbf{C}_{PN_\alpha^{[1]}}(N_\alpha^{[1]})$. Then Q is characteristic in $\mathbf{C}_{PN_\alpha^{[1]}}(N_\alpha^{[1]})$, and hence Q is normal in N_α. This implies that $\mathbf{O}_p(N_\alpha) \neq 1$, where $\mathbf{O}_p(N_\alpha)$ is the maximal normal p-subgroup of N_α. Since $N_\alpha \trianglelefteq G_\alpha$, we have $\mathbf{O}_p(N_\alpha) \trianglelefteq G_\alpha$. Recalling that $(p, |N_\alpha^{[1]}|) = 1$, we deduce that $\mathbf{O}_p(N_\alpha)$ acts faithfully on $\Gamma(\alpha)$. Since G acts 2-transitively on $\Gamma(\alpha)$, the action of $\mathbf{O}_p(N_\alpha)$ on $\Gamma(\alpha)$ is transitive. Noting that $\mathbf{O}_p(N_\alpha) \leqslant P$ is abelian, it follows that $\mathbf{O}_p(N_\alpha)$ is regular on $\Gamma(\alpha)$. Then $|\mathbf{O}_p(N_\alpha)| = |\Gamma(\alpha)| = p^k$, and hence $\mathbf{O}_p(N_\alpha) = P \cong \mathbb{Z}_p^k$. We have $N_\alpha = P{:}N_{\alpha\beta} = (P \times N_\alpha^{[1]}).H$. Then part (1) or (2) of the lemma follows. □

Theorem 4.2. *Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to G, and $G^* = \langle G_\alpha, G_\beta \rangle$ for an edge $\{\alpha, \beta\} \in E$. If Γ is not a complete bipartite graph then either $\mathrm{soc}(G)$ is a nonabelian simple group, or every simple direct factor of $\mathrm{soc}(G)$ is semiregular on V.*

Proof. Assume that Γ is not a complete bipartite graph. Then G^* is faithful on each of its orbits on V. Let $N = \mathrm{soc}(G)$. By Theorem 3.3, $N = \mathrm{soc}(G)$ is the unique minimal normal subgroup of G. If part (1), (2) or (4) of Theorem 3.3 occurs, then our result is true. Thus, in the following, we suppose that part (3) of Theorem 3.3 occurs, that is, G^* is a quasiprimitive permutation group of type PA on each G^*-orbit on V. By [17, III(b)(i)], N is the unique minimal normal subgroup of G^*.

Write $N = T_1 \times \cdots \times T_l$, where $l \geqslant 2$ and T_i are isomorphic nonabelian simple groups. Then $N_\alpha \neq 1$, and N_α has no composition factor isomorphic to T_1, see [17, III(b)(i)]. We next show that every T_i is semiregular on V.

Let U be the G^*-orbit on V with $\alpha \in U$, and let $W = V \setminus U$ if Γ is bipartite. Clearly, U is an N-orbit, and if Γ is bipartite then W is also an N-orbit. Recall that N is a minimal normal subgroup of both G and G^*. Since $G^* = NG_\gamma$ for $\gamma \in V$, it follows that both G and G_γ act transitively on $\Omega := \{T_1, \ldots, T_l\}$ by conjugation. Let

$$\mathcal{C}_\gamma = \{(T_i)_\gamma \mid 1 \leqslant i \leqslant l\},\ \mathcal{C} = \cup_{\gamma \in V} \mathcal{C}_\gamma.$$

For $1 \leqslant i \leqslant l$ and $x \in G$, we have $T_i^x \in \Omega$, and so

$$(T_i)_\gamma^x = (T_i \cap G_\gamma)^x = T_i^x \cap G_{\gamma^x} = (T_i^x)_{\gamma^x} \in \mathcal{C},\ \forall \gamma \in V.$$

We deduce that G_γ acts transitively on $\mathcal{C}_\gamma$ by conjugation, and $\mathcal{C}$ is a conjugacy class of subgroups in G. In particular, all orbits of each T_i on V have the same length $|T_1 : (T_1)_\alpha|$. Thus, if T_1 is semiregular on V then every T_i is semiregular on V.

Case 1. Assume that $N_\alpha^{\Gamma(\alpha)}$ is primitive on $\Gamma(\alpha)$. For any $\gamma \in V$, letting $\gamma = \alpha^g$ for some $g \in G$, we have

$$\Gamma(\gamma) = \Gamma(\alpha)^g,\ N_\gamma = N \cap G_{\alpha^g} = (N \cap G_\alpha)^g = N_\alpha^g.$$

It follows that N_γ acts primitively on $\Gamma(\gamma)$. Thus N is locally-primitive on Γ. Suppose that T_1 is transitive on one of the G^*-orbits, say U. Since T_l centralizes T_1, by [4, Theorem 4.2A], T_l is semiregular on U. This implies that both T_1 and T_l are regular on U. Then $N = T_lN_\alpha$, and so

$$T_1 \times \cdots \times T_{l-1} \cong N/T_l = T_lN_\alpha/T_l \cong N_\alpha/(T_l \cap N_\alpha) = N_\alpha/(T_l)_\alpha.$$

It follows that N_α has a composition factor isomorphic to T_1, a contradiction. Therefore, T_1 is intransitive on every G^*-orbit, and hence T_1 is semiregular on V, see [13, Lemma 2.6]. Then every T_i is semiregular on V, and our result is true.

Case 2. Assume that $(T_1)_\alpha \leqslant G_\alpha^{[1]}$. Then $(T_1)_\alpha \leqslant (T_1)_\beta$, where $\beta \in \Gamma(\alpha)$. Recalling that $\mathcal{C}$ is a conjugacy class in G, it follows that $|(T_1)_\gamma| = |(T_1)_\alpha|$ for all $\gamma \in V$. In particular, $|(T_1)_\alpha| = |(T_1)_\beta|$, and so $(T_1)_\alpha = (T_1)_\beta$. Note that N_β acts transitively on $\Gamma(\beta)$, see [13, Lemma 2.5] for example. Since $(T_1)_\alpha = (T_1)_\beta \trianglelefteq N_\beta$, all $(T_1)_\alpha$-orbits on $\Gamma(\beta)$ have the same length. It follows that $(T_1)_\alpha$ fixes $\Gamma(\beta)$ point-wise, i.e., $(T_1)_\beta = (T_1)_\alpha \leqslant G_\beta^{[1]}$. We deduce from the connectedness of Γ that $(T_1)_\gamma = (T_1)_\alpha$ for all $\gamma \in V$. This forces that $(T_1)_\alpha = 1$. Then our result is true in this case.

Case 3. Now we suppose that $(T_1)_\alpha \nleqslant G_\alpha^{[1]}$ and $N_\alpha^{\Gamma(\alpha)}$ is not primitive on $\Gamma(\alpha)$, and produce a contradiction. Recall that G_α acts transitively on $\mathcal{C}_\alpha$ by conjugation. This implies that G_α acts transitively on $\{(T_1)_\alpha^{[1]}, \ldots, (T_l)_\alpha^{[1]}\}$, $(T_1)_\alpha \times \cdots \times (T_l)_\alpha \trianglelefteq G_\alpha$, and $(T_i)_\alpha \nleqslant G_\alpha^{[1]}$ for $1 \leqslant i \leqslant l$. By Lemma 2.3, we have that

$$\mathrm{soc}(((T_1)_\alpha \times \cdots \times (T_l)_\alpha)^{\Gamma(\alpha)}) = \mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(N_\alpha^{\Gamma(\alpha)}) \cong \mathbb{Z}_p^k,$$

and a Sylow p-subgroup of N_α has order p^k, where p is a prime and $k \geqslant 2$. By Lemma 4.1, $N_\alpha^{[1]}$ has order coprime to p, and thus $(p, (T_i)_\alpha^{[1]}) = 1$ for $1 \leqslant i \leqslant l$.

Let P_i be a Sylow p-subgroup of $(T_i)_\alpha$, where $1 \leqslant i \leqslant l$. Then $P = P_1 \times \cdots \times P_l$ is a Sylow p-subgroup of N_α, and thus

$$P \cong P^{\Gamma(\alpha)} = \mathrm{soc}(N_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(G_\alpha^{\Gamma(\alpha)}),$$

and $\mathbf{O}_p((T_i)_\alpha^{\Gamma(\alpha)}) = P_i^{\Gamma(\alpha)} \cong P_i$ for each i. It follows that

$$\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) = P_1^{\Gamma(\alpha)} \times \cdots \times P_l^{\Gamma(\alpha)}.$$

Let K_i be the preimage of $P_i^{\Gamma(\alpha)}$ in $(T_1)_\alpha \times \cdots \times (T_l)_\alpha$. Then $K_i = (T_i)_\alpha^{[1]} P_i$ for $1 \leqslant i \leqslant l$. It is easily shown that G_α acts transitively on $\{K_1, \ldots, K_l\}$ by conjugation. Then $G_\alpha^{\Gamma(\alpha)}$ acts transitively on $\{P_1^{\Gamma(\alpha)}, \ldots, P_l^{\Gamma(\alpha)}\}$ by conjugation, which is impossible by Lemma 2.4. This completes the proof of the theorem. □

We are now ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\Gamma = (V, E)$ be a basic 2-arc-transitive graph with respect to a group G. Assume that Γ is not a complete bipartite graph. Fix an edge $\{\alpha, \beta\} \in E$, and let $G^* = \langle G_\alpha, G_\beta \rangle$ and $N = \mathrm{soc}(G^*)$. By Theorem 3.3, $N = \mathrm{soc}(G)$ is the unique minimal normal subgroup of G, desired as in part (1) of Theorem 1.1. By Theorem 4.2, we have part (2) of Theorem 1.1.

Let γ be an arbitrary vertex of Γ. Since G acts transitively on V, we write $\gamma = \alpha^g$ for some $g \in G$. Then $\Gamma(\gamma) = \Gamma(\alpha)^g$. Since N is normal in G, we deduce that $N_\gamma = N_\alpha^g$. It follows that $N_\gamma^{\Gamma(\gamma)}$ and $N_\alpha^{\Gamma(\alpha)}$ are permutation isomorphic. Then N

is locally-primitive on Γ if and only if $N_\alpha^{\Gamma(\alpha)}$ is primitive on $\Gamma(\alpha)$. If $N_\alpha^{\Gamma(\alpha)}$ is not primitive on $\Gamma(\alpha)$ then either $N_\alpha = 1$, or N_α is described as in part (1) or (2) of Lemma 4.1. Thus we obtain part (3) of Theorem 1.1. This completes the proof. □

Finally, we give a proof of Theorem 1.4.

Proof of Theorem 1.4. Assume that G is a 2-arc-transitive group of $\Gamma = (V, E)$. Let $G^* = \langle G_\alpha, G_\beta\rangle$ for $\{\alpha, \beta\} \in E$. If Γ is not bipartite and G is primitive on V then $\mathrm{soc}(G)$ is either simple or regular on V by [16, Theorem A], and the result is true.

Assume next that Γ is a bipartite graph with two parts U and W, and that G^* acts primitively on both U and W. If G^* is unfaithful on U or W then Γ is a complete bipartite graph. Thus we assume further that G^* is faithful on both U and W. Let $\alpha \in U$ and $\beta \in W$.

Case 1. Assume that $\mathrm{soc}(G) \leqslant G^*$. If Γ has valency 2 then Γ is a cycle of length $2p$ for some prime p, and $G \cong \mathrm{D}_{4p}$; in this case, the center of G is not contained in G^*, and so $\mathrm{soc}(G) \nleqslant G^*$. Thus Γ has valency at least 3, and hence Γ is a basic 2-arc-transitive graph with respect to G. By Theorem 3.3, $\mathrm{soc}(G) = \mathrm{soc}(G^*)$, and either part (2) of Theorem 1.4 holds or G^* is a primitive permutation group of type PA on U. For the latter case, every simple direct factor of $\mathrm{soc}(G^*)$ is not semiregular on U, refer to [15, Page 391, III(b)(i)]. Then part (2) of Theorem 1.4 occurs by Theorem 4.2.

Case 2. Assume that $\mathrm{soc}(G) \nleqslant G^*$. Let M be a minimal normal subgroup of G with $M \nleqslant G^*$. Then, noting that $|G : G^*| = 2$, we have $G = G^* \times M$ and $|M| = 2$. This implies that $\mathrm{soc}(G) = \mathrm{soc}(G^*) \times M$. Set $M = \langle x\rangle$. Then $G_{\alpha^x} = G_\alpha^x = G_\alpha$, and so G_α acts 2-transitively on $\Gamma(\alpha^x)$. Note that $\Gamma(\alpha^x) \subset U$. Considering the (faithful) action of G^* on U, by [16, Theorem A], $\mathrm{soc}(G)^*$ is either simple or regular on U. Similarly, $\mathrm{soc}(G)^*$ is either simple or regular on W. Then part (3) of Theorem 1.4 follows. This completes the proof. □

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Conflicts of interests/Competing interests

We declare that we do not have any commercial or associative interests that represents a conflict of interests in connection with the work reported in this paper.

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