Extremal P_8-free/P_9-free planar graphs

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November 23, 2020

Abstract

An H-free graph is a graph containing no the given graph H as a subgraph. It is well-known that the Turán number $ex(n, H)$ is the maximum number of edges in an H-free graph on n vertices. Based on this definition, we define $ex_{\mathcal{P}}(n, H)$ to restrict the graph classes to planar graphs, that is, $ex_{\mathcal{P}}(n, H) = \max\{|E(G)| : G \in \mathcal{G}\}$, where $\mathcal{G}$ is a family of all H-free planar graphs on n vertices. Obviously, we have $ex_{\mathcal{P}}(n, H) = 3n - 6$ if the graph H is not a planar graph. The study is initiated by Dowden [J. Graph Theory 83 (2016) 213–230]. And Dowden obtained some results when H is considered as C_4 or C_5. In this paper, we determine the values of $ex_{\mathcal{P}}(n, P_k)$ with $k \in \{8, 9\}$, where P_k is a path with k vertices.

AMS Classification: 05C10; 05C35.

Keywords: Turán number; extremal graph; planar graph

1 Introduction

Only simple graphs are considered in this paper. Let k be a positive integer. If a cycle has k vertices, then we call it C_k. Similarly, we say P_k is a path with k vertices. For convenient, let $[k] := \{1, 2, \ldots, k\}$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $x \in V(G)$, let $N_G(x)$ denote the neighbours of x in G and $d_G(x)$ denote the degree of x in G. The minimum degree of the graph G is $\delta(G)$, that is, $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. Given a vertex set S, we use $G[S]$ to denote the subgraph of G induced on S and use $G\backslash S$ to denote the subgraph of G induced on $V(G)\backslash S$. For two vertex sets $S, S' \subseteq V(G)$, the set consisting of all vertices belong to S' but not S is denoted by $S'\backslash S$ or $S' - S$. In particular, if $S = \{s\}$, then we simply write $S'\backslash s$ and replace $G\backslash S$ with $G\backslash s$. We say that S is *complete to* (resp. *anti-complete to*) S' if for each $a \in S$ and each $b \in S'$, there is an edge $ab \in E(G)$ (resp. $ab \notin E(G)$). And we simply say a is complete to (resp. anti-complete to) S' if $S = \{a\}$. For two vertex disjoint graphs G and H, the *join* $G + H$ is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$;

and the *union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For a positive integer t, we use tH to denote disjoint union of t copies of a graph H. If G is isomorphism to H, then we write $G = H$. A graph H is a *spanning subgraph* of a graph G if H is a subgraph of G with $V(H) = V(G)$. Given a graph H, let $\chi(H)$ denote the chromatic number of H.

Given a family of graphs $\mathcal{H}$, we call a graph is $\mathcal{H}$-free if it contains no graph in $\mathcal{H}$ as a subgraph. If $\mathcal{H} = \{H\}$, then we simply say the graph is H-free. In 1941, Turán showed that the graph with more than the edges of the Turán graph $T_{n,r}$(balanced complete r-partite graph) must contain a K_{r+1} as a subgraph. This theorem is the well-known Turán theorem. Later, Erdős-Stone showed that, for any graph H, an H-free graph on n vertices has at most $(1+o(1))(\frac{\chi(H)-2}{\chi(H)-1})\,n^2$ edges. Turán problems are one of the oldest questions in extremal combinatoric. Some special classes of host graphs are investigated. When the host graphs are hypergraphs, this problems draw the attention of many researchers, see [3, 2, 8]). We refer to [7] for a survey on Turán-type problems.

In 2015, Dowden [1] introduced the Turán-type problem with planar graphs as host graphs. Given a family of planar graph $\mathcal{H}$, the *planar Turán number of* $\mathcal{H}$, denoted by $ex_{\mathcal{P}}(n, \mathcal{H})$, is the maximum number of edges in an $\mathcal{H}$-free planar graph on n vertices. If $\mathcal{H} = \{H\}$, then $ex_{\mathcal{P}}(n, \mathcal{H})$ can be simply written as $ex_{\mathcal{P}}(n, H)$. When H is a special class of graphs, such as complete graphs and cycle graphs, the corresponding planar Turán number have been determined by Dowden. The following are some of his results. Note that each bound is tight.

Theorem 1.1 ([1]) *Let n be a positive integer.*

(a) $ex_{\mathcal{P}}(n, K_3) = 2n - 4$ *for all* $n \geq 3$*;*

(b) $ex_{\mathcal{P}}(n, K_4) = 3n - 6$ *for all* $n \geq 3$*;*

(c) $ex_{\mathcal{P}}(n, C_4) \leq 15(n-2)/7$ *for all* $n \geq 4$*;*

(d) $ex_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$ *for all* $n \geq 11$*.*

It seems quite non-trivial to determine $ex_{\mathcal{P}}(n, C_k)$ for all $k \geq 6$. In [5], together with Song, the authors proved that $ex_{\mathcal{P}}(n, C_6) \leq 18(n-2)/7$ for all $n \geq 6$, where this bound is not tight. Furthermore, several sufficient conditions on H which yield $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$ were obtained in [6]. This partially answers a question of Dowden [1]. In [4], we study the case of short paths and determine the planar Turán number of paths P_k with $k \in \{6, 7, 10, 11\}$.

In this paper, we consider the planar Turán number $ex_{\mathcal{P}}(n, \mathcal{H})$ for some special classes $\mathcal{H}$. And we promote the idea of determining the maximum number of edges in a P_k-free planar graph on $n \geq 3$ vertices when $k \in \{8, 9\}$.

2 Main Results

We need to introduce more notation. For a positive integer t, let ε_t be the remainder of t when divided by 2, and let $M_t = \lfloor t/2 \rfloor K_2 \cup \varepsilon_t K_1$. Let $\mathcal{T}_t$ denote the family of all planar triangulations on t vertices and let $\mathcal{T}_t^* \subseteq \mathcal{T}_t$ be the family of planar triangulations with a hamiltonian cycle. For integer $k \geq 9$, let $\mathcal{F}_{k-5,n}$ be the family of graphs obtained from $T \cup M_{n-k+5}$ by joining every vertex of M_{n-k+5} to the two adjacent vertices of one fixed hamiltonian cycle of T, where $T \in \mathcal{T}_{k-5}^*$. One can easily see that every graph in $\mathcal{F}_{k-5,n}$ is P_k-free and contains a path on $k-1$ vertices. Finally, a graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. If xy is an edge in a graph G, we denote by G/xy the graph obtained from G by contracting the edge xy into a single vertex and deleting all resulting parallel edges and loops.

It is worth noting that if P is a longest path with ends u, v in a graph G, then $N_G(u) \subseteq V(P)$ and $N_G(v) \subseteq V(P)$. We shall make use of the following Lemma 2.1. The proof of Lemma 2.1(a, b, d) is straightforward and is omitted here. The proof of Lemma 2.1(c) can be obtained by applying the key idea in the proof of the classical result of Dirac.

Lemma 2.1 *Let G be a connected graph and let P be a longest path in G with vertices $v_1, v_2, \ldots, v_\ell$ in order, where $\ell = |P|$ and $|G| > \ell \geq 3$. Then*

(a) $G[V(P)]$ has no spanning cycle. In particular, $v_1v_\ell \notin E(G)$, and if $v_1v_s \in E(G)$ for some $s \in \{2, \ldots, \ell-1\}$, then $v_{s-1}v_\ell \notin E(G)$. Similarly, if $v_\ell v_s \in E(G)$ for some $s \in \{2, \ldots, \ell-1\}$, then $v_1v_{s+1} \notin E(G)$.

(b) $v_{s-1}v_{t+1} \notin E(G)$ if $v_1v_s \in E(G)$ and $v_\ell v_t \in E(G)$, where $s, t \in [\ell]$ with $2 \leq s \leq t \leq \ell-1$. Similarly, v_{t-1} is anti-complete to $\{v_{s-1}, v_{s+1}\}$ if $v_1v_s \in E(G)$ and $v_\ell v_t \in E(G)$, where $s, t \in [\ell]$ with $4 \leq t+2 \leq s \leq \ell-1$.

(c) $2\delta(G) \leq d_G(v_1) + d_G(v_\ell) \leq \ell - 1$.

(d) v_ℓ (resp. v_1) is not adjacent to any two consecutive vertices in $\{v_2, v_3, \ldots, v_{\ell-1}\}$ if $v_1v_{\ell-1} \in E(G)$ (resp. $v_\ell v_2 \in E(G)$).

We first study the the maximum number of edges possible in a P_8-free planar graph on $n \geq 3$ vertices. Clearly, $ex_{\mathcal{P}}(n, P_8) = 3n-6$ when $n \in \{3, 4, \ldots, 7\}$.

Theorem 2.2 *Let $n \geq 3$ be an integer. Let G be a P_8-free planar graph on n vertices. Then $e(G) \leq 15n/7$, with equality exactly when $n = 7t$ for some positive integer t and $G = T_1 \cup \cdots \cup T_t$, where $T_i \in \mathcal{T}_7$ for all $i \in [t]$.*

Proof. Let G, n be given as in the statement. We shall prove that $e(G) \leq 15n/7$ by induction on n. Since any graph on at most 7 vertices is P_8-free and $|G| \geq 3$, we see that $e(G) \leq 3n-6 \leq 15n/7$, with equality when $n = 7$ and $G \in \mathcal{T}_7$. So we may assume that $n \geq 8$. Let $x \in V(G)$ be a vertex with $d_G(x) = \delta(G)$. Then $G - x$ is a P_8-free planar graph on $n-1$ vertices. By the induction hypothesis, $e(G-x) \leq 15(n-1)/7$ and so $e(G) = e(G-x) + d_G(x) < 15n/7$ when $d_G(x) \leq 2$. So we may assume that $d_G(x) \geq 3$. Assume next that G is disconnected. Let H be a component of G. Then $|H| \geq 4$ and $|G\backslash V(H)| \geq 4$ because $\delta(G) \geq 3$. By the induction hypothesis, $e(H) \leq 15|H|/7$ and $e(G\backslash V(H)) \leq 15|G\backslash V(H)|/7$. Hence, $e(G) = e(H) + e(G\backslash V(H)) \leq 15|H|/7 + 15|G\backslash V(H)|/7 \leq 15n/7$, with equality when both H and $G\backslash V(H)$ are disjoint union of planar triangulations on 7 vertices. Hence, when G is disconnected, $e(G) \leq 15n/7$, with equality when $n = 7t$ for some integer $t \geq 2$ and $G = T_1 \cup \cdots \cup T_t$, where for all $i \in [t]$, $T_i \in \mathcal{T}_7$, as desired. So we may assume that G is connected. Let P be a longest path in G with vertices $v_1, v_2, \ldots, v_\ell$ in order. Since G is P_8-free, we see that $\ell \leq 7$. By Lemma 2.1(c), $6 \leq 2\delta(G) \leq d_G(v_1) + d_G(v_\ell) \leq 7 - 1 = 6$, which implies that $\ell = 7$, $\delta(G) = 3$ and $d_G(v_1) = d_G(v_7) = 3$. We say that a vertex of degree 3 in G is *good* if it is an end of a path on 7 vertices. Since $|G| \geq 8$, we see that

($*$) the ends of every path in G on 7 vertices must be non-adjacent good vertices.

Let $N_G(v_1) = \{v_2, v_i, v_j\}$ with $3 \leq i < j \leq 6$ and $N_G(v_7) = \{v_{i'}, v_{j'}, v_6\}$ with $2 \leq i' < j' \leq 5$. We next show that either $j = i+1$ or $j' = i'+1$. Suppose that $j \geq i+2$ and $j' \geq i'+2$. By Lemma 2.1(a), $j = i+2$ and $j' = i'+2$. Then $i \in \{3, 4\}$ because $v_1v_7 \notin E(G)$. If $i = 3$, then by Lemma 2.1(a), $i' = i = 3$ and $j' = j = 5$. By ($*$), all of v_2, v_4, v_6 must be good vertices with all their neighbors on P. Then either $v_2v_4 \in E(G)$ or $v_4v_6 \in E(G)$, contrary to Lemma 2.1(b). Thus $i = 4$. Then $N_G(v_1) = \{v_2, v_4, v_6\}$. By Lemma 2.1(a), $N_G(v_7) = \{v_2, v_4, v_6\}$. By ($*$), both v_3 and v_5 are good vertices with all their neighbors on P. By Lemma 2.1(b), $v_3v_5 \notin E(G)$. Thus $v_3v_6 \in E(G)$ and $v_2v_5 \in E(G)$. But then $\{v_1, v_3, v_7\}$ is complete to $\{v_2, v_4, v_6\}$ in G, a contradiction. This proves that either $j = i+1$ or $j' = i'+1$. We may assume that $j = i+1$. By Lemma 2.1(d), $v_2v_7 \notin E(G)$. By Lemma 2.1(a), $N_G(v_1) = \{v_2, v_3, v_4\}$ and $N_G(v_7) = \{v_4, v_5, v_6\}$. One can easily check that all of v_1, v_2, v_3 are good vertices in G. By the induction hypothesis, $e(G\backslash\{v_1, v_2, v_3\}) \leq 15(n-3)/7 = 15n/7 - 45/7$. Hence, $e(G) = e(G\backslash\{v_1, v_2, v_3\}) + 6 < 15n/7$. ■

We are ready to prove a result on the maximum number of edges possible in a P_9-free planar graph on $n \geq 3$ vertices.

Theorem 2.3 *Let $n \geq 3$ be an integer. Let G be a P_9-free planar graph on n vertices. Then $e(G) \leq \max\{\frac{9n}{4}, \frac{5n}{2} - 4\}$, with equality exactly when $G \in \mathcal{T}_8$ or when $G = T_1 \cup T_2$ with $T_1, T_2 \in \mathcal{T}_8$ or when $n \geq 16$ is even and $G \in \mathcal{F}_{4,n}$.*

Proof. Let G, n be given as in the statement. Note that $\max\{\frac{9n}{4}, \frac{5n}{2} - 4\} = \frac{5n}{2} - 4$ when $n \geq 16$ and $\max\{\frac{9n}{4}, \frac{5n}{2} - 4\} = \frac{9n}{4}$ when $n \leq 16$. We shall prove the statement by induction on n. Since any graph on at most 8 vertices is P_9-free and $|G| \geq 3$, we see that $e(G) \leq 3n-6 \leq \frac{9n}{4}$, with equality when $n=8$ and $G \in \mathcal{T}_8$. So we may assume that $n \geq 9$. Let $x \in V(G)$ be a vertex with $d_G(x) = \delta(G)$. Then $G-x$ is a P_9-free planar graph on $n-1$ vertices. By the induction hypothesis, $e(G-x) \leq \max\{\frac{9}{4}(n-1), \frac{5}{2}(n-1)-4\}$ and so $e(G) = e(G-x) + d_G(x) < \max\{\frac{9n}{4}, \frac{5n}{2}-4\}$ when $d_G(x) \leq 2$. So we may assume that $d_G(x) \geq 3$. Assume next that G is disconnected. Let H be a component of G. Then $|H| \geq 4$ and $|G \backslash V(H)| \geq 4$ because $\delta(G) \geq 3$. By the induction hypothesis, $e(H) \leq \max\{\frac{9}{4}|H|, \frac{5}{2}|H|-4\}$ and $e(G\backslash V(H)) \leq \max\{\frac{9}{4}|G\backslash V(H)|, \frac{5}{2}|G\backslash V(H)|-4\}$. Hence, $e(G) = e(H) + e(G\backslash V(H)) \leq \max\{\frac{9}{4}|H|, \frac{5}{2}|H|-4\} + \max\{\frac{9}{4}|G\backslash V(H)|, \frac{5}{2}|G\backslash V(H)| - 4\} \leq \max\{\frac{9n}{4}, \frac{5n}{2}-4\}$, with equality when both H and $G\backslash V(H)$ are planar triangulations on 8 vertices. Hence, when G is disconnected, $e(G) \leq \max\{\frac{9n}{4}, \frac{5n}{2}-4\}$, with equality when $n=16$ and $G = T_1 \cup T_2$, where $T_1, T_2 \in \mathcal{T}_8$. So we may assume that G is connected. Let P be a longest path in G with vertices $v_1, v_2, \ldots, v_\ell$ in order. We may assume that $d_G(v_1) \leq d_G(v_\ell)$. Since G is P_9-free, $\ell \leq 8$. By Lemma 2.1(c), $6 \leq 2\delta(G) \leq d_G(v_1) + d_G(v_\ell) \leq \ell - 1 \leq 7$, which implies that $\delta(G) = 3$. Then $\ell \in \{7, 8\}$. Assume that $\ell = 7$. Then G is P_8-free. By Theorem 2.2, $e(G) \leq \frac{15n}{7} < \frac{9n}{4} \leq \max\{\frac{9n}{4}, \frac{5n}{2}-4\}$, as desired. So we may assume that $\ell = 8$. Then $d_G(v_1) = 3$ and $d_G(v_8) \in \{3, 4\}$. A vertex of degree 3 in G is *good* if it is an end of a path on 8 vertices. Since $|G| \geq 9$, by Lemma 2.1(a), the ends of every path in G on 8 vertices must be non-adjacent and one of them is good.

Let $N_G(v_1) = \{v_2, v_i, v_j\}$ with $3 \leq i < j \leq 7$. We first consider the case when $d_G(v_8) = 4$. Let $N_G(v_8) = \{v_{i'}, v_{j'}, v_{\ell'}, v_7\}$ with $2 \leq i' < j' < \ell' \leq 6$. Since $d_G(v_8) = 4$, by Lemma 2.1(d), $v_1v_7 \notin E(G)$. We next show that $j = i+1$. Suppose that $j \geq i+2$. If $j \geq i+3$, then by Lemma 2.1(a, d), $i = i' = 3$, $j' = 4$, $j = \ell' = 6$. Since $G[V(P)]$ has a path on 8 vertices with ends v_2, v_8 (resp. v_5, v_8), we see that both v_2 and v_5 must be good vertices with all their neighbors on P. By Lemma 2.1(b), v_5 is anti-complete to $\{v_2, v_3, v_7\}$ in G. But then $d_G(v_5) = 2$, a contradiction. Thus $j = i+2$. Then $i \in \{3, 4\}$ because $v_1v_7 \notin E(G)$. If $i = 3$, then $N_G(v_8) = \{v_3, v_5, v_6, v_7\}$ by Lemma 2.1(a). Then $G[V(P)]$ has a path on 8 vertices with ends v_2, v_8 (resp. v_4, v_8). Thus both v_2 and v_4 must be good vertices with all their neighbors on P. By Lemma 2.1(b), v_4 is anti-complete to $\{v_2, v_6, v_7\}$ in G. But then $d_G(v_4) = 2$, a contradiction. Thus $i = 4$. By Lemma 2.1(a), $N_G(v_8) = \{v_2, v_4, v_6, v_7\}$. Then $G[V(P)]$ has a path on 8 vertices with ends v_3, v_8 (resp. v_5, v_8). Thus both v_3 and v_5 must be good vertices with all their neighbors on P. By Lemma 2.1(b), v_5 is anti-complete to $\{v_3, v_7\}$ in G. Thus $v_5v_2 \in E(G)$. But then $\{v_1, v_5, v_8\}$ is complete to $\{v_2, v_4, v_6\}$ and so G contains $K_{3,3}$ as a subgraph, contrary to the fact that G is planar. This proves that $j = i+1$. By Lemma 2.1(d), $v_2v_8 \notin E(G)$. Since $d_G(v_8) = 4$, by Lemma 2.1(a), $N_G(v_1) = \{v_2, v_3, v_4\}$ and so $N_G(v_8) = \{v_4, v_5, v_6, v_7\}$. Then all of v_1, v_2, v_3 must be good vertices in G. By the induction hypothesis, $e(G\backslash\{v_1, v_2, v_3\}) \leq \max\{\frac{9(n-3)}{4}, \frac{5(n-3)}{2}-4\}$. Hence, $e(G) = e(G\backslash\{v_1, v_2, v_3\}) + 6 < \max\{\frac{9n}{4}, \frac{5n}{2}-4\}$, as desired. So we may assume that $d_G(v_8) = 3$ and we may further assume that

($*$) the ends of every path in G on 8 vertices are non-adjacent good vertices.

Let $N_G(v_8) = \{v_{i'}, v_{j'}, v_7\}$ with $2 \le i' < j' \le 6$. We next show that $j \le i+3$. Suppose $j \ge i+4$. Since $v_1v_8 \notin E(G)$, we have $j = i+4$. Then $i = 3$ and $j = 7$. By Lemma 2.1(a,d), $i' = 3$ and $j' = 5$. Then $G[V(P)]$ has a path on 8 vertices with one end in $\{v_2, v_4, v_6\}$, by ($*$), all of v_2, v_4, v_6 are good vertices with all their neighbors on P. By Lemma 2.1(b), $\{v_2, v_4, v_6\}$ is an independent set in G. Since G is connected and $|G| > |P|$, let $w \in V(G)\backslash V(P)$ be such that w is adjacent to some vertex on P in G. Since all of v_1, v_2, v_4, v_6, v_8 are good vertices with all their neighbors on P, we see that w can only be adjacent to v_3, v_5 or v_7 on P. Note that $G[\{v_1, v_2 \ldots, v_7\}]$ has a spanning cycle. It follows that $d_G(w) = 3$ and w is complete to $\{v_3, v_5, v_7\}$ in G. Since G is $K_{3,3}$-free, $v_6v_3 \notin E(G)$, else $\{v_6, v_8, w\}$ is complete to $\{v_3, v_5, v_7\}$ in G. But then $d_G(v_6) = 2$, a contradiction. This proves that $j \le i+3$. By symmetry, $j' \le i'+3$.

Assume next that $j = i+3$. Then $i \in \{3, 4\}$. We next show that $i = 3$. Suppose $i = 4$. Then $j = 7$. By Lemma 2.1(a, d), $i' = 2$ and $j' \in \{4, 5\}$. If $j' = 4$, then by ($*$), all of v_3, v_5, v_6 are good vertices with all their neighbors on P, because $G[V(P)]$ has a path on 8 vertices with one end in $\{v_3, v_5, v_6\}$. By Lemma 2.1(b), v_3 is anti-complete to $\{v_5, v_6\}$ in G. Thus $v_3v_7 \in E(G)$. But then $\{v_1, v_3, v_8\}$ is complete to $\{v_2, v_4, v_7\}$ and so G contains $K_{3,3}$ as a subgraph, contrary to the fact that G is planar. Thus $j' = 5$. By Lemma 2.1(b), v_6 is anti-complete to $\{v_3, v_4\}$ in G. By ($*$), both v_3 and v_6 are good vertices with all their neighbors on P. Thus $v_6v_2 \in E(G)$. But then G has a spanning cycle on 8 vertices $v_2, v_6, v_5, v_8, v_7, v_1, v_4, v_3$ in order, contrary to Lemma 2.1(a). This proves that $i = 3$ and so $j = 6$. By Lemma 2.1(a), $i' \in \{3, 4\}$ and $j' \in \{4, 6\}$. We next show that $(i', j') = (3, 6)$. Suppose $(i', j') \ne (3, 6)$. If $i' = 4$, then $j' = 6$. By ($*$), all of v_2, v_5, v_7 are good vertices with all their neighbors on P. By Lemma 2.1(b), v_5 is anti-complete to $\{v_2, v_3, v_7\}$ in G. But then $d_G(v_5) = 2$, a contradiction. Thus $i' = 3$ and so $j' = 4$ because $(i', j') \ne (3, 6)$. Then $d_G(v_3) \ge 4$. But then G has a path on 8 vertices with vertices $v_3, v_2, v_1, v_6, v_7, v_8, v_4, v_5$ in order, contrary to ($*$). This proves that $(i', j') = (3, 6)$. By ($*$), all of v_2, v_4, v_5, v_7 are good vertices with all their neighbors on P. By Lemma 2.1(b), v_4 is anti-complete to $\{v_2, v_7\}$ in G. Thus $v_4v_6 \in E(G)$. By symmetry, $v_5v_3 \in E(G)$. By Lemma 2.1(b), $v_2v_7 \notin E(G)$. Thus $v_2v_6 \in E(G)$. By symmetry, $v_7v_3 \in E(G)$. Since G is connected and $|G| > |P|$, let $w \in V(G)\backslash V(P)$ be such that w is adjacent to some vertex on P in G. Since all of $v_1, v_2, v_4, v_5, v_7, v_8$ are good vertices with all their neighbors on P, we see that w can only be adjacent to v_3 or v_6 on P. Note that $G[\{v_1, v_2 \ldots, v_6\}]$ has a spanning cycle. Since $\delta(G) \ge 3$ and G is P_9-free, it follows that for any $w \in V(G)\backslash V(P)$, $d_G(w) = 3$, w is complete to $\{v_3, v_6\}$ in G, and every component of $G\backslash V(P)$ is isomorphic to K_2. This is possible when $n \ge 10$ is even. It follows that $G[\{v_3, v_4, v_5, v_6\}] = K_4$ when $v_3v_6 \in E(G)$, $\{v_3, v_6\}$ is complete to $V(G)\backslash\{v_3, v_4, v_5, v_6\}$ in G, and $G\backslash\{v_3, v_4, v_5, v_6\} = \frac{n-4}{2}K_2$. Hence, when $(i, j) = (i', j') = (3, 6)$ and $n \ge 10$ is even, $e(G) \le \frac{5n}{2} - 4$, with equality when $v_3v_6 \in E(G)$, $\{v_3, v_6\}$ is complete to

$V(G)\backslash\{v_3, v_4, v_5, v_6\}$ in G and $G\backslash\{v_3, v_4, v_5, v_6\} = \frac{n-4}{2}K_2$, that is, when $G \in \mathcal{F}_{4,n}$.

So we may assume that $j \le i+2$. By symmetry, $j' \le i'+2$. We next show that either $j = i+1$ or $j' = i'+1$. Suppose $j = i+2$ and $j' = i'+2$. Then $i \in \{3, 4, 5\}$ and $j' \in \{4, 5, 6\}$. If $i = 3$, then by Lemma 2.1(a), $i' = 3$ and so $j = j' = 5$. By $(*)$, all of v_2, v_4, v_6 must be good vertices with all their neighbors on P. By Lemma 2.1(b), $v_4v_2, v_4v_6, v_6v_2 \notin E(G)$. Thus $v_4v_7, v_6v_3 \in E(G)$. But then $\{v_4, v_6, v_8\}$ is complete to $\{v_3, v_5, v_7\}$, a contradiction. Thus $i \ne 3$. By symmetry, $j' \ne 6$. If $i = 4$, then $i' = 2$ because $j' \ne 6$. By $(*)$, both v_3 and v_5 must be good vertices with all their neighbors on P. By Lemma 2.1(b), $v_3v_5 \notin E(G)$. Hence, either $v_3v_6 \in E(G)$ or $v_3v_7 \in E(G)$. But then $G[V(P)]$ contains $K_{3,3}$ as a minor, because $\{v_1, v_3, v_8\}$ is complete to $\{v_2, v_4, w\}$ in G/v_6v_7, where w is the new vertex in G/v_6v_7, a contradiction. Thus $i \ne 4$. By symmetry, $j' \ne 5$. Thus $i = 5$, but then $j' = 4$ because $j' \notin \{5, 6, 7\}$, contrary to Lemma 2.1(a). This proves that either $j = i+1$ or $j' = i'+1$. We may assume that $j = i+1$. Note that $j' \le i'+2$. By Lemma 2.1(d), $v_2v_8 \notin E(G)$. We next show that $i \in \{3, 4\}$. Suppose $i \in \{5, 6\}$. By Lemma 2.1(a), v_8 is anti-complete to $\{v_{i-1}, v_i\}$ in G. If $i = 5$, then $i' = 3$ and $j' = 6$ because $v_2v_8 \notin E(G)$, contrary to the fact that $j' \le i'+2$. If $i = 6$, then $v_1v_7 \in E(G)$, $i' = 3$ and $j' = 4$ because $v_2v_8 \notin E(G)$, contrary to Lemma 2.1(d). Hence $i \in \{3, 4\}$. Assume first that $i = 3$. Then $N_G(v_1) = \{v_2, v_3, v_4\}$. One can easily check that all of v_1, v_2, v_3 must be good vertices in G. By the induction hypothesis, $e(G\backslash\{v_1, v_2, v_3\}) \le \max\{\frac{9}{4}(n-3), \frac{5}{2}(n-3)-4\}$. Hence $e(G) = e(G\backslash\{v_1, v_2, v_3\}) + 6 < \max\{\frac{9n}{4}, \frac{5n}{2} - 4\}$, as desired. It remains to consider the case $i = 4$. Since $j' \le i'+2$, by Lemma 2.1(a), $N_G(v_8) = \{v_5, v_6, v_7\}$ and so all of v_6, v_7, v_8 must be good vertices in G. By symmetry, $e(G) = e(G\backslash\{v_6, v_7, v_8\}) + 6 < \max\{\frac{9n}{4}, \frac{5n}{2} - 4\}$.

This completes the proof of Theorem 2.3. ■

Acknowledgments

The authors would like to thank Zi-Xia Song for her discussion. Yongtang Shi was partially supported by the National Natural Science Foundation of China (11922112, 11771221, 11811540390), and the Fundamental Research Funds for the Central Universities. Yongxin Lan was partially supported by National Natural Science Foundation of China (No.12001154) and Special Funds for Jointly Building Universities of Tianjin (No.280000307).

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