Some interlacing results on weighted adjacency matrices of graphs with degree-based edge-weights*

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Abstract

Let G be a graph and d_i denote the degree of a vertex v_i in G, and let $f(x,y)$ be a real symmetric function. Then one can get an edge-weighted graph in such a way that for each edge v_iv_j of G, the weight of v_iv_j is assigned by the value $f(d_i,d_j)$. Hence, we have a weighted adjacency matrix $A_f(G)$ of G, in which the ij-entry is equal to $f(d_i,d_j)$ if $v_iv_j \in E(G)$ and 0 otherwise. In this paper, we obtain uniform interlacing inequalities for the weighted adjacency eigenvalues under some kinds of graph operations including edge subdivision, vertex deletion and vertex contraction. In addition, if $f(x,y)$ is increasing in the variable x, then some examples are given to show that the interlacing inequalities are the best possible for each type of the operations. This paper attempts to unify the study of spectral properties for the weighted adjacency matrices of graphs with degree-based edge-weights.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. For notation and terminology not defined here, we refer to [6]. Let $G = (V(G), E(G))$ be a graph with vertex-set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge-set $E(G)$. For a vertex v_i in G, let $N_G(v_i)$ denote the set of neighbors of a vertex v_i in G and $d_i = |N_G(v_i)|$ be the degree of v_i, and let $N_G[v_i] = N_G(v_i) \cup \{v_i\}$. If the vertex-set $V(G)$ of G admits a partition into two classes such that the two ends of its every edge are in different classes (or, vertices in the same partition class must not be adjacent), then G is called a bipartite graph. A bipartite graph in which any two vertices from different partition classes are adjacent is called a complete bipartite graph, denoted by $K_{s,t}$, where $s + t = n$. As usual, we denote by K_n, C_n and S_n, respectively, the complete graph, the cycle and the star of order n.

In chemical graph theory, graphical or topological indices in chemistry are used to represent the structural properties of molecular graphs. The general form of these indices is $\sum\limits_{v_iv_j \in E(G)} f(d_i, d_j)$, where $f(x, y)$ is a real symmetric function, called the edge-weight function, and $f(d_i, d_j)$ is the edge-weight of an edge v_iv_j of G. Gutman [8] collected many important and well-studied indices; see them in Table 1. In order to study the discrimination property, Rada [19] introduced the exponentials of the best known degree-based topological indices; see them in Table 2. Each index maps a molecular graph into a single number, obtained by summing up the edge-weights in a molecular graph with edge-weights defined by the function $f(x, y)$.

In spectral graph theory, matrices associated with a graph G play an important role. Thus, using a matrix to represent the structure of a molecular graph with edge-weights separately on its pairs of adjacent vertices, it would much better keep the structural information of the graph. In other words, a matrix keeps much more structural information than just a single number, the value of an index. So, the algebraic properties of these structural matrices are worth thoroughly studying. In 2015, this idea was first proposed by one of the authors Li in [12]. Since then various studies on matrices defined by topological indices from algebraic viewpoint were reported, such as the misbalance degree (Albertson) matrix [1], inverse sum indeg matrix [2], ABC matrix [4], Radić matrix [15], AG matrix [20], Zagreb matrix [18] and GA matrix [21], because many interesting properties of graphs are reflected in the study of these matrices.

In 2018, Das et al. [7] gave the following formal definition of the weighted adjacency matrix for a graph with degree-based edge-weights. Let $A_f(G)$ denote the weighted adjacency matrix of a graph G with edge-weight function $f(x, y)$, whose

Edge-weight function f(x,y)	The corresponding index
$x+y$	first Zagreb index
xy	second Zagreb index
$(x+y)^2$	first hyper-Zagreb index
$(xy)^2$	second hyper-Zagreb index
$x^{-3}+y^{-3}$	modified first Zagreb index
$\|x-y\|$	Albertson index
$(x/y+y/x)/2$	extended index
$(x-y)^2$	sigma index
$1/\sqrt{xy}$	Randić index
$\sqrt{xy}$	reciprocal Randić index
$1/\sqrt{x+y}$	sum-connectivity index
$\sqrt{x+y}$	reciprocal sum-connectivity index
$2/(x+y)$	harmonic index
$\sqrt{(x+y-2)/(xy)}$	atom-bond-connectivity (ABC) index
$(xy/(x+y-2))^3$	augmented Zagreb index
x^2+y^2	forgotten index
$x^{-2}+y^{-2}$	inverse degree
$2\sqrt{xy}/(x+y)$	geometric-arithmetic (GA) index
$(x+y)/(2\sqrt{xy})$	arithmetic-geometric (AG) index
$xy/(x+y)$	inverse sum index
$x+y+xy$	first Gourava index
$(x+y)xy$	second Gourava index
$(x+y+xy)^2$	first hyper-Gourava index
$((x+y)xy)^2$	second hyper-Gourava index
$1/\sqrt{x+y+xy}$	sum-connectivity Gourava index
$\sqrt{(x+y)xy}$	product-connectivity Gourava index
$\sqrt{x^2+y^2}$	Sombor index

Table 1: Some well-studied chemical indices

Edge-weight function f(x,y)	The corresponding index
e^{x+y}	exponential first Zagreb index
e^{xy}	exponential second Zagreb index
$e^{1/\sqrt{xy}}$	exponential Randić index
$e^{\sqrt{(x+y-2)/(xy)}}$	exponential ABC index
$e^{2\sqrt{xy}/(x+y)}$	exponential GA index
$e^{2/(x+y)}$	exponential harmonic index
$e^{1/\sqrt{x+y}}$	exponential sum-connectivity index
$e^{(xy/(x+y-2))^3}$	exponential augmented Zagreb index

Table 2: Some well-known exponential chemical indices

ij-entry is defined as

$$(A_f(G))_{ij} = \begin{cases} f(d_i, d_j) & \text{if } v_iv_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

That is, for any graphical or topological index, one can define the corresponding weighted adjacency matrix of an edge-weighted graph by the edge-weight function $f(x, y)$ of the index. Unfortunately, although the matrix $A_f(G)$ was proposed in a general setting, it was studied still one by one separately for a concrete graphical or topological index or function $f(x, y)$, not as a whole for a general function $f(x, y)$. This lost the sense for us to introduce the general weighted adjacency matrix $A_f(G)$. In 2021, Li and Wang [13] attempted to study the extremal spectral radius of the weighted adjacency matrices in a general setting. They obtained some nice results on extremal spectral radius of weighted adjacency matrices among trees when the edge-weight function $f(x, y)$ has some functional properties. This is the beginning of the study of spectral properties by function classification. In 2022, Li and Yang [14] obtained uniform interlacing inequalities for the weighted adjacency eigenvalues under edge deletion. They also established a uniform equivalent condition for a connected graph G to have m distinct weighted adjacency eigenvalues, from which people can directly get the results in [4, 15, 16, 20]. As one can see from the existing literature, only a tip of an iceberg was excavated for $A_f(G)$, and there are still a lot of properties of $A_f(G)$ waiting to be explored in the future when $f(x, y)$ has some functional properties. This will eventually unify the approaches for spectral properties of the weighted adjacency matrices of an edge-weighted graph by the edge-weight function $f(x, y)$ of graphical or topological indices.

We will simply call the eigenvalues of the $n \times n$ matrix $A_f(G)$ as *weighted adjacency eigenvalues* of a graph G with edge-weight function $f(x, y)$. Since $f(x, y)$ is a real symmetric function and G is an undirected graph. $A_f(G)$ is a real symmetric matrix, and therefore all its eigenvalues are real numbers. We may adopt the convention that the eigenvalues λ_i are always arranged in a decreasing order. i.e.,

$$\lambda_{max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = \lambda_{min}. \tag{1.1}$$

If each of the edges bears the weight 1, i.e., the function $f(x, y) \equiv 1$, then the adjacency matrix of G is simply denoted by $A(G)$.

Because we study the interlacing results of weighted adjacency eigenvalues under some graph perturbations, firstly, we define some operations on graphs.

Definition 1.1 *(Edge deletion) The deletion of an edge $e \in E(G)$ produces a graph $G - e$, where $V(G - e) = V(G)$ and $E(G - e) = E(G) - \{e\}$.*

Definition 1.2 *(Edge subdivision) The subdivision of an edge $e = v_i v_j \in E(G)$ produces a graph G_e, where $V(G_e) = V(G) \cup \{v_{n+1}\}$, such that $v_{n+1} \notin V(G)$, and $E(G_e) = \{E(G) - e\} \cup \{v_i v_{n+1}, v_j v_{n+1}\}$.*

Subdividing the edge $v_i v_j$ in a graph G means that a new vertex v_{n+1} is added to $V(G)$ and the edge $v_i v_j$ is replaced in $E(G)$ by an edge $v_i v_{n+1}$ and an edge $v_j v_{n+1}$.

Definition 1.3 *(Vertex deletion) The deletion of a vertex $v \in V(G)$ produces a graph $G - v$, where $V(G - v) = V(G) - \{v\}$ and $E(G - v) = E(G) - \{uv : u \in N_G(v)\}$.*

Definition 1.4 *(Vertex contraction) The contraction of a pair of vertices $u, v \in V(G)$ produces a graph $G_{\{u,v\}}$, where $V(G_{\{u,v\}}) = (V(G) - \{u, v\}) \cup \{x_{uv}\}$, x_{uv} is a new vertex with $N_{G_{\{u,v\}}}(x_{uv}) = [N_G(u) \cup N_G(v)] - \{u, v\}$, and $E(G_{\{u,v\}}) = [E(G) - (\{uz : z \in N_G(u)\} \cup \{vz : z \in N_G(v)\})] \cup \{x_{uv} z : z \in N_{G_{\{u,v\}}}(x_{uv})\}$.*

The contraction of a pair of vertices u and v produces a graph in which the two vertices u and v are replaced with a single vertex x_{uv} such that x_{uv} is adjacent to the union of the vertices to which v and u were originally adjacent.

The eigenvalue interlacing provides a handy tool for obtaining inequalities and regularity results concerning the structure of graphs in terms of eigenvalues of the adjacency matrix and the Laplacian matrix. There have been many investigations of this field. For a survey of literature, we refer to Haemers [9]. The problem of studying the behaviors on different kind of eigenvalues of graphs under perturbations is of interest. In many papers, such as [3, 5, 10, 17, 22, 24], the interlacing relation under graph operations of the spectra of matrix representation of graphs are studied. Here, we first restate some known results of eigenvalue interlacing for the well-known adjacency matrix $A(G)$ of graphs.

Theorem 1.5 *[17] Let G be a graph of order n and $H = G_e$, where $e = uv$ is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n+1}$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, \ldots, n+1,$$

where $\lambda_i = +\infty$ for each $i \leq 0$ and $\lambda_i = -\infty$ for each $i \geq n+1$.

Let $H = G - v$, where v is a vertex of graph G. Since $A(H)$ is an $(n-1) \times (n-1)$ principal submatrix of $A(G)$, it is not difficult to get its interlacing result of eigenvalues.

Theorem 1.6 *[10] Let G be a graph of order n and $H = G - v$, where v is a vertex of G. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1}$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, \ldots, n-1.$$

Theorem 1.7 *[10] Let G be a graph of order n and $H = G_{\{u,v\}}$, where u and v be two distinct vertices of G. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1}$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2} \quad \text{for each } i = 1, 2, \ldots, n-1,$$

where $\lambda_0 = +\infty$ and $\lambda_{n+1} = -\infty$.

If we assume that $N_G(u) \cap N_G[v] = \emptyset$, then depending on the sign of θ_i, the above inequalities can be strengthened in one of two ways. Let k be such that $\theta_k \geq 0$ and $\theta_{k+1} < 0$. Then

$$\theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, \ldots, k$$

and

$$\lambda_i \geq \theta_i \quad \text{for each } i = k+1, k+2, \ldots, n-1.$$

Very recently, Li and Yang in [14] presented the following result of the weighted adjacency eigenvalues for a graph under edge deletion.

Theorem 1.8 *[14] Let G be a graph of order n and $H = G - e$, where e is an edge of G. Let $f(x, y)$ is any symmetric real function. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$$

are the eigenvalues of $A_f(G)$ and $A_f(H)$, respectively, then

$$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+2} \quad \text{for each } i = 1, 2, \ldots, n,$$

where $\lambda_i = +\infty$ for each $i \leq 0$ and $\lambda_i = -\infty$ for each $i \geq n + 1$.

They said that this result cannot be improved if the edge-weighted function $f(x, y)$ be a symmetric polynomial with nonnegative coefficients and zero constant term.

It is natural to ask what behaviors of the weighted adjacency eigenvalues will be if the graph is perturbed in different ways. In the following paragraphs, we will study interlacing results for the weighted adjacency eigenvalues under some graph operations, including edge subdivision, vertex deletion and vertex contraction.

The structure of this paper is arranged as follows. In the next section, we introduce some necessary notation and terminology and list several previous known results that will be used in the subsequent sections. In Section 3, we first obtain the weighted adjacency eigenvalues for some well-known families of graphs. Then, the interlacing results associated with the edge subdivision, vertex deletion and vertex contraction of the weighted adjacency eigenvalues are presented, respectively. Examples are given to show that the interlacing inequalities are the best possible for their type when $f(x, y)$ is increasing in the variable x. This covers the edge-weight functions $f(x, y)$ of almost a half of the indices listed in Tables 1 and 2.

2 Preliminaries

At the very beginning, we state some fundamental results on matrix theory, which will be used in the sequel. An $n \times n$ complex square matrix M is called *Hermitian* if $M^* = M$, where M^* is the conjugate transpose of M. The eigenvalues of M are defined as:

$$\rho_1(M) \geq \rho_2(M) \geq \cdots \geq \rho_n(M).$$

Suppose the rows and columns of

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1m} \\ M_{21} & M_{22} & \cdots & M_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mm} \end{bmatrix}.$$

are partitioned according to a partitioning $X_1,\ X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$ with characteristic matrix S (that is, the ij entry is 1 if $i \in X_j$, and 0 otherwise). The *quotient matrix* is the matrix B whose entries are the average row sums of the blocks of M. More precisely,

$$(B)_{ij} = \frac{1}{|X_i|}\mathbf{1}^T M_{ij}\mathbf{1} = \frac{1}{|X_i|}(S^T M S)_{ij},$$

where $\mathbf{1}$ denotes the all-one vector. If each block M_{ij} of M has a constant row (and column) sum, then the partition is called *regular (or equitable)* and B is called the *equitable quotient matrix* of M.

Lemma 2.1 *[25] Let B be the equitable quotient matrix of M. The respective eigenvalues of B and M be*

$$\rho_1(B) \geq \rho_2(B) \geq \cdots \geq \rho_m(B) \text{ and } \rho_1(M) \geq \rho_2(M) \geq \cdots \geq \rho_n(M).$$

Then

$$\{\rho_1(B), \rho_2(B), \ldots, \rho_m(B)\} \subseteq \{\rho_1(M), \rho_2(M), \ldots, \rho_n(M)\}.$$

If we delete several rows and the corresponding columns from an Hermitian matrix, the remaining matrix is a principal submatrix of the original matrix. In [11], there is a conclusion as follows.

Lemma 2.2 *[11] Let M be a Hermitian matrix of order n, partitioned as*

$$M = \begin{bmatrix} B_{m\times m} & C_{m\times(n-m)} \\ (C_{m\times(n-m)})^* & D_{(n-m)\times(n-m)} \end{bmatrix}.$$

Let

$$\rho_1(B) \geq \rho_2(B) \geq \cdots \geq \rho_m(B) \text{ and } \rho_1(M) \geq \rho_2(M) \geq \cdots \geq \rho_n(M)$$

be the eigenvalues of B and M, respectively. Then the inequalities

$$\rho_i(M) \geq \rho_i(B) \geq \rho_{n-m+i}(M),$$

hold for each $i = 1, 2, \ldots, m$.

In 1912, Weyl [23] stated a very useful result.

Lemma 2.3 *[23] Let M,N be Hermitian matrices of order n. And let the respective eigenvalues of M, N and $M+N$ be*

$$\rho_1(M) \geq \rho_2(M) \geq \cdots \geq \rho_n(M),\ \rho_1(N) \geq \rho_2(N) \geq \cdots \geq \rho_n(N)$$
$$\text{and } \rho_1(M+N) \geq \rho_2(M+N) \geq \cdots \geq \rho_n(M+N).$$

Then

$$\rho_i(M+N) \leq \rho_j(M) + \rho_{i-j+1}(N), \qquad (n \geq i \geq j \geq 1). \tag{2.1}$$

Also,

$$\rho_j(M) + \rho_{i-j+n}(N) \leq \rho_i(M+N), \qquad (1 \leq i \leq j \leq n). \tag{2.2}$$

This is the root of a great many inequalities involving the sum of two Hermitian matrices and their eigenvalues, for which we refer to Section 3 of Chapter 4 in [11].

3 Main results

In this section, we first get a result that helps us in finding some weighted adjacency eigenvalues, provided G has some special structures.

A subset I of the vertex-set $V(G)$ is said to be an *independent set* if no two vertices of I are adjacent in G, while it is said to be a *clique* if every two vertices of I are adjacent in G.

Theorem 3.1 *Let G be a connected graph with vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $I = \{v_1, v_2, \ldots, v_m\}$ be a subset of $V(G)$, such that $N_G(v_i) - v_j = N_G(v_j) - v_i$ for all $i, j \in \{1, 2, \ldots, m\}$. Then for any symmetric real function $f(x, y)$, we have the following statements.*

(i) *If I is a clique of G, then $-f(d_1, d_1)$ is an eigenvalue of $A_f(G)$ with multiplicity at least $m-1$, where d_1 is the degree of the vertex v_1;*

(ii) *If I is an independent set of G, then 0 is an eigenvalue of $A_f(G)$ with multiplicity at least $m-1$.*

Proof. (i) We first suppose that the vertices of I form a clique in graph G. Since $N_G(v_i) - v_j = N_G(v_j) - v_i$ for all $i, j \in \{1, 2, \ldots, m\}$, it follows that $d_1 = d_2 = \cdots = d_m$. We first index the vertices of I, so that the weighted adjacency matrix $A_f(G)$ can be written as:

$$A_f(G) = \begin{bmatrix} 0 & f(d_1,d_1) & f(d_1,d_1) & \cdots & f(d_1,d_1) & \\ f(d_1,d_1) & 0 & f(d_1,d_1) & \cdots & f(d_1,d_1) & \\ f(d_1,d_1) & f(d_1,d_1) & 0 & \cdots & f(d_1,d_1) & B_{m\times(n-m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ f(d_1,d_1) & f(d_1,d_1) & f(d_1,d_1) & \cdots & 0 & \\ & & (B_{m\times(n-m)})^T & & & C_{(n-m)\times(n-m)} \end{bmatrix},$$

where $(B_{m\times(n-m)})^T$ is the transpose of $B_{m\times(n-m)}$.

For $2 \leq i \leq m$, let $x_{i-1} = (-1, x_{i2}, x_{i3}, \cdots, x_{im}, 0, 0, 0, \cdots, 0)^T$ be the vector in R^n such that $x_{ij} = 1$ if $i = j$ and 0 otherwise. If $x_1, x_2, \ldots, x_{m-1}$ are linearly dependent vectors, then there exist numbers $a_1, a_2, \ldots, a_{m-1}$ not all zero, such that

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero vector.

This means that

$$(-\sum_{i=1}^{m-1} a_i, a_1, a_2, \cdots, a_{m-1}, 0, 0, 0, \cdots, 0) = \mathbf{0},$$

and it follows that $a_1 = a_2 = \cdots = a_{m-1} = 0$. Therefore, the vectors $x_1, x_2, \ldots, x_{m-1}$ cannot be linearly dependent.

Since $N_G(v_i) - v_j = N_G(v_j) - v_i$ for all $i, j \in \{1, 2, \ldots, m\}$. Noting that the rows of B are identical, we see that

$$A_f(G)x_i = -f(d_1,d_1)x_i$$

for $1 \leq i \leq m-1$. Thus $-f(d_1,d_1)$ is the weighted adjacency eigenvalue of G with multiplicity at least $m-1$.

(ii) Next, we suppose I is an independent set of G. Let us label the vertices of G in such a way that the first m vertices are the vertices of I. Thus the weighted adjacency matrix $A_f(G)$ can be written as:

$$A_f(G) = \begin{bmatrix} 0_{m\times m} & B_{m\times(n-m)} \\ (B_{m\times(n-m)})^T & C_{(n-m)\times(n-m)} \end{bmatrix}.$$

Using the same set of vectors $x_1, x_2, \ldots, x_{m-1}$, we can verify that 0 is an eigenvalue of $A_f(G)$ with multiplicity at least $m-1$. Hence, the proof of the theorem is complete. □

From Theorem 3.1, we can obtain the weighted adjacency eigenvalues for some well-known families of graphs.

Proposition 3.2 *Let G be a connected graph of order n and $f(x,y)$ be any real function. Then the following statements hold.*

(i) *If $G = K_n$, then the eigenvalues of the weighted adjacency matrix $A_f(G)$ are $f(n-1,n-1)(n-1)$ and $-f(n-1,n-1)$ with multiplicity $n-1$.*

(ii) *If $G = K_{s,t}$, with $n = s+t$ and $s,t \geq 1$, then the eigenvalues of the weighted adjacency matrix $A_f(G)$ are $f(s,t)\sqrt{st}$, $-f(s,t)\sqrt{st}$ and 0 with multiplicity $n-2$. In particular, if $G = S_n$, then $f(1,n-1)\sqrt{n-1}$, $-f(1,n-1)\sqrt{n-1}$ and 0 with multiplicity $n-2$ are the eigenvalues of $A_f(G)$.*

(iii) *If $G = K_n - e$, then the eigenvalues of the weighted adjacency matrix $A_f(G)$ are 0, $-f(n-1,n-1)$ with multiplicity $n-3$ and the zeros of the following polynomial*

$$\rho^2 - (n-3)f(n-1,n-1)\rho - 2(n-2)f^2(n-1,n-2).$$

(iv) *If $G = S_n + e$, then the eigenvalues of the weighted adjacency matrix $A_f(G)$ are $-f(2,2)$, 0 with multiplicity $n-4$ and the zeros of the following polynomial*

$$\rho^3 - f(2,2)\rho^2 - (2f^2(n-1,2) + (n-3)f^2(n-1,1))\rho + (n-3)f^2(n-1,1)f(2,2).$$

(v) *If $G = K_{n-1}$, then the eigenvalues of the weighted adjacency matrix $A_f(G_e)$ are 0, $-f(n-2,n-2)$ with multiplicity $n-4$ and the zeros of the following polynomial*

$$\rho^3 - (n-4)f(n-2,n-2)\rho^2 - 2(f^2(n-2,2) + (n-3)f^2(n-2,n-2))\rho$$

$$+2(n-4)f^2(n-2,2)f(n-2,n-2).$$

Proof. (i) From Theorem 3.1(i), it follows that $-f(n-1,n-1)$ is an eigenvalue of the weighted adjacency matrix $A_f(K_n)$ with multiplicity at least $n-1$. Since $trace(A_f(K_n)) = 0$, the remaining eigenvalue is $f(n-1,n-1)(n-1)$.

(ii) As graph $K_{s,t}$ consists of two independent sets of cardinalities s and t, where any two vertices from the same independent set share the same neighborhood. Using Theorem 3.1(ii), 0 is an eigenvalue of the weighted adjacency matrix $A_f(K_{s,t})$ with multiplicity at least $(s-1)+(t-1) = n-2$.

Let $X_1 = \{v_1, v_2, \ldots, v_s\}$ be the first partition class and $X_2 = \{v_{s+1}, v_{s+2}, \ldots, v_n\}$ be the second partition class. We get the quotient matrix

$$B = \begin{bmatrix} 0 & tf(s,t) \\ sf(s,t) & 0 \end{bmatrix}.$$

It is not difficult to check that this partition is regular. From Lemma 2.1, $f(s,t)\sqrt{st}$ and $-f(s,t)\sqrt{st}$ are the eigenvalues of $A_f(K_{s,t})$. Since $G = S_n$ is a special case of $G = K_{s,t}$. We omit the proof of $G = S_n$ here.

(iii) For convenience of discussion, we suppose $e = v_1v_2$ and $G = K_n - e$. Because v_1 is not adjacent to v_2 in G and $N_G(v_1) = N_G(v_2) = \{v_3, v_4, \ldots, v_n\}$. By Theorem 3.1(ii), 0 is an eigenvalue of the weighted adjacency matrix $A_f(K_n - e)$. In addition, the vertices in $\{v_3, v_4, \ldots, v_n\}$ form a clique of G. From Theorem 3.1(i), thus $-f(n-1, n-1)$ is an eigenvalue of the weighted adjacency matrix $A_f(K_n-e)$ with multiplicity at least $n-3$.

Next, we give a partition $\{v_1, v_2, \ldots, v_n\} = X_1 \cup X_2$, where $X_1 = \{v_1, v_2\}$ and $X_2 = \{v_3, v_4, \ldots, v_n\}$. Then the quotient matrix B of matrix $A_f(K_n - e)$ is

$$B = \begin{bmatrix} 0 & (n-2)f(n-1, n-2) \\ 2f(n-1, n-2) & (n-3)f(n-1, n-1) \end{bmatrix}.$$

This partition is regular. From Lemma 2.1, the other two eigenvalues of $A_f(K_n - e)$ are the eigenvalues of the following polynomial

$$det(\rho I - B) = \rho^2 - (n-3)f(n-1, n-1)\rho - 2(n-2)f^2(n-1, n-2).$$

(iv) Similarly, we let $e = v_1v_2$ and $G = S_n + e$. Without loss of generality, let v_3 is the central vertex of S_n, then $\{v_4, v_5, \ldots, v_n\}$ is an independent set and has a common neighbor v_3. Using Theorem 3.1(ii), we have that 0 is an eigenvalue of the weighted adjacency matrix $A_f(S_n + e)$ with multiplicity at least $n-4$. In addition, v_1 is adjacent to v_2. By Theorem 3.1(i), $-f(2,2)$ is an eigenvalue of the weighted adjacency matrix $A_f(S_n + e)$.

Now, let $X_1 = \{v_3\}$, $X_2 = \{v_1, v_2\}$ and $X_3 = \{v_4, v_5, \ldots, v_n\}$. Then the quotient matrix B of the weighted adjacency matrix $A_f(S_n + e)$ is

$$B = \begin{bmatrix} 0 & 2f(n-1, 2) & (n-3)f(n-1, 1) \\ f(n-1, 2) & f(2,2) & 0 \\ f(n-1, 1) & 0 & 0 \end{bmatrix}.$$

It is not difficult to see that this partition is regular. From Lemma 2.1, the remaining three eigenvalues of $A_f(S_n + e)$ are the eigenvalues of the following polynomial

$$det(\rho I - B) = \rho^3 - f(2,2)\rho^2 - (2f^2(n-1, 2) + (n-3)f^2(n-1, 1))\rho + (n-3)f^2(n-1, 1)f(2,2).$$

(v) Let us suppose that the subdivision of an edge $e = v_1v_2$ of K_{n-1} produces a graph G_e. Since the neighbours of v_1 and v_2 are $\{v_3, v_4, \ldots, v_{n-1}\}$ and $v_1v_2 \notin$

$E(G)$, using Theorem 3.1(ii), $A_f(G_e)$ has an eigenvalue 0. Besides, the vertices in $\{v_3, v_4, \ldots, v_{n-1}\}$ form a clique of G and $N_G(v_i) - v_j = N_G(v_j) - v_i$ for all $i, j \in \{3, 4, \ldots, n-1\}$. By Theorem 3.1(i), $A_f(G_e)$ has an eigenvalue $-f(n-2, n-2)$ with multiplicity at least $n-4$.

Next, we give a partition $\{v_1, v_2, \ldots, v_n\} = X_1 \cup X_2 \cup X_3$, where $X_1 = \{v_1, v_2\}$, $X_2 = \{v_n\}$ and $X_3 = \{v_3, v_4, \ldots, v_{n-1}\}$. Then the quotient matrix B of matrix $A_f(G_e)$ is

$$B = \begin{bmatrix} 0 & f(n-2,2) & (n-3)f(n-2,n-2) \\ 2f(n-2,2) & 0 & 0 \\ 2f(n-2,n-2) & 0 & (n-4)f(n-2,n-2) \end{bmatrix}.$$

Because each block of $A_f(G_e)$ has a constant row (and column) sum, this partition is regular. From Lemma 2.1, the eigenvalues of B are the eigenvalues of $A_f(G_e)$. By calculating, we have

$$det(\rho I - B) = \rho^3 - (n-4)f(n-2,n-2)\rho^2 - 2(f^2(n-2,2) + (n-3)f^2(n-2,n-2))\rho$$
$$+2(n-4)f^2(n-2,2)f(n-2,n-2).$$

This completes the proof. □

Next, we will give the interlacing results of weighted adjacency eigenvalues under edge subdivision, vertex deletion and vertex contraction, respectively.

Theorem 3.3 *Let G be a graph of order n and $H = G_e$, where $e = uv$ is an edge of G. Let $f(x,y)$ be any symmetric real function and the edge-weight $f(d_i, d_j) \geq 0$ for any edge $v_iv_j \in E(G)$. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n+1}$$

are the eigenvalues of $A_f(G)$ and $A_f(H)$, respectively, then

$$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, \ldots, n+1,$$

where $\lambda_i = +\infty$ for each $i \leq 0$ and $\lambda_i = -\infty$ for each $i \geq n+1$.

In particular, depending on the sign of λ_i, some of the above inequalities can be strengthened. Let k be such that $\lambda_k \geq 0$ and $\lambda_{k+1} < 0$. Then

$$\theta_i \geq \lambda_i \quad \text{for each } i = k+1, k+2, \ldots, n.$$

Proof. For the convenience of discussion, suppose $e = v_1v_2$ is an edge of G and $H = G_e$. Removing the row and column associated with vertex v_1 from $A_f(G)$, we get an $(n-1) \times (n-1)$ matrix B. Let

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$$

be the eigenvalues of B. It is clear that B is a principal submatrix of $A_f(G)$. Using Lemma 2.2, we get

$$\lambda_i \geq \beta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, \ldots, n-1.$$

Furthermore, we know $E(H) = \{E(G) - v_1v_2\} \cup \{v_1v_{n+1}, v_2v_{n+1}\}$. If we add the row and column associated with vertex v_{n+1} to $A_f(G)$ and delete the 12, 21-entries in $A_f(G)$, then we can have $A_f(H)$. Thus B is still a principal submatrix of $A_f(H)$. From Lemma 2.2, we obtain

$$\theta_i \geq \beta_i \geq \theta_{i+2} \quad \text{for each } i = 1, 2, \ldots, n-1.$$

From the above inequalities, we directly have

$$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, \ldots, n+1,$$

where $\lambda_i = +\infty$ for each $i \leq 0$ and $\lambda_i = -\infty$ for each $i \geq n+1$.

In particular, let $A_f^{'}(G)$ be an $(n+1) \times (n+1)$ matrix, which is obtained by adding a zero row and a zero column to $A_f(G)$. The eigenvalues of $A_f^{'}(G)$ differ from those of $A_f(G)$ only in that $A_f^{'}(G)$ has an additional zero eigenvalue. Let

$$\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n+1}$$

be the eigenvalues of $A_f^{'}(G)$. By properly labelling the vertices of H, we can get a matrix $B = A_f^{'}(G) - A_f(H)$, written as follows:

$$B = \begin{bmatrix} 0 & f(d_1,d_2) & -f(d_1,2) & 0 & \cdots & 0 \\ f(d_1,d_2) & 0 & -f(d_2,2) & 0 & \cdots & 0 \\ -f(d_1,2) & -f(d_2,2) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By calculating, we have $det(\rho I - B) = \rho^{n-3}(\rho^3 - (f^2(d_1,d_2) + f^2(d_1,2) + f^2(d_2,2))\rho - 2f(d_1,d_2)f(d_1,2)f(d_2,2))$. Because $2f(d_1,d_2)f(d_1,2)f(d_2,2) = \rho_1(B)\rho_2(B)\rho_3(B)$ is a nonnegative number and the trace of B is 0, we can claim that matrix B has at most one positive eigenvalue.

Since $A_f^{'}(G)$ and $A_f(H)$ are symmetric matrices, using the inequalities (2.1) in Lemma 2.3, we can get

$$\xi_i \leq \theta_{i-1} \quad \text{for each } i = 1, 2, \ldots, n+1,$$

where $\theta_0 = +\infty$. Recall that the eigenvalues of $A_f(G)$ and $A_f^{'}(G)$ differ only in that the latter set includes an additional zero eigenvalue. That is $\lambda_i = \xi_{i+1}$ for $i = k+1, k+2, \ldots, n$. Thus we have

$$\theta_i \geq \lambda_i \quad \text{for each } i = k+1, k+2, \ldots, n.$$

The proof is thus complete. $\square$

The interlacing inequalities in Theorem 3.3 are the same as that in Theorem 1.5.

In the above theorem, we cannot improve the gap on the left hand side by reducing it from 2 to 1, when $f(2,2) > 0$. In fact, considering the cycle, let $G = C_{n-1}$ with $n \geq 4$ and $H = C_n$. It is widely known that the adjacency eigenvalues of C_n are $2\cos\frac{2\pi j}{n}$ with $j = 0, 1, \ldots, n-1$. Since the degree of each vertex of C_n is 2. We can directly get the weighted adjacency eigenvalues of C_n: $2f(2,2)\cos\frac{2\pi j}{n}$ with $j = 0, 1, \ldots, n-1$. Based on the distribution of their weighted adjacency eigenvalues, we have $\lambda_2 = 2f(2,2)\cos\frac{2\pi}{n-1}$ and $\theta_3 = 2f(2,2)\cos\frac{2\pi}{n}$, where $f(2,2) > 0$. Since $\pi \geq \frac{2\pi}{n-1} > \frac{2\pi}{n}$ for $n \geq 4$, then $\cos\frac{2\pi}{n-1} < \cos\frac{2\pi}{n}$, thus we have $\lambda_2 \ngeq \theta_3$. For nearly all of the indices in Tables 1 and 2, apart from the Albertson index and sigma index, we cannot improve the gap on the left hand side in Theorem 3.3.

If $f(x,y)$ is increasing in the variable x, we cannot improve the gap on the right hand side by reducing it from 1 to 0. In fact, considering the complete graph, let $G = K_{n-1}$ with $n \geq 5$ and $H = G_e$. From Proposition 3.2 (i), we have $\lambda_1(A_f(K_{n-1})) = (n-2)f(n-2, n-2)$. Recall that the edge-weight $f(d_i, d_j) \geq 0$ for any edge $v_i v_j \in E(G)$ and $f(x,y)$ is increasing in the variable x, we have $f(n-2, 2) > 0$ and $f(n-2, n-2) > 0$. From the proof of Proposition 3.2 (v), since $\rho_1(B)\rho_2(B)\rho_3(B) = det(B) = -2(n-4)f^2(n-2,2)f(n-2,n-2) < 0$ and the trace of B is $(n-4)f(n-2, n-2) > 0$, we can claim that the matrix B has two positive eigenvalues and one negative eigenvalue. Now let

$$g(\rho) = \rho^3 - (n-4)b\rho^2 - 2(a^2 + (n-3)b^2)\rho + 2(n-4)a^2 b,$$

where $a = f(n-2, 2)$ and $b = f(n-2, n-2)$. The polynomial $g(\rho)$ has three zeros such that $\rho_1 > 0, \rho_2 > 0$ and $\rho_3 < 0$. By calculation, we have $g(\lambda_1) = g((n-2)b) = 2nb^3 - 4a^2b - 4b^3 > 2nb^3 - 8b^3 > 0$. Hence we can reduce it to two possibilities: either $\lambda_1 > \rho_1 = \theta_1$ or $\rho_2 > \lambda_1 > 0$. If $\rho_2 > \lambda_1 = (n-2)b$, then $\rho_1 + \rho_2 > 2(n-2)b$. Because $trace(B) = \rho_1 + \rho_2 + \rho_3 = (n-4)b$, we have $\rho_3 < -nb$. Thus $g(-nb) > 0$. However, it is not difficult to calculate that $g(-nb) = 4na^2b - 8a^2b - 2b^3n^3 + 6b^3n^2 - 6b^3n = 2nb(2a^2 - 3b^2) + b^3n^2(6-2n) - 8a^2b < 0$, where $b > a > 0$ and $n \geq 5$. This is a contradiction. We finally have $\theta_1 \ngeq \lambda_1$. Thus this result is good enough for the first Zagreb index, second Zagreb index, first hyper-Zagreb index, second hyper-Zagreb index, reciprocal Randić index, reciprocal sum-connectivity index, forgotten index, inverse sum index, first Gourava index, second Gourava index, first hyper-Gourava index, second hyper-Gourava index, product-connectivity Gourava index and Sombor index in Table 1 and the exponential first Zagreb index and exponential second Zagreb index in Table 2.

When we delete a vertex v from a graph G, the degree of each vertex $v_i \in N_G[v]$ is changed. Thus the $(n-1)\times(n-1)$ principal submatrix of $A_f(G)$ may no longer be the weighted adjacency matrix of a subgraph. We can not get the same interlacing result as that in Theorem 1.6. In general, we can get the following result for the weighted adjacency matrix $A_f(G)$.

Theorem 3.4 *Let G be a graph of order n and $H = G - v_1$. The degree of the vertex v_1 be d_1. Let $f(x,y)$ be any symmetric real function. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1}$$

are the eigenvalues of $A_f(G)$ and $A_f(H)$, respectively, then

$$\lambda_{i-d_1} \geq \theta_i \geq \lambda_{i+d_1+1} \quad \text{for each } i = 1, 2, \ldots, n-1,$$

where $\lambda_i = +\infty$ for each $i \leq 0$ and $\lambda_i = -\infty$ for each $i \geq n+1$.

In particular, depending on the sign of θ_i, the above inequalities can be strengthened in one of two ways. Let k be such that $\theta_k \geq 0$ and $\theta_{k+1} < 0$. Then

$$\theta_i \geq \lambda_{i+d_1} \quad \text{for each } i = 1, 2, \ldots, k$$

and

$$\lambda_{i-d_1+1} \geq \theta_i \quad \text{for each } i = k+1, k+2, \ldots, n-1.$$

Proof. Suppose $H = G - v_1$ and the degree of the vertex v_1 is d_1. Then the matrix $A_f(G)$ and $A_f(H)$ must have a same $(n-d_1-1)\times(n-d_1-1)$ principal submatrix. In addition, the matrix $A'_f(H)$ be obtained by adding a zero row and zero column to $A_f(H)$. The eigenvalues of $A'_f(H)$ are denoted by

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n.$$

Let $B = A'_f(H) - A_f(G)$ be a symmetric matrix, which has an $(n-d_1-1)\times(n-d_1-1)$ zero principal submatrix. Let us firstly label the vertices in $N_G[v_1]$ of G, so that the matrix B can be written as follows:

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} & \cdots & b_{1(d_1+1)} & 0 & \cdots & 0 \\ b_{12} & 0 & b_{23} & b_{24} & \cdots & b_{2(d_1+1)} & b_{2(d_1+2)} & \cdots & b_{2n} \\ b_{13} & b_{23} & 0 & b_{34} & \cdots & b_{3(d_1+1)} & b_{3(d_1+2)} & \cdots & b_{3n} \\ b_{14} & b_{24} & b_{34} & 0 & \cdots & b_{4(d_1+1)} & b_{4(d_1+2)} & \cdots & b_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{1(d_1+1)} & b_{2(d_1+1)} & b_{3(d_1+1)} & b_{4(d_1+1)} & \cdots & 0 & b_{(d_1+1)(d_1+2)} & \cdots & b_{(d_1+1)n} \\ 0 & b_{2(d_1+2)} & b_{3(d_1+2)} & b_{4(d_1+2)} & \cdots & b_{(d_1+1)(d_1+2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{2n} & b_{3n} & b_{4n} & \cdots & b_{(d_1+1)n} & 0 & \cdots & 0 \end{bmatrix},$$

where b_{ij} are real numbers.

Now we assume that $B = B_1 + B_2 + \cdots + B_{d_1}$, where

$$B_i = \left[\begin{array}{ccc} & b_{1(i+1)} & \\ & 0 & \\ 0_{i\times i} & 0 & 0_{i\times(n-i-1)} \\ & \vdots & \\ b_{1(i+1)} \quad 0 \quad 0 \quad \cdots & 0 & b_{(i+1)(i+2)} \quad b_{(i+1)(i+3)} \quad \cdots \quad b_{(i+1)n} \\ & b_{(i+1)(i+2)} & \\ 0_{(n-i-1)\times i} & b_{(i+1)(i+3)} & 0_{(n-i-1)\times(n-i-1)} \\ & \vdots & \\ & b_{(i+1)n} & \end{array}\right].$$

By calculating, we have $det(\rho I - B_i) = \rho^{n-2}(\rho^2 - (b^2_{1(i+1)} + b^2_{(i+1)(i+2)} + \cdots + b^2_{(i+1)n}))$. Thus the eigenvalues of B_i are $\rho_1(B_i) = \sqrt{b^2_{1(i+1)} + b^2_{(i+1)(i+2)} + \cdots + b^2_{(i+1)n}}$, $\rho_n(B_i) = -\sqrt{b^2_{1(i+1)} + b^2_{(i+1)(i+2)} + \cdots + b^2_{(i+1)n}}$ and $\rho_2(B_i) = \rho_3(B_i) = \cdots = \rho_{n-1}(B_i) = 0$, for $i = 1, 2, \ldots, d_1$.

Since $A_f(G)$ and B are real symmetric matrices, from the inequalities (2.1) in Lemma 2.3, we can obtain

$$\begin{aligned} \beta_i = \rho_i(A_f(G) + B) &= \rho_i(A_f(G) + B_1 + B_2 + \cdots + B_{d_1}) \\ &\le \rho_{i-1}(A_f(G) + B_1 + B_2 + \cdots + B_{d_1-1}) \\ &\le \rho_{i-2}(A_f(G) + B_1 + B_2 + \cdots + B_{d_1-2}) \\ &\quad\vdots \\ &\le \rho_{i-d_1}(A_f(G)) \\ &= \lambda_{i-d_1}. \end{aligned}$$

Similarly, using the inequalities (2.2) in Lemma 2.3, we can have $\beta_i \ge \lambda_{i+d_1}$. This means that

$$\lambda_{i-d_1} \ge \beta_i \ge \lambda_{i+d_1}.$$

The eigenvalues of $A'_f(H)$ differ from those of $A_f(H)$ only in that $A'_f(H)$ has an additional zero eigenvalue. Since $\theta_k \ge 0$ and $\theta_{k+1} < 0$, we have that $\theta_i = \beta_i$ for each $i = 1, 2, \ldots, k$ and $\theta_i = \beta_{i+1}$ for each $i = k+1, k+2, \ldots, n-1$. Hence it is not difficult for us to get that

$$\lambda_{i-d_1} \geq \theta_i \geq \lambda_{i+d_1+1} \quad \text{for each } i = 1, 2, \ldots, n-1,$$

where $\lambda_i = +\infty$ for each $i \leq 0$ and $\lambda_i = -\infty$ for each $i \geq n+1$.

In particular, depending on the sign of θ_i, we have

$$\theta_i \geq \lambda_{i+d_1} \quad \text{for each } i = 1, 2, \ldots, k$$

and

$$\lambda_{i-d_1+1} \geq \theta_i \quad \text{for each } i = k+1, k+2, \ldots, n-1.$$

The required result is thus obtained. □

A vertex of degree 0 is called *isolated*. If v_1 is an isolated vertex of G and $H = G - v_1$, then $\lambda_i = \theta_i$ for $i = 1, 2, \ldots, k$ and $\theta_i = \lambda_{i+1}$ for $i = k+1, k+2, \ldots, n-1$. Thus we have $\lambda_i \geq \theta_i \geq \lambda_{i+1}$ for each $i = 1, 2, \ldots, n-1$. This is the case in Theorem 3.4 with $d_1 = 0$.

In the above theorem, when $f(x, y)$ be a symmetric real function that is increasing in the variable x, we cannot improve the gap on the right hand side by reducing it from d_1 to $d_1 - 1$ when $i = 1, 2, \ldots, k$. In fact, considering the star, let $G = S_n$ with $n \geq 4$ and $H = S_{n-1}$. From Proposition 3.2 (ii), if $G = S_n$, then $\lambda_1 = f(1, n-1)\sqrt{n-1}$, $\lambda_n = -f(1, n-1)\sqrt{n-1}$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = 0$. And the weighted adjacency eigenvalues of H are $\theta_1 = f(1, n-2)\sqrt{n-2}$, $\theta_n = -f(1, n-2)\sqrt{n-2}$ and $\theta_2 = \theta_3 = \cdots = \theta_{n-1} = 0$. We can easily have $\theta_1 = f(1, n-2)\sqrt{(n-2)} \ngeq \lambda_1 = f(1, n-1)\sqrt{(n-1)}$.

If $f(x, y)$ be a symmetric real function that is increasing in the variable x, we cannot improve the gap on the left hand side by reducing it from $d_1 - 1$ to $d_1 - 2$ when $i = k, k+1, \ldots, n-1$. In fact, considering the complete graph, let $G = K_n$ with $n \geq 3$ and $H = K_{n-1}$. From Proposition 3.2 (i), if $G = K_n$, the eigenvalues of $A_f(G)$ are $\lambda_1 = (n-1)f(n-1, n-1)$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -f(n-1, n-1)$. And the weighted adjacency eigenvalues of H are $\theta_1 = (n-2)f(n-2, n-2)$ and $\theta_2 = \theta_3 = \cdots = \theta_{n-1} = -f(n-2, n-2)$. We can verify that $\lambda_2 = -f(n-1, n-1) \ngeq \theta_{n-1} = -f(n-2, n-2)$.

Thus for the first Zagreb index, second Zagreb index, first hyper-Zagreb index, second hyper-Zagreb index, reciprocal Randić index, reciprocal sum-connectivity index, forgotten index, inverse sum index, first Gourava index, second Gourava index, first hyper-Gourava index, second hyper-Gourava index, product-connectivity Gourava index and Sombor index in Table 1 and the exponential first Zagreb index and exponential second Zagreb index in Table 2, we cannot improve the gap on the right hand side by reducing it from d_1 to $d_1 - 1$ when $i = 1, 2, \ldots, k$ and the gap on the left hand side by reducing it from $d_1 - 1$ to $d_1 - 2$ when $i = k+1, k+2, \ldots, n-1$.

A vertex v of G is said to be a *pendant vertex* if $d_i = 1$. An edge of G is said to be pendant if one of its end-vertices is a pendant vertex.

Corollary 3.5 *Let G be a graph of order n and $H = G - v_1$, where v_1 is a pendant vertex. Let $f(x,y)$ be any symmetric real function. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1}$$

are the eigenvalues of $A_f(G)$ and $A_f(H)$, respectively, then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2} \quad \text{for each } i = 1, 2, \ldots, n-1,$$

where $\lambda_0 = +\infty$ and $\lambda_{n+1} = -\infty$.

In particular, depending on the sign of θ_i, the above inequalities can be strengthened in one of two ways. Let k be such that $\theta_k \geq 0$ and $\theta_{k+1} < 0$. Then

$$\theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, \ldots, k$$

and

$$\lambda_i \geq \theta_i \quad \text{for each } i = k+1, k+2, \ldots, n-1.$$

If $e = uv$ is a pendant edge in G and v is a pendant vertex, we have $G_e - v = G$. Thus, using Corollary 3.5, we have a special case that e is a pendant edge in Theorem 3.3.

We give the last interlacing result associated with the vertex contraction for the weighted adjacency matrix.

Theorem 3.6 *Let G be a graph of order n and $H = G_{\{u,v\}}$, where u and v be two distinct vertices of G, such that either $N_G(u) \cap N_G(v) = \emptyset$ or $d_i = 2$ for $v_i \in N_G(u) \cap N_G(v)$. Let $f(x,y)$ be any symmetric real function. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1}$$

are the eigenvalues of $A_f(G)$ and $A_f(H)$, respectively, then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2} \quad \text{for each } i = 1, 2, \ldots, n-1,$$

where $\lambda_0 = +\infty$ and $\lambda_{n+1} = -\infty$.

Proof. Firstly, removing the rows and columns associated with u and v from $A_f(G)$, we get a matrix B. Let

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-2}$$

be the eigenvalues of B. Because the matrix B is an $(n-2) \times (n-2)$ principal submatrix of $A_f(G)$, using Lemma 2.2, we get

$$\lambda_i \geq \beta_i \geq \lambda_{i+2} \quad \text{for each } i = 1, 2, \ldots, n-2.$$

Furthermore, in the process of contraction, we know that only the row and column associated with x_{uv} are changed. So, B is still an $(n-2)\times(n-2)$ principal submatrix of $A_f(H)$. By Lemma 2.2, we have

$$\theta_i \geq \beta_i \geq \theta_{i+1} \quad \text{for each } i = 1, 2, \ldots, n-2.$$

Thus, it is not difficult for us to get the following inequalities:

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2} \quad \text{for each } i = 1, 2, \ldots, n-1,$$

where $\lambda_0 = +\infty$ and $\lambda_{n+1} = -\infty$. Hence the theorem holds. $\square$

If $e = uv$ is a pendant edge and v is a pendant vertex, we have $G_{\{u,v\}} = G - v$. Hence we can get the same conclusion as Corollary 3.5 when we contract a pair of vertices $u, v \in V(G)$ such that v is a pendant vertex.

In the above theorem, when $f(x, y)$ is a symmetric real function that is increasing in the variable x, we cannot improve the gap on the right hand side by reducing it from 2 to 1 and the gap on the left hand side by reducing it from 1 to 0. In fact, considering the star, let $G = S_n \cup S_n$ and $H = S_{2n-1}$, that is we contract the two central vertices of the two disjoint stars. From Proposition 3.2 (ii), the eigenvalues of $A_f(G)$ are $\lambda_1 = \lambda_2 = f(1, n-1)\sqrt{n-1}$, $\lambda_{2n-1} = \lambda_{2n} = -f(1, n-1)\sqrt{n-1}$ and $\lambda_3 = \lambda_4 = \cdots = \lambda_{2n-2} = 0$. Similarly, we have $\theta_1 = f(1, 2n-2)\sqrt{2n-2}$, $\theta_{2n-1} = -f(1, 2n-2)\sqrt{2n-2}$ and $\theta_2 = \theta_3 = \cdots = \theta_{2n-2} = 0$ are the eigenvalues of $A_f(H)$. Now, we can say that $\theta_{2n-1} = -f(1, 2n-2)\sqrt{(2n-2)} \ngeq \lambda_{2n} = -f(1, n-1)\sqrt{(n-1)}$ and $\lambda_1 = f(1, n-1)\sqrt{(n-1)} \ngeq \theta_1 = f(1, 2n-2)\sqrt{(2n-2)}$.

Hence for the first Zagreb index, second Zagreb index, first hyper-Zagreb index, second hyper-Zagreb index, reciprocal Randić index, reciprocal sum-connectivity index, forgotten index, inverse sum index, first Gourava index, second Gourava index, first hyper-Gourava index, second hyper-Gourava index, product-connectivity Gourava index and Sombor index in Table 1 and the exponential first Zagreb index and exponential second Zagreb index in Table 2, we cannot improve the gap in Theorem 3.6.

To end this paper, we summarize our results in the following table to show the difference and similarity of the main eigenvalues interlacing results for the adjacency matrix $A(G)$ and the weighted adjacency matrix $A_f(G)$ under the operations of edge deletion, edge subdivision, vertex deletion and vertex contraction.

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	$H=G-e$	$H=G_e$	$H=G-v$	$H=G_{\{u,v\}}$
$A(G)$	$\lambda_{i-1}\geq\theta_i\geq\lambda_{i+1}$	$\lambda_{i-2}\geq\theta_i\geq\lambda_{i+1}$	$\lambda_i\geq\theta_i\geq\lambda_{i+1}$	$\lambda_{i-1}\geq\theta_i\geq\lambda_{i+2}$
$A_f(G)$	$\lambda_{i-2}\geq\theta_i\geq\lambda_{i+2}$	$\lambda_{i-2}\geq\theta_i\geq\lambda_{i+1}$	$\lambda_{i-d_1}\geq\theta_i\geq\lambda_{i+d_1+1}$	$\lambda_{i-1}\geq\theta_i\geq\lambda_{i+2}$

Table 3: Difference and similarity of the interlacing results for $A(G)$ and $A_f(G)$.

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References

[1] A. Bharali, A. Doley, B. Deka, Misbalance degree matrix and related energy of graphs, *J. Math. Comput. Sci.* 10(2020) 436–447.

[2] A. Bharali, A. Mahanta, I.J. Gogoi, A. Doley, Inverse sum indeg index and ISI matrix of graphs, *J. Discrete. Math. Sci. Cryptogr.* 23(2020) 1315–1333.

[3] S. Butler, Interlacing for weighted graphs using the normalized Laplacian, *Electron. J. Linear Algebra* 16(2007) 90–98.

[4] X. Chen, On ABC eigenvalues and ABC energy, *Linear Algebra Appl.* 544(2018) 141–157.

[5] Y. Chen, D. Li, J. Meng, Nordhaus-Gaddum type inequalities of the second A_α-eigenvalue of a graph, *Linear Algebra Appl.* 602 (2020) 57–72.

[6] D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, New York, 2010.

[7] K. Das, I. Gutman, I. Milovanović, E. Milovanović, B. Furtula, Degree-based energies of graphs, *Linear Algebra Appl.* 554(2018) 185–204.

[8] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* 86(2021) 11–16.

[9] W.H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* 226–228(1995) 593–616.

[10] F. Hall, K. Patel, M. Stewart, Interlacing results on matrices associated with graphs, *J. Combin. Math. Combin. Comput.* 68(2009) 113–127.

[11] R.A. Horn, C.R. Johnson, Matrix Analysis, Universitext, New York, 2013.

[12] X. Li, Indices, polynomials and matrices - a unified viewpoint, *Invited talk at the 8th Slovinian Conf. Graph Theory*, Kranjska Gora, June 21–27, 2015.

[13] X. Li, Z. Wang, Trees with extremal spectral radius of weighted adjacency matrices among trees weighted by degree-based indices, *Linear Algebra Appl.* 620(2021) 61–75.

[14] X. Li, N. Yang, Unified approach for spectral properties of weighted adjacency matrices for graphs with degree-based edge-weights, submitted 2022.

[15] R. Liu, W. Shiu, General Randić matrix and general Randić incidence matrix, *Discrete Appl. Math.* 186(2015) 168–175.

[16] S. Pirzada, B.A. Rather, M. Aouchiche, On eigenvalues and energy of geometric-arithmetic matrix of graphs, *Medit. J. Math.* 19(2022) 115.

[17] G. Porto, L.E. Allem, Eigenvalue interlacing in graphs, *Proc. Ser. Brazil. Soc. Appl. Comput. Math* 5(2017) 1–7.

[18] N.J. Rad, A. Jahanbani, I. Gutman, Zagreb energy and Zagreb Estrada index of graphs, *MATCH Commun. Math. Comput. Chem.* 79(2018) 371–386.

[19] J. Rada, Exponential vertex-degree-based topological indices and discrimination, *MATCH Commun. Math. Comput. Chem.* 82(2019) 29–41.

[20] B.A. Rather, M. Aouchiche, M. Imran, S. Pirzada, On arithmetic-geometric eigenvalues of graphs, *Main Group Metal Chem.* 45(2022) 111–123.

[21] J.M. Rodríguez, J.M. Sigarreta, Spectral properties of geometric-arithmetic index, *Appl. Math. Comput.* 277(2016) 142–153.

[22] J. Wang, F. Belardo, A note on the signless Laplacian eigenvalues of graphs, *Linear Algebra Appl.* 435(2011) 2585–2590.

[23] H. Weyl, Das asymptotische verteilungsgesetz der eigenwerte linearer partieller differentialgleichungen, *Math. Ann.* 71(1912) 441–479.

[24] B. Wu, J. Shao, Y. Liu, Interlacing eigenvalues on some operations of graphs, *Linear Algebra Appl.* 430(2009) 1140–1150.

[25] H. You, M. Yang, W. So, W. Xi, On the spectrum of an equitable quotient matrix and its application, *Linear Algebra Appl.* 577(2019) 21–40.