TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

LAGUERRE INEQUALITIES AND COMPLETE MONOTONICITY FOR THE RIEMANN XI-FUNCTION AND THE PARTITION FUNCTION

LARRY X.W. WANG AND NEIL N.Y. YANG

ABSTRACT. In this paper, we find some conditions under which a sequence $\{\alpha(n)\}$ will satisfy the Laguerre inequality of any order asymptotically. Using this method, we prove that for any r and some constant c, the Maclaurin coefficients $\gamma(n)$ of the Riemann Xi-function satisfy the Laguerre inequality of order r when $n > cr^3$, which provides a necessary condition for the Riemann hyperthesis. We also prove that the partition function satisfies the Laguerre inequality of order $r \geq 5$ when $n \geq 6r^4$. As a consequence, it gives an affirmative answer to Wagner's conjecture on the threshold for the Laguerre inequalities of order no more than 10 for the partition function. Moreover, motivated by the study of Craven and Csordas on the complete monotonicity of the Maclaurin coefficients of entire functions in Laguerre-Pólya class, we consider the complete monotonicity of the sequences $\{\alpha(n)\}$. We give the criteria for the asymptotically complete monotonicity of the sequence $\{\alpha(n)\}$ and $\{\log \alpha(n)\}$, respectively. With this criteria, we show that $(-1)^r\Delta^r\gamma(n) > 0$ for $n > cr^3$ and $(-1)^{r-1}\Delta^r \log\gamma(n) > 0$ for sufficiently large n. Furthermore, we propose some open problems.

1. INTRODUCTION

A real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{1.1}$$

is said to be in the Laguerre-Pólya class, denoted by $\psi(x) \in \mathcal{LP}$, if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} \left(1 + x/x_n\right) e^{-x/x_n}, \tag{1.2}$$

where c, β, x_k are real numbers, $\alpha \geq 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. For more backgrounds on the theory of the $\mathcal{LP}$ class, we refer to [21] and [30]. The $\mathcal{LP}$ class has attracted much attention in view of its connection with the Riemann hypothesis.

Received by the editors 29th October, 2022.
2020 *Mathematics Subject Classification.* Primary 11M26, 11M06, 11P82, 05A20.
Key words and phrases. Riemann Xi-function, Laguerre inequality, partition function, Riemann hypothesis, finite difference, complete monotonicity.
We thank the anonymous referee for helpful comments. This work was supported by the National Science Foundation of China (grant number 12171254).

The Riemann Xi-function is defined as

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

Following [5], the entire function Ξ can be written in Taylor series form as

$$F(z) := \frac{1}{8}\Xi\left(\frac{z}{2}\right) = \sum_{n=0}^{\infty}\frac{\gamma(n)}{n!}\left(-z^2\right)^n.$$

It is known that the Riemann hypothesis is related to the $\mathcal{LP}$ properties of the function $F(z)$.

We say that a polynomial with real coefficients is *hyperbolic* if all of its zeros are real. The *Jensen polynomial of degree d and shift n* of an arbitrary real sequence $\{\alpha(0), \alpha(1), \alpha(2), \ldots\}$ is the polynomial

$$J_{\alpha}^{d,n}(X) := \sum_{j=0}^{d}\binom{d}{j}\alpha(n+j)X^j.$$

Pólya and Schur [29] proved that the Riemann hypothesis holds if and only if the function $F(z)$ belongs to the $\mathcal{LP}$ class, i.e., having only real and negative zeros, or equivalently, all associated Jensen polynomials having only real zeros. Karlin [17] conjectured that the Riemann hypothesis is equivalent to the total positivity of certain matrices involving the coefficients of the Riemann Xi-function. Csordas and Varga [7] showed that the Riemann hypothesis is equivalent to that the function $F(z)$ satisfies all of the Laguerre inequalities of any order.

In a brilliant paper [13], Griffin, Ono, Rolen and Zagier proved that the Jensen polynomials associated with the Riemann Xi-function and the partition function are hyperbolic for sufficiently large n. Griffin, Ono, Rolen, Thorner, Tripp and Wagner [12] made this approach effective and gave a lower bound for the hyperbolicity of Jensen polynomials associated with the Riemann Xi-function. O'Sullivan [25, 26] supplied some details for the proofs in [13]. Wagner [32] extended these results to various L-functions.

For Karlin's conjecture, define

$$\beta(n) := \begin{cases} \frac{\gamma(n)}{n!} & , \quad n \geq 0 \\ 0 & , \quad n < 0 \end{cases},$$

and

$$D(n,r) := \det\left(\beta(n-i+j)\right)_{1\leq i,j\leq r}.$$

Csordas, Norfolk and Varga [5] proved that $D(n,2) > 0$ for all n (see also [6] and [22]). Nuttall [23] proved $D(n,3) > 0$ for all n. Recently, the authors [36] showed that the r-order determinants $\det\left(\beta(n-i+j)\right)_{1\leq i,j\leq r}$ and $\det\left(\gamma(n-i+j)\right)_{1\leq i,j\leq r}$ are positive for $n > ce^{r(r-1)}$ and the determinants associated with the partition function $\det\left(p(n-i+j)\right)_{1\leq i,j\leq r}$ are positive for $n > e^{2r(r-1)}$.

In this paper, we first confine our attention to the Laguerre inequality. Recall that if a function $f(x)$ satisfies

$$f'(x)^2 - f(x)f''(x) \geq 0,$$

then it is called to satisfy *Laguerre inequality.* Laguerre [20] showed that if $f(x)$ is a real function with only real zeros, then the Laguerre inequality holds for $f(x)$ and hence Laguerre inequality is intimately related to Riemann hypothesis. In [16], Jensen generalized Laguerre inequality to higher order. Namely, for each n, we say that $f(x)$ satisfies *Laguerre inequality of order* n if we have

$$L_n(f(x)) := \frac{1}{2}\sum_{k=0}^{2n}(-1)^{n+k}\binom{2n}{k}f^{(k)}(x)f^{(2n-k)}(x) \geq 0$$

for all $x \in \mathbb{R}$, where $f^{(k)}(x)$ denotes the k-th derivative of $f(x)$. It yields the classical Laguerre inequality when $n = 1$. Note that Csordas and Vishnyakova [8] showed that if a function $f(x)$ satisfies $L_n(f(x)) \geq 0$ for all n and for all $x \in R$, then $f(x)$ is in the Laguerre-Pólya class. It means that Laguerre inequality is a characterizing property of functions in the Laguerre-Pólya class. For more backgrounds on Laguerre inequalities, see [10, 27, 28, 31].

The first author and Yang [34] considered whether discrete sequences have similar results with the Laguerre inequalities for functions. Recall that a sequence $\{a_n\}_{n\geq 0}$ satisfies *Laguerre inequality of order* m if we have

$$L_m(a_n) := \frac{1}{2}\sum_{k=0}^{2m}(-1)^{k+m}\binom{2m}{k}a_{n+k}a_{2m-k+n} \geq 0$$

for all $n \geq 0$. They proved that the partition function, the overpartition function, the Bernoulli numbers, the derangement numbers, the Motzkin numbers, the Fine numbers, the Franel numbers and the Domb numbers possess Laguerre inequality of order 2. Wagner [33] proved that $\gamma(n)$ and the partition function $p(n)$ satisfy Laguerre inequalities of any order as $n \to \infty$ and proposed a conjecture on the threshold of the Laguerre inequality of order m of $p(n)$ for $3 \leq m \leq 10$. Dou and the first author [9] gave an explicit bound $N(m)$ such that for $n > N(m)$, $p(n)$ satisfies Laguerre inequality of order m. As a consequence, the cases $3 \leq m \leq 9$ of Wagner's conjecture have been proved.

In this paper, we concern with a general family of sequences that includes $\gamma(n)$ and the partition function $p(n)$. We will prove under some conditions the sequence $\{\alpha(n)\}$ satisfies the Laguerre inequalities of any order r when n is larger than a computable bound $N(r)$. Moreover, we give explicit expression of $N(r)$ for $\gamma(n)$ and $p(n)$.

Theorem 1.1. *Let* $\{\alpha(n)\}, \{\delta(n)\}, \{A_i(n)\}$ *be sequences such that as* $n \to \infty$, $\alpha(n)$ *stays positive,* $\delta(n) \to 0^+$, $\lim_{n\to\infty} A_2(n)/\delta(n)^t < 0$ *and* $A_{2i}(n) = o(\delta(n)^{it})$ $(i \geq 2)$ *for some positive* t, *and for* $-r < j < r$,

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = \sum_{i=1}^{2N} A_i(n)j^i + o\left(\delta(n)^{tr}\right), \tag{1.3}$$

then for sufficiently large n,

$$L_r(\alpha(n)) = \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\alpha(n+k)\alpha(n+2r-k) > 0. \tag{1.4}$$

In the above theorem, the integer N is chosen to ensure that the remaining term is an infinitesimal of higher order, and so are the N's in the other theorems in this section. Note that N may vary with different inequalities, different orders and different sequences. If we can show $L_r(\alpha(n-r)) > 0$ for arbitrary n, then Theorem 1.1 holds since $L_r(\alpha(n)) = L_r(\alpha((n+r)-r)) > 0$. To ensure it, we provide the sufficient condition contained in the following theorem.

Theorem 1.2. *Suppose that* $\{\alpha(n)\}, \{\delta(n)\}, \{A_i(n)\}$ *satisfy the conditions in the above theorem. Define the function*

$$R(n,j) := \log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) - \sum_{i=1}^{2N} A_i(n) j^i. \tag{1.5}$$

In addition, suppose that for some real numbers $c_1 > 0, c_2 > 1$ *and* $t_1 > t > 0$, *we have*

$$\begin{gathered} |R(n,j)| < c_2 \delta(n)^{\frac{t+t_1}{2} r}, \\ -c_2\delta(n)^t \le A_2(n) \le -c_1\delta(n)^t, \quad |A_{2i}(n)| \le c_1 c_2^{i-1} \delta(n)^{t+(i-1)t_1}. \end{gathered} \tag{1.6}$$

Denote

$$c := (48 c_2 c_3)^{-t_2},$$

where

$$c_3 := \max\left\{1, c_1^{-1}\right\}, \quad t_2 := \max\left\{\frac{3}{t}, \frac{2}{t_1 - t}\right\}.$$

Then the inequality (1.4) *holds for* $\delta(n) < c r^{-t_2}$.

Employing these theorems, we will prove the following results.

Theorem 1.3. *For every* r, *there exists a constant* c *such that* $\gamma(n)$ *satisfies the Laguerre inequality of order* r *whenever* $n > cr^3$.

Theorem 1.4. *For every* $r \ge 5$, *the partition function* $p(n)$ *satisfies the Laguerre inequality of order* r *whenever* $n > 6r^4$.

We also consider complete monotonicity of the Maclaurin coefficient $\gamma(n)$ of the Riemann Xi-function. Note that the complete monotonicity is another property related to the $\mathcal{LP}$ class. Denote $\Delta^0\alpha(n) = \alpha(n), \Delta^{r+1}\alpha(n) = \Delta^r\alpha(n+1) - \Delta^r\alpha(n)$, i.e.,

$$\Delta^r \alpha(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{r+k} \alpha(n+k).$$

We say $\{\alpha(n)\}$ satisfies *r-order monotonicity* if $\operatorname{sgn}(\Delta^r\alpha(n))$ is invariant. If for any r, $\alpha(n)$ satisfies r-order monotonicity, then we call $\{\alpha(n)\}$ possesses *complete monotonicity*. We also call $\{\alpha(n)\}$ satisfies *asymptotically complete monotonicity* if for any r, there exists $N(r)$ such that $\{\alpha(n)\}$ satisfies r-order monotonicity for $n > N(r)$.

Recall that a function $\psi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k!, \gamma_k > 0$ is said to be of the first type of $\mathcal{LP}$, denoted by $\psi(x) \in \mathcal{LP}\mathrm{I}$, if

$$\psi(x) = c x^m e^{\beta x} \prod_{k=1}^{\infty} (1 + x/x_n), \tag{1.7}$$

where c, β, x_k are positive real numbers, m is a positive integer and $\sum x_n^{-1} < \infty$. Craven and Csordas [4] proved that if $\psi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{LP}\mathrm{I}$ with γ_k nonnegative and increasing, then $\Delta^r\gamma_k > 0$ for all r and k. Thus it is interesting to consider whether $\gamma(n)$ satisfies this property as well. Unfortunately, $\gamma(n)$ is decreasing since it is log-concave and $\gamma(1) > \gamma(2)$. However, notice that Craven and Csordas [4] actually proved that $\beta \ge 1$ if and only if γ_k is increasing, we consider the case $\beta < 1$ and establish the following theorem.

Theorem 1.5. *If $\psi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{LP}\mathrm{I}$ is of the form (1.7) and $\gamma_k > 0$, then $\beta < 1$ implies $(-1)^r\Delta^r\gamma_k > 0$ for sufficiently large k.*

This result encourages us to consider the positivity of $(-1)^r\Delta^r\gamma(n)$. Another motivation is complete monotonicity of the partition function arising from the conjecture proposed by Good [11] which states that $\Delta^r p(n)$ alternates in sign up to a certain value $n = n(r)$, and then it stays positive. Gupta [14] proved that for any given r, $\Delta^r p(n) > 0$ for sufficiently large n. Odlyzko [24] proved Good's conjecture and gave an asymptotical formula for the threshold $n(r) \sim 6\pi^{-2}r^2\log(r)^2$. Knessl and Keller [18, 19] obtained an approximation $n(r)'$ for $n(r)$ for which $|n(r)' - n(r)| \leq 2$ up to $r = 75$. Almkvist [1, 2] proved that $n(r)$ satisfies certain equations. Moreover, Chen, the first author and Xie [3] considered the logarithm of partition function and deduced that $(-1)^{r-1}\Delta^r\log p(n) > 0$ for sufficiently large n.

Motivated by these results, for the sequence $\{\alpha(n)\}$ given in Theorem 1.1, we prove that the sign of $\Delta^r\alpha(n)$ ($\Delta^r\log\alpha(n)$, respectively) are asymptotically in accordance with the sign of $A_1(n)$ ($A_r(n)$, respectively). We pertain our method to $\gamma(n)$ and prove $(-1)^r\Delta^r\gamma(n) > 0$ and $(-1)^r\Delta^r\log\gamma(n) > 0$ for $n > cr^3$.

Theorem 1.6. *Let $\{\alpha(n)\}, \{\delta(n)\}, \{A_i(n)\}$ be sequences such that as $n \to \infty$, $\alpha(n)$ stays positive, $\delta(n) \to 0^+$, and*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = \sum_{i=1}^{N} A_i(n)j^i + R(n,j),$$

where $A_1(n) = O\left(\delta(n)^t\right)$, $A_i(n) = o\left(\delta(n)^{it}\right)$ and $R(n,j) = o\left(\delta(n)^{rt}\right)$ $(i \geq 2)$ for some positive r and t, then there exists $N_1(r)$ s.t. for $n \geq N_1(r)$,

$$A_1(n)^r\Delta^r\alpha(n) > 0. \tag{1.8}$$

Moreover, if for some real numbers $c_1 > 0, c_2 > 1$ and $t_1 > t > 0$,

$$\begin{aligned} &|R(n,j)| < c_2\delta(n)^{\frac{t+t_1}{2}r}, \\ &c_1\delta(n)^t < |A_1(n)| < c_2\delta(n)^t, \quad |A_i(n)| \leq c_1c_2^{i-1}\delta(n)^{t+(i-1)t_1}, \quad i \geq 2, \end{aligned} \tag{1.9}$$

then (1.8) holds for $\delta(n) < cr^{-t_2}$, where c and t_2 are defined as in Theorem 1.2.

For the complete monotonicity of $\log\alpha(n)$, we also deduce the following result.

Theorem 1.7. *Let $\{\alpha(n)\}, \{\delta(n)\}, \{A_i(n)\}$ be sequences such that as $n \to \infty$, $\alpha(n)$ stays positive, $\delta(n) \to 0^+$, and*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = \sum_{i=1}^{N} A_i(n)j^i + R(n,j),$$

where $A_r(n) = O\left(\delta(n)^{rt}\right)$, $A_i(n) = o\left(\delta(n)^{rt}\right)$ $(i > r)$ and $R(n,j) = o\left(\delta(n)^{rt}\right)$ for some positive r and t, then there exists $N_2(r)$ s.t. for $n \geq N_2(r)$,

$$A_r(n)\Delta^r\log\alpha(n) > 0. \tag{1.10}$$

Moreover, if for some real numbers $c_1 > 0, c_2 > 1$ and $t_1 > t > 0$,

$$\begin{aligned} &|R(n,j)| < c_2\delta(n)^{\frac{t+t_1}{2}r}, \\ &c_1^r\delta(n)^t < |A_r(n)| < c_2^r\delta(n)^t, \quad |A_i(n)| \leq c_1^rc_2^{i-t}\delta(n)^{t+(i-r)t_1}, \quad i > r, \end{aligned} \tag{1.11}$$

then (1.10) holds for $\delta(n) < (3c_2)^{-1}r^{2/(t_1-t)}$.

Using these results, we can deduce the asymptotical positivity of $(-1)^r\Delta^r\gamma(n)$ and $(-1)^r\Delta^r\log\gamma(n)$ as follows.

Theorem 1.8. *For every r, there exists a positive c such that for $n > cr^3$,*

$$(-1)^r\Delta^r\gamma(n) > 0.$$

Theorem 1.9. *For every r, there exists a positive c such that for sufficiently large n,*

$$(-1)^r\Delta^r\log\gamma(n) > 0.$$

It encourages us to propose the following conjectures and problem.

Conjecture 1.10. For all positive integers r and n, the Maclaurin coefficients $\gamma(n)$ of the Riemann Xi-function satisfy

$$(-1)^r\Delta^r\gamma(n) > 0,$$

Conjecture 1.11. For all positive integers r and n, the Maclaurin coefficients $\gamma(n)$ of the Riemann Xi-function satisfy

$$(-1)^{r-1}\Delta^r\log(\gamma(n)) > 0.$$

Open Problem. For a function $\sum_{k=0}^{\infty}\gamma_k x^k/k!$ belonging to Laguerre-Pólya class with γ_n being positive and decreasing, whether $(-1)^r\Delta^r\gamma_n$ is positive for any r.

The remaining of this paper is organized as follows. In Section 2, we give some inequalities which will be used in the proof of our main theorems. In Section 3, we prove Theorem 1.1 and Theorem 1.2. In Section 4, we illustrate that the sequence $\gamma(n)$ satisfies the Laguerre inequalities of order r for $n > cr^3$ and some positive c. We also take a deep look into the partition function and show that $p(n)$ satisfies the laguerre inequality of order r for $n \geq 6r^4$. In Section 5, we focus on the complete monotonicity and prove Theorem 1.5, 1.6, 1.7, 1.8 and 1.9.

2. Preparation

First, let us give an inequality which is the key ingredient in the proof of the two main lemmas of this section.

Lemma 2.1. *For integers $x \geq 2$ and $t \geq x+1$, we have*

$$\left(1+\frac{1}{x^2}\right)^t < \left(1+\frac{1}{x}\right)^{2(t-x)}.$$

Proof. For $x+1 \leq t \leq \sqrt{2}x$ (which implies that $x \geq 3$), we have

$$\binom{t}{k} < x^k, \quad k \geq 2,$$

hence

$$\begin{aligned}\left(1+\frac{1}{x^2}\right)^t =&\sum_{k=0}^{t}\binom{t}{k}x^{-2k} < 1+tx^{-2}+\sum_{k=2}^{t}x^{-k} < 1+(t+2)x^{-2}\\ \leq&1+2(t-x)x^{-1} < \left(1+\frac{1}{x}\right)^{2(t-x)}.\end{aligned}$$

For $t \geq \sqrt{2}x$, since

$$1+\frac{1}{x} > \left(1+\frac{1}{x^2}\right)^{1+\frac{\sqrt{2}}{2}},$$

we have

$$\left(1+\frac{1}{x^2}\right)^{t} < \left(1+\frac{1}{x}\right)^{t(2-\sqrt{2})} < \left(1+\frac{1}{x}\right)^{2(t-x)}.$$

This completes the proof. $\square$

We proceed to prove the main lemmas.

Lemma 2.2. *For positive integers n and t, we have that*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^t = \begin{cases} 0, & t<n \\ (-1)^n n!, & t=n \end{cases}, \tag{2.1}$$

and for $t>n$,

$$\left|\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^t\right| \leq n^{2(t-n)}n!. \tag{2.2}$$

Proof. We first consider the case $t \leq n$. Note that by the inclusion-exclusion principle, we can combinatorially interpret

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^t$$

as the number of the ways of putting t distinct balls into n distinct boxes such that there is no empty box. Thus (2.1) is immediate.

We proceed to use induction on (n,t) to prove (2.2) for the general case. The case $n=1$ is trivial. Suppose the result holds when $n<m$ or $n<t<s$. For $(n,t)=(m,s)$,

$$\begin{aligned}
&\left|\sum_{k=0}^{m}(-1)^k\binom{m}{k}k^s\right| = \left|m\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}(k+1)^{s-1}\right| \\
\leq&\left|m\sum_{i=0}^{s-1}\left(\binom{s-1}{i}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}k^i\right)\right| \leq m\sum_{i=0}^{s-1}\binom{s-1}{i}(m-1)^{2(i-m+1)}(m-1)! \\
=&(m-1)^{2(s-m)}m!\left(1+\frac{1}{(m-1)^2}\right)^{s-1} \\
=&m^{2(s-m)}\left(\frac{m-1}{m}\right)^{2(s-m)}m!\left(1+\frac{1}{(m-1)^2}\right)^{s-1} \\
=&m^{2(s-m)}m!\left(1+\frac{1}{m-1}\right)^{-2(s-m)}\left(1+\frac{1}{(m-1)^2}\right)^{s-1} < m^{2(s-m)}m!.
\end{aligned}$$

The last inequality can be deduced from Lemma 2.1. Hence the proof. $\square$

Now we go ahead to prove Lemma 2.3.

Lemma 2.3. *For positive integers m and $t \leq 2m$,*

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}(m-k)^t = \begin{cases} 0, & t<2m \\ (2m)!, & t=2m \end{cases}.$$

and for $t > 2m$, we have that

$$\left|\sum_{k=0}^{2m}(-1)^{m+k}\binom{2m}{k}(m-k)^t\right| < e\left(4m^2+2m\right)^{t-2m}(2m)!.$$

Proof. For $t \le 2m$, from Lemma 2.2 we deduce that

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}(m-k)^t = \sum_{i=1}^{t}\binom{t}{i}\left(\sum_{k=0}^{2m}(-1)^{i+k}\binom{2m}{k}m^ik^{t-i}\right),$$

where

$$\sum_{k=0}^{2m}(-1)^{i+k}\binom{2m}{k}k^im^{t-i} = \begin{cases}0, & i<2m\\ (2m)!m^{t-2m}, & i=2m\end{cases}.$$

Since $i \le t \le 2m$, the only term which does not vanish is the term with $i = t = 2m$, which equals $(2m)!$.

For $t > 2m$, we deduce from Lemma 2.2 that

$$\begin{aligned}
&\left|\sum_{k=0}^{2m}(-1)^{i+k}\binom{2m}{k}(m-k)^t\right| < \sum_{i=1}^{t}\binom{t}{i}\left|\sum_{k=0}^{2m}(-1)^{m+k}\binom{2m}{k}k^im^{t-i}\right|\\
&<\sum_{i=1}^{t}\binom{t}{i}m^i(2m)^{2(t-i-2m)}(2m)! = (2m)^{2(t-2m)}(2m)!\sum_{i=1}^{t}\binom{t}{i}(2m)^{-i}\\
&<\left(4m^2\right)^{t-2m}(2m)!\left(1+\frac{1}{2m}\right)^t = \left(4m^2\left(1+\frac{1}{2m}\right)\right)^{t-2m}(2m)!\left(1+\frac{1}{2m}\right)^{2m}\\
&<e\left(4m^2+2m\right)^{t-2m}(2m)!,
\end{aligned}$$

as desired. □

3. Laguerre inequality

In this section, we follow the spirit given in our previous paper [36] to prove that for any r, the sequence $\{\alpha(n)\}$ satisfies the Laguerre inequality of order r for sufficiently large n. We also give a computable upper bound on the threshold for these Laguerre inequalities. The main idea is to show under the conditions in Theorem 1.1, $L_r(\alpha(n))$ can be viewed as a polynomial in n whose first few terms vanish.

Proof of Theorem 1.1. For $l \in [0,1]$, set

$$\tilde{\alpha}(n+jl) := \begin{cases}\alpha(n)\exp\left(\sum_{i=1}^{2N}A_i(n)(jl)^i\right) & , \quad jl \notin \mathbb{Z}\\ \alpha(n+jl) & , \quad jl\in\mathbb{Z}\end{cases}, \tag{3.1}$$

and

$$L_{r,l}(\tilde{\alpha}(n)) := \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\tilde{\alpha}(n+kl)\tilde{\alpha}(n+(2r-k)l).$$

Then by the conditions in Theorem 1.1, for all l,

$$\frac{\tilde{\alpha}(n+jl)}{\alpha(n)} = \exp\left(\sum_{i=1}^{2N}A_i(n)(jl)^i + o\left(\delta(n)^{tr}\right)\right)$$

as $n \to \infty$. Let

$$\phi_r(x) := \sum_{i=1}^{2r-1} \frac{x^i}{i!} = \exp(x) + O\left(x^{2r}\right). \tag{3.2}$$

Then for all $l \in [0,1]$ and $n \to \infty$, we get

$$\begin{aligned}
\frac{L_{r,l}(\tilde{\alpha}(n-rl))}{\alpha(n)^2} =& \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\exp\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i} + o\left(\delta(n)^{tr}\right)\right) \\
=& \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\exp\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i}\right)\exp\left(o\left(\delta(n)^{tr}\right)\right) \\
=& \left(\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i}\right) + o\left(\delta(n)^{tr}\right)\right)\exp\left(o\left(\delta(n)^{tr}\right)\right).
\end{aligned}$$

Since $A_{2i}(n) = O\left(\delta(n)^{it}\right)$, the terms in the left brackets are less than 1. It follows that

$$\frac{L_{r,l}(\tilde{\alpha}(n-rl))}{\alpha(n)^2} = \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i}\right) + o\left(\delta(n)^{tr}\right).$$

Now we focus on the expression

$$\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i}\right).$$

The ϕ_r-terms can be viewed as a polynomial in $(r-k)^2l^2$, i.e. there exist coefficients $A'_{2i}(0 \le i \le 2rN)$ such that

$$\phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i}\right) = \sum_{i=0}^{2rN} A'_{2i}(n)l^{2i}(r-k)^{2i}.$$

By Lemma 2.3, one can see that

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i}\right) \\
=& \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\sum_{i=0}^{2rN} A'_{2i}(n)l^{2i}(r-k)^{2i} \\
=& \frac{1}{2}\sum_{i=0}^{2rN} A'_{2i}(n)l^{2i}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}(r-k)^{2i} \\
=& \frac{1}{2}\sum_{i=r}^{2rN} A'_{2i}(n)l^{2i}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}(r-k)^{2i}.
\end{aligned}$$

(The sum of terms with $i < r$ vanishes.) It implies that as $l \to 0$, we have

$$\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i}\right) = O\left(l^{2r}\right).$$

Note that once a term takes l^{2i}, the corresponding coefficient

$$A_{2i}(n) = \begin{cases} O\left(\delta(n)^t\right) & , \quad i=1 \\ o\left(\delta(n)^{it}\right) & , \quad i \geq 2 \end{cases}$$

as $n \to \infty$. It follows that

$$\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}l^{2i}\right) = O\left(\delta(n)^{tr}\right),$$

and only terms with $A_2(n)$ contribute to the main term. We have

$$\begin{aligned}\frac{L_{r,l}(\tilde{\alpha}(n-rl))}{\alpha(n)^2} =& \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(A_2(n)2(r-k)^2l^2\right) + o\left(\delta(n)^{tr}\right) \\ =& \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\frac{1}{r!}\left(2A_2(n)(r-k)^2l^2\right)^r + o\left(\delta(n)^{tr}\right) \\ =& \frac{(2r)!}{r!}2^{r-1}(-A_2(n))^r l^{2r} + o\left(\delta(n)^{tr}\right).\end{aligned} \tag{3.3}$$

The last equality can be derived from Lemma 2.3 which states that

$$\sum_{k=0}^{2r}(-1)^k\binom{2r}{k}(r-k)^{2r} = (2r)!.$$

Set $l=1$ in the above equality, it leads to (1.4). This completes the proof. □

Up to now, we have solved the case for n large enough. Now we admit the additional assumption (1.6) on $A_k(n)$ and $R(n,j)$ in Theorem 1.2 and proceed to find a computable lower bound of n for the positivity of $L_r(\alpha(n))$.

Proof of Theorem 1.2. Let

$$\mathcal{R} := \left|\frac{L_r(\alpha(n-r))}{\alpha(n)^2} - \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}\right)\right|,$$

and for $0 \leq k \leq 2r$, define

$$\mathcal{R}_k := \left|\exp\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i} + R(n,j) + R(n,-j)\right) - \phi_r\left(\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}\right)\right|.$$

Then

$$\mathcal{R} \leq \frac{1}{2}\sum_{k=0}^{2r}\binom{2r}{k}\mathcal{R}_k.$$

We observe that for $\delta(n) < c_2^{-1}(4r)^{-1/t} < cr^{-t_2}$, there are

$$|A_{2i}(n)| \leq c_1 c_2^{i-1}\delta(n)^t\delta(n)^{(i-1)t_1} < (4r)^{-(i-1)}\delta(n)^t,$$

and

$$|R(n,j) + R(n,-j)| < 2c_2\delta(n)^{\frac{t+t_1}{2}r} < 2(4r)^{-r+1}\delta(n)^t.$$

Hence we have

$$\left|\sum_{i=1}^{N} A_{2i}(n)2(r-k)^{2i}\right| + |R(n,j)+R(n,-j)| < 4r^2\delta(n)^t\left(\sum_{i=1}^{N}\frac{2r^{2i}}{(2r)^{2i}} + 2(4r)^{-r}\right) < 4r^2\delta(n)^t < 1.$$

For $|x|+|r|<1$, we have that

$$|\exp(x+r)-\exp(x)|<\exp(|x|+|r|)r<er,$$

and

$$|\phi(x)-e(x)|=\sum_{i=2r}^{\infty}\frac{x^i}{i!}<\frac{x^{2r}}{(2r)!}\left(1+\frac{1}{2r+1}+\frac{1}{(2r+1)^2}+\dots\right)=\frac{x^{2r}}{(2r)!}\left(1+\frac{1}{2r}\right).$$

It follows that

$$\begin{aligned}\mathcal{R}_k\le&\left|\exp\left(\sum_{i=1}^{N}A_{2i}(n)2(r-k)^{2i}+R(n,j)+R(n,-j)\right)-\exp\left(\sum_{i=1}^{N}A_{2i}(n)2(r-k)^{2i}\right)\right|\\&+\left|\phi_r\left(\sum_{i=1}^{N}A_{2i}(n)2(r-k)^{2i}\right)-\exp\left(\sum_{i=1}^{N}A_{2i}(n)2(r-k)^{2i}\right)\right|\\<&e(R(n,j)+R(n,-j))+\frac{(4r^2\delta(n)^t)^{2r}}{(2r)!}\left(1+\frac{1}{2r}\right)\\<&2ec_2\delta(n)^{\frac{t+t_1}{2}r}+2^{2r}r^{3r}\delta(n)^{2tr}.\end{aligned}$$

Therefore,

$$\mathcal{R}\le\frac{1}{2}\sum_{k=0}^{2r}\binom{2r}{k}\mathcal{R}_k<2^{2r-1}\left(2ec_2\delta(n)^{\frac{t+t_1}{2}r}+2^{2r}r^{3r}\delta(n)^{2tr}\right).$$

Since $A_2(n)<-c_1\delta(n)^t$, we have

$$\left|\frac{4\mathcal{R}}{A_2(n)^r}\right|<\frac{2^{2r+2}ec_2}{c_1^r}\delta(n)^{\frac{t_1-t}{2}r}+\frac{2^{4r+1}r^{3r}}{c_1^r}\delta(n)^{tr}. \tag{3.4}$$

Recall that

$$t_2=\max\left\{\frac{3}{t},\frac{2}{t_1-t}\right\},\quad c=(48c_2\max\{1,c_1^{-1}\})^{-t_2}.$$

It can be checked that (3.4) is less than 1 for $\delta(n)<cr^{-t_2}$.

Now denote

$$C_{2i}:=A_{2i}(n)\delta(n)^{-t-(i-1)t_1},$$

then

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N}A_{2i}(n)2(r-k)^{2i}l^{2i}\right)\\=&\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N}C_{2i}\delta(n)^{t+(i-1)t_1}(n)2(r-k)^{2i}l^{2i}\right)\end{aligned}$$

can be viewed as a function on $\delta(n)$ which (in the proof of Theorem 1.1) we actually proved to be of the form

$$f(\delta(n))=\sum_{\substack{u,v\in\mathbb{Z}_+,\ 1\le u\le 2r\\ r\le u+v\le 2Nr}}c_{u,v}\delta(n)^{ut+vt_1}.$$

First, we consider $c_{r,0}$ for $l=1$. By (3.3) and condition (1.6), there is

$$c_{r,0}=\frac{(2r)!}{r!}2^{r-1}(-A_2(n))^r>\frac{(2r)!}{r!}2^{r-1}c_1^r. \tag{3.5}$$

Now we are in a position to estimate $c_{u,v}$ (where $(u,v) \neq (r,0)$). If there exist different pairs (u_1,v_1) and (u_2,v_2) such that $u_1t+v_1t_1 = u_2t+v_2t_1 = w$, we consider

$$(c_{u_1,v_1}+c_{u_2,v_2})\delta(n)^w = c_{u_1,v_1}\delta(n)^{u_1t+v_1t_1}+c_{u_2,v_2}\delta(n)^{u_2t+v_2t_1}$$

as the sum of two different terms. Note that

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\phi_r\left(\sum_{i=1}^{N}C_{2i}\delta(n)^{t+(i-1)t_1}(n)2(r-k)^{2i}l^{2i}\right)\\ =&\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\sum_{s=1}^{r}\frac{1}{s!}\left(\sum_{i=1}^{N}C_{2i}\delta(n)^{t+(i-1)t_1}(n)2(r-k)^{2i}l^{2i}\right)^s,\end{aligned} \tag{3.6}$$

which has been proved to be of $O\left(l^{r(r-1)}\right)$ when $l\to 0$ in Theorem 1.1. It implies that the terms corresponding with $\delta(n)^{ut+vt_1}$ only appears in the right-hand side of (3.6) for $s=u$ and $i=i_1,i_2,\ldots,i_s$ respectively in s brackets with

$$i_1+i_2+\cdots+i_s = u+v.$$

After expanding the right-hand side of (3.6) thoroughly for s and i, there are $\binom{u+v-1}{v}$ terms corresponding with $\delta(n)^{ut+vt_1}$. And by Lemma 2.3, when $l=1$, each term has an absolute value less than

$$\begin{aligned}&\left|(u!)^{-1}\frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}2^u(r-k)^{2u+2v}\prod_{h=1}^{u}C_{2i_h}\right|\\ <&\frac{(2r)!}{u!}2^{u-1}e\left(4r^2+2r\right)^{2(u+v-r)}c_1^uc_2^v.\end{aligned}$$

Thus we have

$$|c_{u,v}| < \binom{u+v-1}{v}\frac{(2r)!}{u!}2^{u-1}e\left(4r^2+2r\right)^{2(u+v-r)}c_1^uc_2^v. \tag{3.7}$$

Combining (3.5) and (3.7) gives

$$\begin{aligned}\frac{|c_{u,v}|}{c_{r,0}} =&\binom{u+v-1}{v}\frac{r!}{u!}2^{u-r}e\left(4r^2+2r\right)^{2(u+v-r)}c_2^vc_1^{u-r}\\ <&(u-1)^v r^{r-u}2^{u-r}e2^{2(u+v-r)}r^{2(u+v-r)}(2r+1)^{2(u+v-r)}c_2^vc_1^{u-r}\\ <&r^{v+(r-u)+2(u+v-r)}2^{u-r+2(u+v-r)}e(2r+1)^{2(u+v-r)}c_2^vc_1^{u-r}\\ <&2^{3u+2v-3r}e(2r+1)^{2u+2v-2r}r^{u+3v-r}c_2^vc_1^{u-r}.\end{aligned}$$

It can be checked that for $\delta(n) < cr^{-t_2}$,

$$\sum_{r-v\le u\le r}\frac{|c_{u,v}|}{c_{r,0}}\delta(n)^{(u-r)t+vt_1} < 3^{-v}. \tag{3.8}$$

On the other hand, from (3.4) we deduce that for $\delta(n) < cr^{-t_2}$,

$$\mathcal{R} < \frac{1}{4}c_{r,0}\delta(n)^d. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$
\begin{aligned}
f(\delta(n))-\mathcal{R} &= \sum_{u,v} c_{u,v}\delta(n)^{ut+vt_1}-\mathcal{R}\\
&= \sum_{v=0}^{2(N-1)r}\sum_{r-v\le u\le r} c_{u,v}\delta(n)^{ut+vt_1}-\mathcal{R}\\
&> c_{r,0}\delta(n)^{tr}-R-\sum_{v=1}^{2(N-1)r}\sum_{r-v\le u\le r}|c_{u,v}|\delta(n)^{rt+vt_2}\\
&> c_{r,0}\delta(n)^{d}\left(1-4^{-1}-\sum_{v=1}^{(N-1)r}3^{-v}\right)>0.
\end{aligned}
\tag{3.10}
$$

It means that $L_r(\alpha(n-r))>0$ for $\delta(n)<cr^{-t_2}$, as desired. $\square$

4. Applications

In this section we apply our main theorems to prove the Maclaurin coefficients $\gamma(n)$ of the Riemann Xi-function and the partition function both asymptotically satisfy the Laguerre inequalities of any order. Following the approach given by Griffin, Ono, Rolen, Thorner, Tripp and Wagner [12], we can write $\gamma(n)$ in form (1.5). For the partition function, we have mentioned in [36] that it can also be written in the same form. We give brief retrospects in the following proofs.

Proof of Theorem 1.3. Denote

$$
f(z):=\int_1^{\infty}(\log t)^z t^{-\frac{3}{4}}\left(\sum_{k=1}^{\infty}e^{-\pi k^2 t}\right)\mathrm{d}t,
$$

then

$$
\gamma(M)=\frac{\Gamma(M+1)}{\Gamma(2M+1)}\frac{32\binom{2M}{2}f(2M-2)-f(2M)}{2^{2M-1}}.
$$

Consider the expansion

$$
\log\left(\frac{\gamma(M+j)}{\gamma(M)}\right)=\sum_{i=1}^{\infty}G_i(M)j^i,
$$

then the regularized function

$$
\begin{aligned}
&(M-2)^{-1}\log\left(\left(\frac{eL_{2M-2}^2M}{4(2M-2)^2}\right)^{\lambda(M-2)}\frac{\gamma(M+\lambda(M-2))}{\gamma(M)}\right)+(1+\lambda)\log(1+\lambda)\\
=&\left(G_1(M)+\log\left(\frac{eL_{2M-2}^2M}{4(2M-2)^2}\right)-1\right)\lambda+\sum_{k=2}^{\infty}\left(G_k(M)(M-2)^{k-1}+(-1)^k\frac{2}{k(k-1)}\right)\lambda^m
\end{aligned}
$$

converges to zero absolutely and uniformly as $M\to\infty$. Hence for any positive c, there exists m_c such that for $M>m_c$,

$$
\left|G_k(M)(M-2)^{k-1}-\frac{(-1)^{k+1}}{k(k-1)}\right|<c. \tag{4.1}
$$

Set $c = 4^{-1}$, the above inequality gives that

$$
\begin{aligned}
G_2(M)(M-2) &< -\frac{1}{4}, \\
\left|G_k(M)(M-2)^{k-1}\right| &< \frac{2}{k(k-1)} + \frac{1}{4}.
\end{aligned}
\tag{4.2}
$$

Furthermore, for $M > \max\{m_{4^{-1}}, r^3\}$,

$$
\begin{aligned}
|R(M,j)| &= \left|\log\left(\frac{\gamma(M+j)}{\gamma(M)}\right) - \sum_{i=1}^{3r} G_i(M) j^i\right| = \left|\sum_{i=3r+1}^{\infty} G_i(M) j^i\right| < \sum_{i=3r+1}^{\infty} |G_i(M)| r^i \\
&< \sum_{i=3r+1}^{\infty} |G_i(M)|^{2/3} < \sum_{i=3r+1}^{\infty} (M-2)^{-2/3i} < (M-2)^{-2r}.
\end{aligned}
$$

Setting $\delta(n) = \frac{1}{n-2}, t = 1, t_1 = 2$, we find out that $\gamma(n)$ satisfies the conditions in Theorem 1.2. Hence we conclude that for any r, the Laguerre inequality of order r holds when $n > cr^3$ for some positive c. $\square$

Now we confine our attention to the partition function $p(n)$. To get a better result, we give more precise estimations when determining the constants.

Theorem 4.1 (Wang and Yang)**.** *Let $\delta(n) = 6(24n+1)^{-1/2}/\pi$, then for $r \geq 5$ and $\delta(n)^{-1} \geq 8e(e-1)^{-1} r \log(8r)$,*

$$
\log\left(\frac{p(n+j)}{p(n)}\right) = \sum_{k=1}^{4r} \Omega_k(n) j^k + R(n,j),
$$

where

$$
\begin{aligned}
\Omega_2(n) &< -5\delta(n)^3, \\
|\Omega_k(n)| &< 2^{2k-3} 3^{-k} 5^k \delta(n)^{2k-1}, \quad k \geq 3, \\
|R(n,j)| &< \delta(n)^{7r/2}.
\end{aligned}
$$

Sketch of proof. Following the method given in our previous paper [35], one may find that for $\mu(n) > 8e(e-1)^{-1} r \log(8r)$, there is

$$
\frac{\sqrt{12}\pi^2 e^{\mu(n)}}{36\mu(n)^2}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^{4r}}\right) < p(n) < \frac{\sqrt{12}\pi^2 e^{\mu(n)}}{36\mu(n)^2}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^{4r}}\right),
\tag{4.3}
$$

where $\mu(n) := \pi\sqrt{24n-1}/6$. Now

$$
\mu(n+j) - \mu(n) = \sum_{k=1}^{4r} \omega_k(n) j^k + R_1(n,j),
$$

where

$$
\begin{aligned}
\omega_1(n) &= \frac{\pi^2}{3\mu(n)}, \quad \omega_2(n) = \frac{-\pi^4}{18\mu(n)^3}, \\
|\omega_k(n)| &< \frac{2^{k-3}\pi^{2k}}{3^k \mu(n)^{2k-1}}, \quad k \geq 3,
\end{aligned}
$$

and $|R_1(n,j)| < \mu(n)^{-4r}$. Similarly,

$$
\log\left(\frac{\mu(n+j)}{\mu(n)}\right) = \sum_{k=1}^{4r} \omega'_k(n) j^k + R_2(n,j),
$$

with $|\omega_k'(n)| < \mu(n)^{-2k}$ and $|R_2(n,j)| < \mu(n)^{-4r}$.

For $-r+1 \le j \le r-1$, we have that

$$\log\left(\frac{p(n+j)}{p(n)}\right) = (\mu(n+j)-\mu(n))+3\log\left(\frac{\mu(n)}{\mu(n+j)}\right)+\log\left(\frac{\mu(n+j)+1}{\mu(n)+1}\right)+R_3(n,j),$$

where $|R_3(n,j)| < \mu(n)^{-4r}$. Set $\delta(n) = \mu(n)^{-1}$. Expanding the right-hand side of the above equation leads to

$$\log\left(\frac{p(n+j)}{p(n)}\right) = \sum_{k=1}^{4r} \Omega_k(n) j^k + R(n,j),$$

where

$$\Omega_2(n) < \omega_2(n) - 4|\omega_2'(n)| < -5\delta(n)^3,$$

$$|\Omega_k(n)| < |\omega_k(n)| + 4|\omega_k'(n)| < 2^{2k-3}3^{-k}5^k\delta(n)^{2k-1}, \quad k \ge 3,$$

and

$$|R(n,j)| < \delta(n)^{7r/2}.$$

Hence the proof. □

Now we are ready to prove the Laguerre inequality for the partition function.

Proof of Theorem 1.4. Set $t = 3, t_1 = 4$. Then it follows from Theorem 1.2 and Theorem 4.1 that there exists some constant C such that for any $r \ge 5$ and $\delta(n)^{-1} \ge \max\{Cr^2, 8e(e-1)^{-1}r\log(8r)\}$, we have

$$L_r(p(n)) = \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}p(n+k)p(n+2r-k) > 0.$$

Now we proceed to determine the constant C. In (3.4) we have that for $\delta(n)^{-1} > 6r^2$,

$$\left|\frac{256R}{A_2(n)^r}\right| < \frac{2^{2r+8}5e}{c_1^r}\delta(n)^{\frac{t_1-t}{2}r} + \frac{2^{4r+7}r^{3r}}{c_1^r}\delta(n)^{tr} < 1.$$

While in (3.8),

$$\begin{aligned}\frac{|c_{u,v}|}{c_{r,0}} &< \binom{u+v-1}{v}\frac{r!}{u!}2^{u-r}e\left(4r^2+2r\right)^{2(u+v-r)}\frac{2^{4v-3}3^{-v}5^{u+v}}{5^r}\\ &< r^{u+3v-r}(2r+1)^{2u+2v-2r}2^{3u+6v-3r}3^{-v}5^{u+v-r},\end{aligned}$$

which means that for $\delta(n)^{-1} \ge 6r^2$, we get

$$\sum_{r-v\le u\le r}\frac{|c_{u,v}|}{c_{r,0}}\delta(n)^{3u+4v-3r} < \sum_{r-v\le u\le r}\left(\frac{9}{4}\right)^{-v}\left(\frac{27r^5}{5(2r+1)^2}\right)^{r-u-v} < 1.2\times 2.25^{-v}.$$

Mirroring the proof of (3.10), we conclude that for $r \ge 5$ and $n \ge 6r^4 > 54\pi^{-2}r^4$ (which implies that $\delta(n)^{-1} \ge \max\{6r^2, 8e(e-1)^{-1}r\log(8r)\} = 6r^2$), the Laguerre inequality of order r holds for $p(n)$. □

5. Asymptotically complete monotonicity

In this section we consider the complete monotonicity, and we first prove the general theorem.

Proof of Theorem 1.5. Recall that with the conditions of the theorem,

$$\psi(x) = cx^m e^{\beta x} \prod_{k=1}^{\infty}(1 + x/x_n) = \sum_{k=1}^{\infty} \gamma_k \frac{x^k}{k!}, \tag{5.1}$$

where $\beta < 1$ and $\gamma_k > 0$. We aim to show that $(-1)^r \Delta^r \gamma_k > 0$ for any r and large k.

When $r = 0$, $\Delta^0 \gamma_k = \gamma_k > 0$ by condition. Next we handle the case when $r \geq 1$. Suppose $\{x_n\}$ is in incremental arrangement. Consider the function

$$e^{-x}\psi^{(k)}(x) = \sum_{r=0}^{\infty} \frac{g^*_{r,k}}{r!} x^r,$$

where

$$g^*_{r,k} = \sum_{i=1}^{r} \binom{r}{i} \gamma_{k+i}(-1)^{r-i} = \Delta^r \gamma_k.$$

It suffices to show that $(-1)^r g^*_{r,k} > 0$ for sufficiently large k. One can refer to [4] and [15] to see that the k-th derivative of ψ is

$$\psi^{(k)}(x) = \gamma_k x^{\max\{0,m-k\}} e^{\beta x} \prod_{n=1}^{\infty}(1 + x/x_{n,k}),$$

which belongs to $\mathcal{LP}$I as well. Note that $x_{n,0} = x_n$ for all n. Without loss of generality, suppose that $m = 0$, then we have

$$\psi^{(k+1)}(x) = \psi^{(k)}(x) \left(\beta + \sum_{n=1}^{\infty} (x + x_{n,k})^{-1} \right).$$

If $x_{n,k} < x_{n+1,k}$, from the monotonicity of $\beta + \sum(x + x_{n,k})^{-1}$ in $[-x_{n+1,k}, -x_{n,k}]$ we know that $\psi^{(k+1)}$ has exactly one root in $(-x_{n+1,k}, -x_{n,k})$. If $x_{n,k} = x_{n,k+1}$, then it is also a root of $\psi^{(k+1)}$. Thus we get

$$x_{n,k} \leq x_{n,k+1} \leq x_{n+1,k}. \tag{5.2}$$

Note that $\sum x_n^{-1} < \infty$. We claim that there are infinitely many integer N's satisfying

$$\sharp \left\{ n : x_n \leq \frac{6r}{1-\beta} N \right\} \leq N. \tag{5.3}$$

Otherwise, choose the largest odd N that does not satisfy (5.3). Then for any $t \geq 0$, there is

$$\sharp \left\{ n : x_n \leq \frac{6r}{1-\beta} 2^t(N+1) \right\} > 2^t(N+1).$$

Denote

$$A_t = \left\{ \text{the smallest } 2^{t-1}(N+1)\ n\text{'s} : \forall s < t, n \notin A_s \ \&\ x_n \leq \frac{6r}{1-\beta} 2^t(N+1) \right\}.$$

Since

$$\sum_{s<t} \sharp A_s = \sum_{s<t-1} 2^s(N+1) = \left(2^{t-1} - \frac{1}{2}\right)(N+1) < 2^t(N+1) - 2^{t-1}(N+1),$$

The construction of A_t is well-defined. Now we have that

$$\sum_{n=1}^{\infty} x_n^{-1} > \sum_{t=0}^{\infty}\sum_{n\in A_t} x_n^{-1} > \sum_{t=0}^{\infty}\sum_{n\in A_t} \frac{1-\beta}{6r}2^{-t}(N+1)^{-1} > \sum_{t=0} \frac{1-\beta}{12r} = \infty,$$

a contradiction. Therefore, there are infinitely many integer N's satisfying (5.3). On the other hand, since $x_n > 0$ and that $\sum_{n=1}^{\infty} x_n^{-1} < \infty$, there exists M such that for any $m > M$,

$$\sum_{n=m}^{\infty} x_n^{-1} < \frac{1-\beta}{6r}.$$

It implies that there exists N_0 satisfying condition (5.3) and that

$$\sum_{n=N_0+1}^{\infty} x_n^{-1} < (1-\beta)/(6r),$$

which also implies that

$$x_n > \frac{6r}{1-\beta}, \quad n > N_0.$$

Next we (backward) inductively construct $\{k_i\}_{1\le i\le N_0+1}$ which satisfy

$$x_{i,k_i} > \frac{5r}{1-\beta}N_0 - 2(N_0 - i + 1). \tag{5.4}$$

Set $k_{N_0+1} := 0$. Suppose k_{i+1} has been constructed. Arbitrarily choose $j \in \mathbb{N}$. If $k_i = k_{i+1} + j$ does not satisfy the condition (5.4), then $x_{i,k_{i+1}+j} < x_{i+1,k_{i+1}+j} - 2$. When

$$x \in (-x_{i,k_{i+1}+j} - N_0^{-1}, -x_{i,k_{i+1}+j}),$$

one has

$$\begin{aligned}
&\beta + \sum_{n=1}^{\infty}(x + x_{n,k_{i+1}+j})^{-1} \\
=&\beta + \sum_{n=1}^{i-1}(x + x_{n,k_{i+1}+j})^{-1} + (x + x_{i,k_{i+1}+j})^{-1} \\
&+ \sum_{n=i+1}^{N_0}(x + x_{n,k_{i+1}+j})^{-1} + \sum_{n=N_0+1}^{\infty}(x + x_{n,k_{i+1}+j})^{-1} \\
<&\beta + N_0 + \sum_{n=i+1}^{N_0}(2 - N_0^{-1})^{-1} + \sum_{n=N_0+1}^{\infty}(x + x_{n,k_{i+1}+j})^{-1} \\
<&\beta - N_0 + (N_0 - i) + \sum_{n=N_0+1}^{\infty}\left(-\frac{5r}{1-\beta}N_0 + x_n\right)^{-1} \\
<&\beta - N_0 + (N_0 - i) + \sum_{n=N_0+1}^{\infty}\left(-\frac{5}{6}x_n + x_n\right)^{-1} \\
=&-i + \beta + 6\sum_{n=N_0+1}^{\infty} x_n^{-1} < -1 + \beta + \frac{1-\beta}{r} \le 0,
\end{aligned}$$

and hence we get that $x_{i,k_{i+1}+j+1} > x_{i,k_{i+1}+j} + N_0^{-1}$. Denote

$$k_i := k_{i+1} + N_0 \left(\frac{5r}{1-\beta} N_0 - 2(N_0 - i + 1) - x_{i,k_{i+1}} \right),$$

then it follows that

$$\begin{aligned} x_{i,k_i} >& x_{i,k_{i+1}} + N_0^{-1} N_0 \left(\frac{5r}{1-\beta} N_0 - 2(N_0 - i + 1) - x_{i,k_{i+1}} \right) \\ =& \frac{5r}{1-\beta} N_0 - 2(N_0 - i + 1), \end{aligned}$$

which satisfies the condition (5.4). Hence for $k \geq k_1$ and all n, from the monotonicity of $x_{n,k}$ in both n and k ((5.5)) we have

$$x_{n,k} \geq x_{1,k_1} > \frac{5r}{1-\beta} N_0 - 2N_0 \geq \frac{3r}{1-\beta} N_0 \geq \frac{3r}{1-\beta}.$$

Therefore,

$$\sum_{n=1}^{\infty} x_{n,k}^{-1} = \left(\sum_{n=1}^{N_0} + \sum_{n=N_0+1}^{\infty} \right) x_{n,k}^{-1} < \frac{1-\beta}{3r} + \frac{1-\beta}{6r} = \frac{1-\beta}{2r},$$

Since

$$\sum_{r=0}^{\infty} \frac{g_{r,k}^*}{r!} x^r = e^{-x} \psi^{(k)}(x) = \gamma_k e^{(\beta-1)x} \prod_{k=1}^{\infty} (1 + x/x_{n,k}),$$

comparing the coefficients of x^r, it gives

$$g_{r,k}^* = \gamma_k \sum_{i=1}^{r} \frac{(\beta-1)^{r-i}}{(r-i)!} \left(\sum_{n=1}^{\infty} x_{n,k}^{-1} \right)^i = \gamma_k (\beta-1)^r \sum_{i=1}^{r} \frac{1}{(2r)^i (r-i)!},$$

which has the same sign with $(\beta-1)^r$. Hence, $\mathrm{sgn}(g_{r,k}^*) = (-1)^r$ and the proof is completed. □

The proofs of Theorem 1.6 and 1.7 are similar to those of Theorem 1.1 and 1.2. Thus we just sketch the proof and omit some details.

Proof of Theorem 1.6. First we prove for the case without additional assumptions (1.9). Denote $\tilde{\alpha}(n+jl)$ and $\phi_r(x)$ as in (3.1) and (3.2), and define

$$\Delta_l^r \tilde{\alpha}(n) := \sum_{k=0}^{n} \binom{n}{k} (-1)^{r+k} \tilde{\alpha}(n+kl).$$

As $n \to \infty$, we have

$$\begin{aligned} \frac{\Delta_l^r \tilde{\alpha}(n)}{\alpha(n)} =& \sum_{k=1}^{n} \binom{n}{k} (-1)^{r+k} \exp \left(\sum_{i=1}^{N} A_i(n) k^i l^i + o\left(\delta(n)^r\right) \right) \\ =& \sum_{k=0}^{n} \binom{n}{k} (-1)^{r+k} \phi_r \left(\sum_{i=1}^{N} A_i(n) k^i l^i \right) + o\left(\delta(n)^r\right). \end{aligned}$$

By Lemma 2.2, the terms with $i < r$ vanish after summation through k. As $l \to 0$ we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{r+k} \phi_r \left(\sum_{i=1}^{N} A_i(n) k^i l^i \right) = O\left(l^r\right),$$

Note that once a term takes l^i, the corresponding coefficients

$$A_i(n) = \begin{cases} O(\delta(n)) & , \quad i = 1 \\ o\left(\delta(n)^i\right) & , \quad i \geq 2 \end{cases}$$

as $n \to \infty$. It follows that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{r+k} \phi_r \left(\sum_{i=1}^{N} A_i(n) k^i l^i \right) = O\left(\delta(n)^r\right), \tag{5.5}$$

and only terms with $A_1(n)$ contributes to the main term. Setting $l = 1$ we deduce that

$$\Delta^r \alpha(n) = A_1(n)^r + o\left(\delta(n)^r\right),$$

and has the same sign with $A_1(n)^r$.

Now if (1.9) is satisfied, substituting $A_i(n)$ with $C_i(n)\delta(n)^{t+(i-1)t_1}$ in the left of (5.5) we get a polynomial

$$f(\delta(n)) = \sum_{\substack{u,v \in \mathbb{Z}_+,\ 1 \leq u \leq r \\ r \leq u+v \leq Nr}} c_{u,v} \delta(n)^{ut+vt_1},$$

where

$$c_{r,0} = |c_2^r| > |c_1^r|, \quad |c_{u,v}| < \binom{u+v-1}{v} \frac{r!}{u!} r^{2(u+v-r)} c_1^u c_2^v.$$

For $\delta(n) < cr^{-t_2}$, we also have

$$\sum_{r-v \leq u \leq r} \frac{|c_{u,v}|}{c_{r,0}} \delta(n)^{(u-r)t+vt_1} < 3^{-v},$$

$$|\Delta_l^r \alpha(n) - f(\delta(n))| < \frac{1}{4} c_{r,0} \delta(n)^d.$$

By the same approach as in (3.10) we deduce that

$$|f(\delta(n))| > \left| \frac{1}{4} c_{r,0} \delta(n)^d \right|.$$

Hence, we have

$$\operatorname{sgn}\left(\Delta_l^r \alpha(n)\right) = \operatorname{sgn}(f(\delta(n))) = \operatorname{sgn}(c_1^r) = \operatorname{sgn}(A_1(n)^r),$$

as desired. □

Now we proceed to prove Theorem 1.7.

Proof of Theorem 1.7. By Lemma 2.2, as $n \to \infty$ we get that

$$\begin{aligned}
\Delta^r \log(\alpha(n)) = &\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) \\
= &\sum_{i=1}^{N} (-1)^{r-j} A_i(n) \sum_{j=1}^{r} \binom{r}{j} j^i + o(A_r(n)) \\
= &A_r(n) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^r + \sum_{i=1}^{r-1} A_i(n) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^i + o(A_r(n)) \\
= &A_r(n) r! + o(A_r(n)).
\end{aligned}$$

Thus, for large n, we have

$$A_r(n)\Delta^r \log(\alpha(n)) = A_r^2(n)r! + o\left(\delta(n)^{2r}\right) > 0.$$

For the additional assumption (1.11), we have

$$f(\delta(n)) = \sum_{1\le v\le n} c_{u,v}\delta(n)^{rt+vt_1},$$

where

$$c_{r,0} > r!c_1^r, \quad |c_{u,v}| < r!r^{2v}c_1^r c_2^v.$$

It is easily seen that $A_r(n)\Delta^r \log(\alpha(n)) = A_r^2(n)r! + o\left(\delta(n)^{2r}\right) > 0$ for $\delta(n) < (3c_2)^{-1}r^{2/(t-t_1)}$. □

We conclude this paper with the proofs of Theorem 1.8 and Theorem 1.9.

Proof of Theorem 1.8. For the Maclaurin coefficients $\gamma(n)$ of the Riemann Xi-function, Griffin, Ono, Rolen and Zagier [13] have made an estimation which illustrated that $G_1(M)$ satisfies the condition in Theorem 1.6 (in their paper the notion $A(n)$ is equal to $-G_1(n)$). For the other terms, (4.2) and Theorem 1.6 give that

$$(-1)^r\Delta^r\gamma(n) > 0, \quad n > cr^3$$

for some constant c. □

Proof of Theorem 1.9. Setting $c = r^{-2}$ in (4.1) gives

$$\frac{1}{r^2(r-1)} \le (-1)^{r-1}G_r(M)(M-2)^{r-1} < \frac{2}{r(r-1)},$$

which implies that

$$(-1)^{r-1}\Delta^r \log(\gamma(n)) > 0$$

for sufficiently large n. □

References

1. G. Almkvist, *Exact asymptotic formulas for the coefficients of nonmodular functions*, J. Number Theory, **38** (1991), no. 2, 145–160.
2. G. Almkvist, *On the differences of the partition function*, Acta Arith., **61** (1992), no. 2, 173–181.
3. W.Y.C. Chen, L.X.W. Wang and G.Y.B. Xie, *Finite differences of the logarithm of the partition function*, Math. Comp., **85** (298) (2016), 825–847.
4. T. Craven and G. Csordas, *The Gauss-Lucas theorem and jensen polynomoials*, Trans. Amer. Math. Soc., **278(1)** (1983), 415–429.
5. G. Csordas, T.S. Norfolk and R.S. Varga, *The Riemann hypothesis and the Turán inequalities*, Trans. Amer. Math. Soc. **296 (2)** (1986), 521–541.
6. G. Csordas and R.S. Varga, *Moment inequalities and the Riemann hypothesis*, Constr. Approx., **4** (1988), 175–198.
7. G. Csordas and R.S. Varga, *Necessary and sufficient conditions and the Riemann hypothesis*, Adv. Appl. Math., **11 (3)** (1990), 328–357.
8. G. Csordas and A. Vishnyakova. *The generalized Laguerre inequalities and functions in the Laguerre-Pólya class*, Cent. Eur. J. Math., **11** (2013), no. 9, 1643–1650.
9. L-M. Dou and L.X.W. Wang, *Higher order Laguerre inequalities for the partition function*, submitted.
10. G. Gasper, *Positivity and special functions, Proceedings of an Advanced Seminar Sponsored by the Mathematics Research Center*, the University of Wisconsin-Madison, March 31-April **2**, (1975), 375–433.
11. I. J. Good, *The difference of the partition function*, Problem 6137, Amer. Math. Monthly *84* (1997), 141.
12. M. Griffin, K. Ono, L. Rolen, J. Thorner, Z. Tripp and I. Wagner, *Jensen polynomials for the Riemann xi function*, Adv. Math., **397** (2022), 108186.

13. M. Griffin, K. Ono, L. Rolen and D. Zagier, *Jensen polynomials for the Riemann zeta function and other sequences*, Proc. Nat. Acad. Sci. USA, **116 (23)** (2019), 11103–11110.
14. H. Gupta, *Finite differences of the partition function*, Math. Comp. **32** (1978), no. 144, 1241–C1243.
15. S. Hellerstein and J. Williamson, *Successive derivatives of entire functions*, Proc. Amer. Math. Soc., **66(1)** (1977), 105-108
16. J.L.W.V. Jensen, *Recherches sur la théorie des équations*, Acta Math., **36(1)** (1913), 181–195.
17. S. Karlin, *Total Positivity.* Stanford University Press, Stanford (1968).
18. C. Knessl and J.B. Keller, *Partition asymptotics from recursion equations*, SIAM J. Appl. Math., **50** (1990), no. 2, 323–338.
19. C. Knessl and J.B. Keller, *Asymptotic behavior of high-order differences of the partition function*, Comm. Pure Appl. Math., **44** (1991), no. 8-9, 1033–1045.
20. E. Laguerre, *Oeuvres*, vol.1, (Paris: Gaauthier-Villars, 1989).
21. B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956, 632 pp..
22. Y.V. Matiyasevich, *Yet another machine experiment in support of Riemann's conjecture*, Cybernetics, **18** (1983), 705–707.
23. J. Nuttall, *Wronskians, cumulants, and the Riemann hypothesis*, Constr. Approx., **38** (2013), 193–212.
24. A.M. Odlyzko, *Differences of the partition function*, Acta Arithmetica, 1988
25. C. O'Sullivan, *Zeros of Jensen polynomials and asymptotics for the Riemann xi function*, Res. Math. Sci., **8(3)** (2021) Paper No.46.
26. C. O'Sullivan, *Limits of Jensen polynomials for partitions and other sequences*, Monatsh. Math., (2022).
27. M.L. Patrick, *Some inequalities concerning Jacobi polynomials*, SIAM J. Math. Anal., **2 (2)** (1971), 213–220.
28. M.L. Patrick, *Extensions of inequalities of the Laguerre and Turán type.* Pacific J. Math., **44 (2)** (1973), 675–682.
29. G. Pólya and J. Schur, *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen*, J. Reine Angew. Math., **144** (1914), 89–113.
30. Q.I. Rahman and G. Schmeisser, *Analytic theroy of polynomials*, Oxford University Press, Oxford, 2002. xiv+742 pp.
31. H. Skovgaard, *On inequalities of the Turán type*, Math. Scand., **2** (1954),p 65–73.
32. I. Wagner, *The Jensen-Pólya program for various L-functions*, Forum Math., **32(2)** (2020), 525–539.
33. I. Wagner, *On a new class of laguerree-Pólya type functions with applications in number theory*, Pac. J. Math., **320** (2022), 177–192.
34. L.X.W. Wang and E.Y.Y. Yang, *Laguerre inequalities for combinatorial sequences*, Adv. Appl. Math., **139** (2022), 120357.
35. L.X.W. Wang and N.N.Y. Yang, *Positivity of the determinants of the partition function and the overpartition function*, Math. Comp., **92** (2023), 1383–1402.
36. L.X.W. Wang and N.N.Y. Yang, *Asymptotical positivity of the determinant associated with the Riemann Xi-function and other sequences*, submitted.

Center for Combinatorics, LMPC, Nankai University, Tianjin, 300071, China
Current address: Center for Combinatorics, LMPC, Nankai University, Tianjin, 300071, China
Email address: wsw82@nankai.edu.cn

Department of Mathematics, Nankai University, Tianjin, 300071, China
Current address: Department of Mathematics, Nankai University, Tianjin, 300071, China
Email address: 1910132@mail.nankai.edu.cn