Higher order Laguerre inequalities for partition function

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Abstract. The Laguerre inequality and their higher order generalizations have been proved to be closely relative with the Laguerre-Pólya class and Riemann hypothesis. Wang and Yang concerned with the relation between discrete sequences and higher order Laguerre inequality and show the Laguerre inequality of order 2 holds for the partition function, the overpartition function, the Bernoulli numbers, the derangement numbers, the Motzkin numbers, the Fine numbers, the Franel numbers and the Domb numbers. Wagner proved the partition function satisfies the Laguerre inequality of any order as $n \to \infty$ and conjectured the thresholds for order no more than 10. In this paper, we will give $N(m)$ such that for $3 \leq m \leq 10$ and $n > N(m)$, the partition function satisfies the Laguerre inequality of order m. As consequences, we affirm Wagner's conjecture for $3 \leq m \leq 9$.

Keywords: partition function, higher order Laguerre inequality, Laguerre inequality, Hardy-Ramanujan-Rademacher formula

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1 Introduction

The main objective of this paper is to find the thresholds $N(m)$ such that for $n > N(m)$, the partition function satisfies the Laguerre inequality of order m for $m \leq 10$.

The Laguerre inequality [13] arises in the study of the real polynomials with only real zeros and the Laguerre-Pólya class. Recall that a real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{1.1}$$

is said to be in the Laguerre-Pólya class, denoted $\psi(x) \in \mathcal{LP}$, if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} (1 + x/x_k)\, e^{-x/x_k}, \tag{1.2}$$

where c, β, x_k are real numbers, $\alpha \geq 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. For more background on the theory of the $\mathcal{LP}$ class, we refer to [16] and [20].

Recall that if a polynomial $f(x)$ satisfies

$$f'(x)^2 - f(x)f''(x) \geq 0, \tag{1.3}$$

then it is called to satisfy Laguerre inequality. Laguerre [13] stated that if $f(x)$ is a polynomial with only real zeros, then the Laguerre inequality holds for $f(x)$. Jensen [12] found a higher order generalization of the Laguerre inequality, namely, for f belonging to Laguerre-Pólya class,

$$L_n(f(x)) := \frac{1}{2}\sum_{k=0}^{2n} (-1)^{n+k} \binom{2n}{k} f^{(k)}(x) f^{(2n-k)}(x) \geq 0, \tag{1.4}$$

for all real x, and $f^{(k)}(x)$ denotes the kth derivative of $f(x) = f^{(0)}(x)$. It yields the classical Laguerre inequality for $n = 1$. Note that Csordas and Vishnyakova [6] showed that if a function $f(x)$ satisfies $L_n(f(x)) \geq 0$ for all n and all $x \in R$, then $f(x)$ is in the Laguerre-Pólya class. It means that Laguerre inequality is a characterizing property of functions in the Laguerre-Pólya class. For more background on Laguerre inequality, see [2, 3, 4, 5, 8, 9, 10, 18, 19, 21]

Recently, Wang and Yang [23] considered whether the discrete sequence $\{a_n\}_{n\geq 0}$ has the similar results with higher order Laguerre inequality. Recall that a sequence $\{a_n\}$ satisfies Laguerre inequality of order m if

$$L_m(a_n) := \frac{1}{2}\sum_{k=0}^{2m} (-1)^{k+m} \binom{2m}{k} a_{n+k} a_{2m-k+n} \geq 0, \tag{1.5}$$

These inequalities can be simply obtained from equation (1.4) by specializing $x = 0$ where we can chose the function $f(x)$ to have Taylor coefficients a_{n+m}. For $m = 1$, the above inequality reduces to

$$a_n^2 - a_{n-1}a_{n+1} > 0,$$

i.e., the log-concavity of $\{a_n\}_{n\geq 0}$. Nicolas [17], DeSalvo and Pak [7] independently prove the partition function satisfies the log-concavity for $n \geq 26$. Chen, Jia and Wang [1] proved the partition function possesses the higher order Turán inequality. These results lead to the resurgence of the study of combinatorial inequalities with a focus on partitions. Wang and Yang [23] proved that the partition function, the overpartition function, the Bernoulli numbers, the derangement numbers, the Motzkin numbers, the Fine numbers, the Franel numbers and the Domb numbers possess Laguerre inequality of order 2. Wagner [22] showed the partition function satisfies the Laguerre inequality of any order as $n \to \infty$ and conjectured the thresholds for order no more than 10.

In this paper, we will prove Wagner's conjecture holds for order 3 through 9. The remaining of this paper is organized as follows. In Section 2, we shall show the partition function satisfies the Laguerre inequality of order 3 for $n \geq 531$ by generalizing an approach mentioned in [24]. In Section 3, we will find $N(m)$ such that for $4 \leq m \leq 10$ and $n > N(m)$, the partition function $p(n)$ satisfies the Laguerre inequality of order m.

2 Partition function

In this section, we will show the Laguerre inequality of order 3 holds for partition function. Recall that an integer partition of a positive integer n is a nonincreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Denote $p(n)$ the number of integer partitions of n. Hardy-Ramanujan-Rademacher formula for $p(n)$ states that for $n \geq 1$,

$$\begin{aligned} p(n) =& \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] \\ &+ R_2(n, N), \end{aligned} \tag{2.1}$$

where $A_k(n)$ is an arithmetic function, $R_2(n, N)$ is the remainder term and

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}. \tag{2.2}$$

Lehmer [14, 15] gave an error bound for $R_2(n, N)$.

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)}\right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2 \right], \tag{2.3}$$

which is valid for all positive integers n and N. Let $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$. Using this error bound, Wang and Yang [24] gave the following helpful estimate of $p(n)$.

Lemma 2.1. *For any given integer t, there exists $N(t)$ such that for all $n \geq N(t)$,*

$$\frac{\sqrt{12}\pi^2 e^{\mu(n)}}{36\mu(n)^2}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^t}\right) < p(n) < \frac{\sqrt{12}\pi^2 e^{\mu(n)}}{36\mu(n)^2}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^t}\right). \tag{2.4}$$

This lemma is main tool in our proof. Now we are in a position to prove the following theorem which affirms Wagner's conjecture for $m = 3$.

Theorem 2.2. *For $n \geq 531$, $p(n)$ satisfies the Laguerre inequality of order 3, i.e.,*

$$10p(n+3)^3 - 15p(n+2)p(n+4) + 6p(n+1)p(n+5) - p(n)p(n+6) > 0. \tag{2.5}$$

Proof. To prove it, setting $t = 10$ in Lemma 2.1 gives that for $n \geq 1520$,

$$e^{\mu}\frac{\sqrt{12}\beta(\mu)\pi^2}{36\mu^{12}} < p(n) < e^{\mu}\frac{\sqrt{12}\alpha(\mu)\pi^2}{36\mu^{12}}, \tag{2.6}$$

where

$$\alpha(t) = t^{10} - t^9 + 1, \quad \beta(t) = t^{10} - t^9 - 1. \tag{2.7}$$

Let

$$f(n) := e^{\mu}\frac{\sqrt{12}\beta(\mu)\pi^2}{36\mu^{12}} \tag{2.8}$$

and

$$g(n) := e^{\mu}\frac{\sqrt{12}\alpha(\mu)\pi^2}{36\mu^{12}}. \tag{2.9}$$

Then, in the remaining of this section, we aim to show that

$$10f(n+3)^3 - 15g(n+2)g(n+4) + 6f(n+1)f(n+5) - g(n)g(n+6) > 0. \tag{2.10}$$

For convenience,we denote

$$\begin{aligned} &y = \mu(n), z = \mu(n+1), w = \mu(n+2), r = \mu(n+3) \\ &j = \mu(n+4), k = \mu(n+5), i = \mu(n+6). \end{aligned} \tag{2.11}$$

The left-hand side of the inequality (2.10) can be simplified to

$$\frac{-h_1 e^{y-2r+i} + 6h_2 e^{z-2r+k} - 15h_3 e^{w-2r+j} + 10h_4}{h_4}, \tag{2.12}$$

where

$$h_1 = \alpha(i)\alpha(y)r^{24}w^{12}j^{12}z^{12}k^{12}, \tag{2.13}$$

$$h_2 = \beta(z)\beta(k)r^{24}w^{12}j^{12}y^{12}i^{12}, \tag{2.14}$$

$$h_3 = \alpha(w)\alpha(j)r^{24}z^{12}k^{12}y^{12}i^{12}, \tag{2.15}$$

$$h_4 = \beta(r)^2w^{12}j^{12}z^{12}k^{12}y^{12}i^{12}. \tag{2.16}$$

Now we proceed to prove the numerator of (2.12) is positive for $n \geq 2$. Since h_4 is positive for all $n \geq 1$, we only need to prove

$$-h_1e^{y-2r+i} + 6h_2e^{z-2r+k} - 15h_3e^{w-2r+j} + 10h_4 > 0. \tag{2.17}$$

For this aim, we need to estimate $h_1, h_2, h_3, h_4, e^{y-2r+i}, e^{z-2r+k}$ and e^{w-2r+j}. We prefer to give the estimates of y, z, w, j, k and i by the following equalities. For $n \geq 2$,

$$y = \sqrt{r^2 - 2\pi^2}, z = \sqrt{r^2 - \frac{4\pi^2}{3}}, w = \sqrt{r^2 - \frac{2\pi^2}{3}}, \tag{2.18}$$

$$j = \sqrt{r^2 + \frac{2\pi^2}{3}}, k = \sqrt{r^2 + \frac{4\pi^2}{3}}, i = \sqrt{r^2 + 2\pi^2}.$$

By the expansions of y, z, w, j, k, i, we have that for $n \geq 41$,

$$y_1 < y < y_2,\ z_1 < z < z_2,\ w_1 < w < w_2, \tag{2.19}$$

$$j_1 < j < j_2,\ k_1 < k < k_2,\ i_1 < i < i_2.$$

where

$$y_1 = r - \frac{\pi^2}{r} - \frac{\pi^4}{2r^3} - \frac{\pi^6}{2r^5} - \frac{5\pi^8}{8r^7} - \frac{8\pi^{10}}{8r^9},$$

$$y_2 = r - \frac{\pi^2}{r} - \frac{\pi^4}{2r^3} - \frac{\pi^6}{2r^5} - \frac{5\pi^8}{8r^7} - \frac{7\pi^{10}}{8r^9},$$

$$z_1 = r - \frac{2\pi^2}{3r} - \frac{2\pi^4}{9r^3} - \frac{4\pi^6}{27r^5} - \frac{10\pi^8}{81r^7} - \frac{29\pi^{10}}{243r^9},$$

$$z_2 = r - \frac{2\pi^2}{3r} - \frac{2\pi^4}{9r^3} - \frac{4\pi^6}{27r^5} - \frac{10\pi^8}{81r^7} - \frac{28\pi^{10}}{243r^9},$$

$$w_1 = r - \frac{\pi^2}{3r} - \frac{\pi^4}{18r^3} - \frac{\pi^6}{54r^5} - \frac{5\pi^8}{648r^7} - \frac{8\pi^{10}}{1944r^9},$$

$$w_2 = r - \frac{\pi^2}{3r} - \frac{\pi^4}{18r^3} - \frac{\pi^6}{54r^5} - \frac{5\pi^8}{648r^7} - \frac{7\pi^{10}}{1944r^9},$$

$$j_1 = r + \frac{\pi^2}{3r} - \frac{\pi^4}{18r^3} + \frac{\pi^6}{54r^5} - \frac{5\pi^8}{648r^7},$$

$$j_2 = r + \frac{\pi^2}{3r} - \frac{\pi^4}{18r^3} + \frac{\pi^6}{54r^5} - \frac{5\pi^8}{648r^7} + \frac{7\pi^{10}}{1944r^9},$$

$$k_1 = r + \frac{2\pi^2}{3r} - \frac{2\pi^4}{9r^3} + \frac{4\pi^6}{27r^5} - \frac{10\pi^8}{81r^7},$$

$$k_2 = r + \frac{2\pi^2}{3r} - \frac{2\pi^4}{9r^3} + \frac{4\pi^6}{27r^5} - \frac{10\pi^8}{81r^7} + \frac{28\pi^{10}}{243r^9},$$

$$i_1 = r + \frac{\pi^2}{r} - \frac{\pi^4}{2r^3} + \frac{\pi^6}{2r^5} - \frac{5\pi^8}{8r^7},$$

$$i_2 = r + \frac{\pi^2}{r} - \frac{\pi^4}{2r^3} + \frac{\pi^6}{2r^5} - \frac{5\pi^8}{8r^7} + \frac{7\pi^{10}}{8r^9}.$$

Applying (2.19) to the definition (2.7) of $\alpha(t)$ and $\beta(t)$ and substituting these $\alpha(t)$ and $\beta(t)$ into h_1, h_2, h_3, we get

$$h_1 < \left(i^{10} - i_1 i^8 + 1\right)\left(y^{10} - y_1 y^8 + 1\right) r^{24} w^{12} j^{12} z^{12} k^{12}, \tag{2.20}$$
$$h_2 > \left(z^{10} - z_1 z^8 - 1\right)\left(k^{10} - k_2 k^8 - 1\right) r^{24} y^{12} w^{12} j^{12} i^{12},$$
$$h_3 < \left(w^{10} - w_1 w^8 + 1\right)\left(w^{10} - w_1 w^8 + 1\right) r^{24} z^{12} k^{12} y^{12} i^{12},$$

Next we turn to estimate e^{w-2r+j}, e^{z-2r+k} and e^{y-2r+i}. By (2.19), one can see that for $n \geq 41$,

$$w_1 - 2r + j_1 < w - 2r + j < w_2 - 2r + j_2, \tag{2.21}$$
$$z_1 - 2r + k_1 < z - 2r + k < z_2 - 2r + k_2,$$
$$y_1 - 2r + i_1 < y - 2r + i < y_2 - 2r + i_2$$

which implies that

$$e^{w_1-2r+j_1} < e^{w-2r+j} < e^{w_2-2r+j_2}, \tag{2.22}$$
$$e^{z_1-2r+k_1} < e^{z-2r+k} < e^{z_2-2r+k_2}$$
$$e^{y_1-2r+i_1} < e^{y-2r+i} < e^{y_2-2r+i_2}.$$

In order to give a feasible bound for e^{w-2r+j}, e^{z-2r+k} and e^{y-2r+i}, we define

$$\Phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}$$

and

$$\phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120}.$$

It can be checked that for $t < 0$,

$$\phi(t) < e^t < \Phi(t). \tag{2.23}$$

To apply this result to (2.22), it suffices to show that w_2-2r+j_2, z_2-2r+k_2 and $y_2 - 2r + i_2$ are negative. By straightforward calculation, one can get that

$$w_2 - 2r + j_2 = -\frac{\pi^4 \left(5\pi^4 + 36r^4\right)}{324r^7},$$

$$z_2 - 2r + k_2 = -\frac{4\pi^4 \left(9\pi^4 + 5r^4\right)}{81r^7},$$

$$y_2 - 2r + i_2 = -\frac{4\pi^4 \left(\pi^4 + 5r^4\right)}{4r^7}.$$

Obviously, for $n \geq 2$, w_2-2r+j_2, z_2-2r+k_2 and y_2-2r+i_2 are negative. Thus, applying (2.23) to (2.22), we obtain that for $n \geq 41$,

$$\phi(w_1 - 2r + j_1) < e^{w-2r+j} < \Phi(w_2 - 2r + j_2), \tag{2.24}$$

$$\phi(z_1 - 2r + k_1) < e^{z-2r+k} < \Phi(z_2 - 2r + k_2),$$

$$\phi(y_1 - 2r + i_1) < e^{y-2r+i} < \Phi(y_2 - 2r + i_2).$$

Now, we proceed to prove (2.17). For convenience, let

$$A(r) = -h_1 e^{y-2r+i} + 6h_2 e^{z-2r+k} - 15h_3 e^{w-2r+j} + 10h_4, \tag{2.25}$$

we need to show the positivity of $A(r)$. Using (2.20) and (2.24), we obtain that for $n \geq 41$,

$$\begin{aligned} A(r) >& -\left(i^{10} - i_1 i^8 + 1\right)\left(y^{10} - y_1 y^8 + 1\right) r^{24} w^{12} j^{12} z^{12} k^{12} \Phi(y_2 - 2r + i_2) \\ &+ 6\left(z^{10} - z_2 z^8 + 1\right)\left(k^{10} - k_2 k^8 + 1\right) r^{24} w^{12} j^{12} y^{12} i^{12} \phi(z_1 - 2r + k_1) \\ &- 15\left(w^{10} - w_1 w^8 + 1\right)\left(j^{10} - j_1 j^8 + 1\right) r^{24} z^{12} k^{12} y^{12} i^{12} \Phi(w_2 - 2r + j_2) \end{aligned}$$

$$+ 10\beta(r)^2 w^{12} j^{12} z^{12} k^{12} y^{12} i^{12}.$$

Denote the right-hand side of the above inequality by $A_1(r)$. Using (2.18) to $A_1(r)$, one can see $A_1(r)$ can be rewritten as a polynomial of r. We use software Mathematica to expand $A_1(r)$ as a polynomials of r by elementary commands. Note that all the coefficients of $A_1(r)$ can be known. To ensure $A_1(r) > 0$, it suffices to let r be larger than the sum of absolute value of the coefficients of other terms divided by the coefficient of the terms with degree 235. We also make this approach effective using the elementary commands in Mathematica. For details, by Mathematica we can quickly simplify $A_1(r)$ as

$$A_1(r) = \frac{\sum_{k=0}^{122} a_k r^k}{2^{13} 3^{55} 5 r^{39}}, \tag{2.26}$$

where a_k are known real numbers, and the first few terms $a_{122},\ a_{121}, a_{120}$ are given below,

$$a_{122} = 2^{14} 3^{50} 5 \pi^{10} (10\pi^2 - 87), \quad a_{121} = 2^{14} 3^{50} 5 (-215\pi^{12} + 174\pi^{10} - 7776),$$
$$a_{120} = 2^{15} 3^{50} 5 (875\pi^{12} - 108\pi^{10} + 3888).$$

Thus, for $n \geq 41$, we have

$$A(r) > \frac{\sum_{k=0}^{122} a_k r^k}{2^{13} 3^{55} 5 r^{39}}. \tag{2.27}$$

Since r is positive for $n \geq 1$, we have that

$$\sum_{k=0}^{122} a_k r^k > \sum_{k=0}^{121} -|a_k| r^k + a_{122} r^{122}. \tag{2.28}$$

Thus, to get (2.27), we only need to show that for $n \geq 39839$,

$$\sum_{k=0}^{121} -|a_k| r^k + a_{122} r^{122} > 0. \tag{2.29}$$

For $0 \leq k \leq 120$, we find that for $r \geq 8$

$$-|a_k| r^k > -a_{120} r^{120}. \tag{2.30}$$

It follows that for $r \geq 8$,

$$\sum_{k=0}^{122} a_k r^k > \sum_{k=0}^{121} -|a_k| r^k + a_{122} r^{122} > (-121 a_{120} + a_{121} r + a_{122} r^2) r^{120}. \tag{2.31}$$

Combing (2.27) and (2.31), $A(r)$ is positive provided

$$-121a_{120}+a_{121}r+a_{122}r^2>0, \tag{2.32}$$

which is true for $r>511$, or equivalently, $n\geq 39839$. Thus, we arrive at that (2.17) is true for $n\geq 39839$, which implies (2.10). On the other hand, it can be checked that (2.2) is also true for $531\leq n\leq 39838$. Thus the proof is completed. ∎

3 For $4\leq m\leq 10$

In this section, we will concern with the Laguerre inequalities of any order. In fact, the approach given in Section 2 still works for any order. We only need to find an appropriate t in Lemma 2.1, adjust the terms of the Taylor expansion of exponential function and the terms of the Taylor expansion of the upper and lower bound of $\sqrt{r^2-2n\pi^2}$ and $\sqrt{r^2+2n\pi^2}$, where $n=1,2,\cdots,m$. In the rest of this section, for $4\leq m\leq 10$, we will give $N(m)$ such that for $n>N(m)$, $p(n)$ satisfies the Laguerre inequality of order m. Since the procedure is similar with that in Section 2, we omit tedious formulae and just give the value of t, the terms of the Taylor expansion of e^t and the terms of the Taylor expansion of the upper and lower bound of $\sqrt{r^2-2n\pi^2}$ and $\sqrt{r^2+2n\pi^2}$, where $n=1,2,\cdots,m$.

For $m=4$, set $t=18$ in Lemma 2.1. Then for $n\geq 5720$ we have

$$e^y\frac{\sqrt{12}\beta(y)\pi^2}{36y^{20}}<p(n)<e^y\frac{\sqrt{12}\alpha(y)\pi^2}{36y^{20}}. \tag{3.1}$$

where

$$\alpha(t)=t^{18}-t^{17}+1,\quad \beta(t)=t^{18}-t^{17}-1. \tag{3.2}$$

We use the first several terms of the Taylor expansion to approximate the exponential function. Specially, for $t<0$, $\phi(t)<e^t<\Phi(t)$ where

$$\Phi(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}, \tag{3.3}$$

$$\phi(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}. \tag{3.4}$$

It also requires more precise estimates for y,z,w,j,k,i,s,q as in the proof of Theorem 2.2, where

$$s=\sqrt{r^2-\frac{8\pi^2}{3}},y=\sqrt{r^2-2\pi^2},z=\sqrt{r^2-\frac{4\pi^2}{3}},w=\sqrt{r^2-\frac{2\pi^2}{3}},$$

$$j=\sqrt{r^2+\frac{2\pi^2}{3}},k=\sqrt{r^2+\frac{4\pi^2}{3}},i=\sqrt{r^2+2\pi^2},q=\sqrt{r^2-\frac{8\pi^2}{3}}.$$

We use the first 7 and 8 terms of the Taylor expansion to approximate y, z, w, j, k, i, s, q. It can be checked that for $n \geq 22$,

$$\begin{aligned} &y_1 < y < y_2, z_1 < z < z_2, w_1 < w < w_2, j_1 < j < j_2, \\ &k_1 < k < k_2, i_1 < i < i_2, s_1 < s < s_2, q_1 < q < q_2. \end{aligned} \tag{3.5}$$

where

$$y_1 = r - \frac{4\pi^2}{3r} - \frac{8\pi^4}{9r^3} - \frac{32\pi^6}{27r^5} - \frac{160\pi^8}{81r^7} - \frac{896\pi^{10}}{243r^9} - \frac{1972\pi^{12}}{243r^{11}} - \frac{13000\pi^{14}}{729r^{13}},$$

$$y_2 = r - \frac{4\pi^2}{3r} - \frac{8\pi^4}{9r^3} - \frac{32\pi^6}{27r^5} - \frac{160\pi^8}{81r^7} - \frac{896\pi^{10}}{243r^9} - \frac{1972\pi^{12}}{243r^{11}} - \frac{11264\pi^{14}}{729r^{13}},$$

$$z_1 = r - \frac{\pi^2}{r} - \frac{\pi^4}{2r^3} - \frac{\pi^6}{2r^5} - \frac{5\pi^8}{8r^7} - \frac{7\pi^{10}}{8r^9} - \frac{21\pi^{12}}{16r^{11}} - \frac{40\pi^{14}}{16r^{13}},$$

$$z_2 = r - \frac{\pi^2}{r} - \frac{\pi^4}{2r^3} - \frac{\pi^6}{2r^5} - \frac{5\pi^8}{8r^7} - \frac{7\pi^{10}}{8r^9} - \frac{21\pi^{12}}{16r^{11}} - \frac{33\pi^{14}}{16r^{13}},$$

$$w_1 = r - \frac{2\pi^2}{3r} - \frac{2\pi^4}{9r^3} - \frac{4\pi^6}{27r^5} - \frac{10\pi^8}{81r^7} - \frac{28\pi^{10}}{243r^9} - \frac{28\pi^{12}}{243r^{11}} - \frac{100\pi^{14}}{729r^{13}},$$

$$w_2 = r - \frac{2\pi^2}{3r} - \frac{2\pi^4}{9r^3} - \frac{4\pi^6}{27r^5} - \frac{10\pi^8}{81r^7} - \frac{28\pi^{10}}{243r^9} - \frac{28\pi^{12}}{243r^{11}} - \frac{88\pi^{14}}{729r^{13}},$$

$$j_1 = r - \frac{\pi^2}{3r} - \frac{\pi^4}{18r^3} - \frac{\pi^6}{54r^5} - \frac{5\pi^8}{648r^7} - \frac{7\pi^{10}}{1944r^9} - \frac{7\pi^{12}}{3888r^{11}} - \frac{12\pi^{14}}{11664r^{13}},$$

$$j_2 = r - \frac{\pi^2}{3r} - \frac{\pi^4}{18r^3} - \frac{\pi^6}{54r^5} - \frac{5\pi^8}{648r^7} - \frac{7\pi^{10}}{1944r^9} - \frac{7\pi^{12}}{3888r^{11}} - \frac{11\pi^{14}}{11664r^{13}},$$

$$k_1 = r + \frac{\pi^2}{3r} - \frac{\pi^4}{18r^3} + \frac{\pi^6}{54r^5} - \frac{5\pi^8}{648r^7} + \frac{7\pi^{10}}{1944r^9} - \frac{7\pi^{12}}{3888r^{11}},$$

$$k_2 = r + \frac{\pi^2}{3r} - \frac{\pi^4}{18r^3} + \frac{\pi^6}{54r^5} - \frac{5\pi^8}{648r^7} + \frac{7\pi^{10}}{1944r^9} - \frac{7\pi^{12}}{3888r^{11}} + \frac{11\pi^{14}}{11664r^{13}},$$

$$i_1 = r + \frac{2\pi^2}{3r} - \frac{2\pi^4}{9r^3} + \frac{4\pi^6}{27r^5} - \frac{10\pi^8}{81r^7} + \frac{28\pi^{10}}{243r^9} - \frac{28\pi^{12}}{243r^{11}},$$

$$i_2 = r + \frac{2\pi^2}{3r} - \frac{2\pi^4}{9r^3} + \frac{4\pi^6}{27r^5} - \frac{10\pi^8}{81r^7} + \frac{28\pi^{10}}{243r^9} - \frac{28\pi^{12}}{243r^{11}} + \frac{88\pi^{14}}{729r^{13}},$$

$$s_1 = r + \frac{\pi^2}{r} - \frac{\pi^4}{2r^3} + \frac{\pi^6}{2r^5} - \frac{5\pi^8}{8r^7} + \frac{7\pi^{10}}{8r^9} - \frac{21\pi^{12}}{16r^{11}},$$

$$s_2 = r + \frac{\pi^2}{r} - \frac{\pi^4}{2r^3} + \frac{\pi^6}{2r^5} - \frac{5\pi^8}{8r^7} + \frac{7\pi^{10}}{8r^9} - \frac{21\pi^{12}}{16r^{11}} + \frac{33\pi^{14}}{16r^{13}},$$

$$q_1 = r + \frac{4\pi^2}{3r} - \frac{8\pi^4}{9r^3} + \frac{32\pi^6}{27r^5} - \frac{160\pi^8}{81r^7} + \frac{896\pi^{10}}{243r^9} - \frac{1972\pi^{12}}{243r^{11}},$$

$$q_1 = r + \frac{4\pi^2}{3r} - \frac{8\pi^4}{9r^3} + \frac{32\pi^6}{27r^5} - \frac{160\pi^8}{81r^7} + \frac{896\pi^{10}}{243r^9} - \frac{1972\pi^{12}}{243r^{11}} + \frac{11264\pi^{14}}{729r^{13}}.$$

With the similar arguments as in Section 2, we have $A(r)$ is a certain polynomial of degree 235 with positive coefficient of the first term. One can easily see that for $n > 20701$, $A(r)$ is positive. It follows that for $n > 20701$ the Laguerre inequality of order 4 holds. A direct calculation reveals that for $1102 \leq n \leq 20701$, $p(n)$ satisfies the Laguerre inequality of order 4. Hence, we proved Wagner's conjecture for $m = 4$.

For $m = 5$, we also set $t = 18$ in Lemma 2.1 and then for $n \geq 5720$, (3.1) holds. We use the first 7 and 8 terms of the Taylor expansion to approximate the exponential function, i.e., for $t < 0$, $\phi(t) < e^t < \Phi(t)$ where

$$\Phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720}, \tag{3.6}$$

$$\phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040}. \tag{3.7}$$

We use the first 9 and 10 terms of the Taylor expansion to approximate $\sqrt{r^2 - 2n\pi^2}$ and $\sqrt{r^2 + 2n\pi^2}$, where $n = 1, 2, 3, 4, 5$. With the similar arguments as in Section 2, we have $A(r)$ is a certain polynomial of degree 334 with positive coefficient of the first term. It can be checked that for $n > 35612$, $A(r)$ is positive. It affirms that $p(n)$ satisfies the Laguerre inequality of order 5 for $n > 35612$. Numerical evidence gives that for $1923 \leq n \leq 35612$, $p(n)$ satisfies the Laguerre inequality of order 5. Hence, we proved Wagner's conjecture for $m = 5$.

For $m = 6$, we still set $t = 18$ in Lemma 2.1 and use the same Taylor expansion of e^t as that in the case $m = 5$. We use the first 10 and 11 terms of the Taylor expansion to approximate $\sqrt{r^2 - 2n\pi^2}$ and $\sqrt{r^2 + 2n\pi^2}$, where $n = 1, 2, \cdots, 6$. With the similar argument, we have $A(r)$ is a certain polynomial of degree 387 with positive first coefficient. It can be checked that for $n > 223615$, $A(r)$ is positive. It means the positivity of the Laguerre inequality of order 6. A direct calculation gives that for $3014 \leq n \leq 223615$, $p(n)$ satisfies the Laguerre inequality of order 6. Hence, we proved Wagner's conjecture for $m = 6$.

For $m = 7$, set $t = 22$ in Lemma 2.1, then for $n \geq 6084$ we have

$$e^y \frac{\sqrt{12}\beta(y)\pi^2}{36y^{24}} < p(n) < e^y \frac{\sqrt{12}\alpha(y)\pi^2}{36y^{24}}. \tag{3.8}$$

where

$$\alpha(t) = t^{22} - t^{21} + 1, \quad \beta(t) = t^{22} - t^{21} - 1. \tag{3.9}$$

Let $\Phi(t)$ and $\phi(t)$ be the first 9 and 10 terms in the Taylor expansion of e^t. Then, for $t<0$, we have $\phi(t)<e^t<\Phi(t)$.We use the first 12 and 13 terms of the Taylor expansion to approximate $\sqrt{r^2-2n\pi^2}$ and $\sqrt{r^2+2n\pi^2}$, where $n=1,2,\cdots,7$. With the similar argument, we have $A(r)$ is a certain polynomial of degree 541 with positive coefficient of the first term. It can be verified that for $n>711707$, $A(r)$ is positive. It leads to the positivity of the Laguerre inequality of order 7. The case for $4391\le n\le 711707$ can be checked by Mathematica. Hence, we affirm Wagner's conjecture for $m=7$.

For $m=8$, set $t=24$ in Lemma 2.1, then for $n\ge 11327$ we have

$$e^y\frac{\sqrt{12}\beta(y)\pi^2}{36y^{26}}<p(n)<e^y\frac{\sqrt{12}\alpha(y)\pi^2}{36y^{26}}. \tag{3.10}$$

where

$$\alpha(t)=t^{24}-t^{23}+1,\quad \beta(t)=t^{24}-t^{23}-1. \tag{3.11}$$

We also set $\Phi(t)$ and $\phi(t)$ be the first 9 and 10 terms in the Taylor expansion of e^t. We use the first 13 and 14 terms of the Taylor expansion to approximate $\sqrt{r^2-2n\pi^2}$ and $\sqrt{r^2+2n\pi^2}$, where $n=1,2,\cdots,8$. Then $A(r)$ is a certain polynomial of degree 663 with positive first coefficient. It is easily seen that for $n>31423081$, $A(r)$ is positive. Then $p(n)$ satisfies the Laguerre inequality of order 8 for $n>31423081$. A direct calculation reveals that for $6070\le n\le 31423081$, $p(n)$ satisfies the Laguerre inequality of order 8. Hence, we proved Wagner's conjecture for $m=8$.

For $m=9$, set $t=28$ in Lemma 2.1, then for $n\ge 16350$ we have

$$e^y\frac{\sqrt{12}\beta(y)\pi^2}{36y^{30}}<p(n)<e^y\frac{\sqrt{12}\alpha(y)\pi^2}{36y^{30}}, \tag{3.12}$$

where

$$\alpha(t)=t^{28}-t^{27}+1,\quad \beta(t)=t^{28}-t^{27}-1. \tag{3.13}$$

$\Phi(t)$ and $\phi(t)$ are the same as those in $m=8$. We use the first 15 and 16 terms of the Taylor expansion to approximate $\sqrt{r^2-2n\pi^2}$ and $\sqrt{r^2+2n\pi^2}$, where $n=1,2,\cdots,9$. It leads that $A(r)$ is a certain polynomial of degree 835 with positive first coefficient. One can see that for $n>68197175$, $A(r)$ is positive. It means the positivity of the Laguerre inequality of order 9. We use mathematica to verify the case for $8063\le n\le 68197175$. It takes about ten minutes. Thus $p(n)$ satisfies the Laguerre inequality of order 9 for $n\ge 8063$, which proves Wagner's conjecture for $m=9$.

For $m=10$, with the similar argument, we can deduce that for $n>218573927203706866261$, $A(r)$ is positive. It means the positivity of the Laguerre inequality of order 10. But there exists a gap between Wagner's conjecture and this bound. We try to use mathematica to verify it, but it takes few hours and do not give the result.

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