Proper Orientation, Proper Biorientation and Semi-proper Orientation Numbers of Graphs

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Abstract

An orientation D of G is proper if for every $xy \in E(G)$, we have $d^-_D(x) \neq d^-_D(y)$. An orientation D is a p-orientation if the maximum in-degree of a vertex in D is at most p. The minimum integer p such that G has a proper p-orientation is called the proper orientation number $\text{pon}(G)$ of G (introduced by Ahadi and Dehghan in 2013). We introduce a proper biorientation of G, where an edge xy of G can be replaced by either arc xy or arc yx or both arcs xy and yx. Similarly to $\text{pon}(G)$, we can define the proper biorientation number $\text{pbon}(G)$ of G using biorientations instead of orientations. Clearly, $\text{pbon}(G) \leq \text{pon}(G)$ for every graph G. We compare $\text{pbon}(G)$ with $\text{pon}(G)$ for various classes of graphs. We show that for trees T, the tight bound $\text{pon}(T) \leq 4$ extends to the tight bound $\text{pbon}(T) \leq 4$ and for cacti G, the tight bound $\text{pon}(G) \leq 7$ extends to the tight bound $\text{pbon}(G) \leq 7$. We also prove that there is an infinite number of trees T for which $\text{pbon}(T) < \text{pon}(T)$.

Let (H, w) be a weighted digraph with a weight function $w : A(H) \rightarrow \mathbb{Z}_+$. The *in-weight* $w^-_H(v)$ of a vertex v of H is the sum of the weights of arcs towards v. A *semi-proper p-orientation* (D, w) of an undirected graph G is an orientation D of G together with a weight function $w : A(D) \rightarrow \mathbb{Z}_+$, such that the in-weight of any adjacent vertices are distinct and $w^-_D(v) \leq p$ for every $v \in V(D)$. The *semi-proper orientation number* $\text{spon}(G)$ of a graph G (introduced by Dehghan and Havet in 2021) is the minimum p such that G has a semi-proper p-orientation (D, w) of G. We prove that $\text{spon}(G) \leq \text{pbon}(G)$ and characterize graphs G for which $\text{spon}(G) = \text{pbon}(G)$.

1 Introduction

In this paper we introduce a new graph parameter, the proper biorientation number, and show some of its basic properties. The introduction of this parameter was motivated by a recent paper by Dehghan and Havet [9] on semi-proper orientations of graphs. To define these notions, we need some basic notation. For a digraph D and vertex $x \in V(D)$, the *in-neighborhood* of x is $N^-_D(x) = \{y \in V(D) : \ yx \in A(D)\}$ and the *in-degree* of x is $d^-_D(x) = |N^-_D(x)|$. We will often omit the subscript D when D is clear from the context.

An orientation D of G is *proper* if for every $xy \in E(G)$, we have $d^-_D(x) \neq d^-_D(y)$. An orientation D is a *p-orientation* if the maximum in-degree of a vertex in D is at most p. The minimum integer p such that G has a proper p-orientation is called the *proper orientation number* $\mathrm{pon}(G)$ of G. This graph parameter was introduced by Ahadi and Dehghan [1]. They observed that this parameter is well-defined for any graph G since one can always obtain a proper $\Delta(G)$-orientation, by sorting all vertices by degree and orienting all edges forward. (Here $\Delta(G)$ is the maximum degree of G). The parameter has been widely investigated, see e.g. [1, 2, 3, 4, 5, 8, 12, 13].

Let (H, w) be a weighted digraph with a weight function $w : A(H) \to \mathbb{Z}_+$. The *in-weight* $w^-_H(v)$ of a vertex v of H is the sum of the weights of arcs towards v. A *semi-proper p-orientation* (D, w) of an undirected graph G is an orientation D of G together with a weight function $w : A(D) \to \mathbb{Z}_+$, such that the in-weight of any adjacent vertices are distinct and $w^-_D(v) \leq p$ for every $v \in V(D)$. The *semi-proper orientation number* $\mathrm{spon}(G)$ of a graph G is the minimum p such that G has a semi-proper p-orientation (D, w) of G. This parameter was introduced by Dehghan and Havet [9] and studied also in [10, 11]. It was proved in [9] that for every graph G there is a semi-proper $\mathrm{spon}(G)$-orientation in which the weight of each edge in G is 1 or 2. This shows that there is an equivalent definition of a semi-proper orientation, where we can only replace an edge xy of G either by one arc with end-vertices x and y or by two arcs between x and y, both directed either from x to y or from y to x.

Dehghan's theorem and the equivalent definition of a semi-proper orientation above lead us to the following natural extension of a proper orientation. Let G be a graph. A *biorientation* of G is a digraph D obtained from G by replacing every edge xy by arc xy either arc yx, or two mutually opposite arcs xy, yx [7]. An arc xy of D is called *single* if there is no arc yx. Thus, a biorientation D is an *orientation* if all arcs of D are single. One can define a *proper biorientation* and *p-biorientation* in absolutely the same way as a proper orientation and a p-orientation. The minimum integer p such that G has a proper p-biorientation is called the *proper biorientation number* $\mathrm{pbon}(G)$ of G. Note that for any graph G, $\mathrm{pbon}(G) \leq \mathrm{pon}(G)$.

In Section 2, we compare $\mathrm{pbon}(G)$ with $\mathrm{pon}(G)$ for various classes of graphs. In Subsection 2.1, we compare $\mathrm{pbon}(T)$ with $\mathrm{pon}(T)$ for trees T. Araújo et al. [4] proved that for every tree T, we have $\mathrm{pon}(T) \leq 4$ and this bound is tight. It follows from $\mathrm{pbon}(G) \leq \mathrm{pon}(G)$ that $\mathrm{pbon}(T) \leq 4$. We prove that the last bound is also tight. The fact that for trees the tight upper bound on proper orientation number coincides with that on proper biorientation number does not mean that the two numbers are equal on trees. We show that there is a tree T^* such that $\mathrm{pbon}(T^*) = 3$ and $\mathrm{pon}(T^*) = 4$. We extend this result by showing that there is an infinite number of trees for which the two numbers are not equal.

Araujo et al. [5] proved that $\mathrm{pon}(G) \leq 7$ for every cactus G and the bound is tight. In Subsection 2.2 we prove that the bound remains tight for proper biorientation number on cacti as well. It is natural to ask when $\mathrm{pbon}(G) = \mathrm{pon}(G)$. While we are unable to give a complete answer, in Subsection 2.3 we show that $\mathrm{pbon}(G) = \mathrm{pon}(G)$ for every graph with $\mathrm{pon}(G) \leq 3$. The

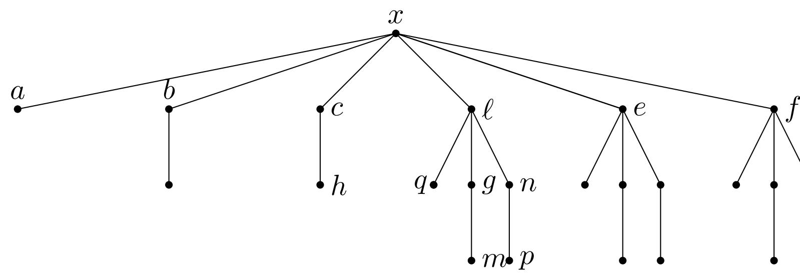

Figure 1: R_3

previously discussed result on T^* shows that we cannot replace 3 by 4 in the last inequality.

In Section 3, we prove that $\text{spon}(G) \leq \text{pbon}(G)$ for every graph G. Thus, for every graph G,

$$\text{spon}(G) \leq \text{pbon}(G) \leq \text{pon}(G). \quad (1)$$

We also characterize when $\text{spon}(G) = \text{pbon}(G)$ for a graph G.

We conclude the paper by discussing a number of open problems.

2 pbon(G) vs pon(G)

The following simple observation will be useful in some proofs below.

Observation 1. *Let D be a proper biorientation of G and let xy and yx be two mutually opposite arcs. If $d^-(y) = 1$, then we can always obtain a new proper biorientation such that $d^-(y) = 0$ by removing the arc xy.*

We will use the notion of an x-pendant subgraph. Let H be an induced proper subgraph of G and let $x \in V(H)$. The subgraph H is *x-pendant* if there is no edges between $V(H) - \{x\}$ and $V(G) - V(H)$.

2.1 pbon(G) vs pon(G) for trees

Theorem 1. *[4] If T is a tree, then* $\text{pon}(T) \leq 4$*, and this bound is tight.*

Araujo et al. [4] proved that for the tree R_3 in Fig. 1, $\text{pon}(R_3) = 3$. In R_3, x is called its *root*.

Lemma 1. *[4] Let G be a graph with an x-pendant subgraph R_3. In any proper 3-orientation D of G, for every $z \in N_{G-R_3}(x)$ we have $xz \in A(D)$.*

Theorem 2. *If T is a tree, then* $\text{pbon}(T) \leq 4$*, and this bound is tight.*

This theorem follows from the assertion that $\text{pon}(T) \leq 4$ for every tree [4] and that $\text{pbon}(T_3) = 4$ for the tree T_3 obtained from two copies of R_3 with roots x and x' by adding the edge xx'. Araujo et al. [4] showed that $\text{pon}(T_3) = 4$. We will show a slightly stronger result.

Lemma 2. $\text{pbon}(T_3) = 4$.

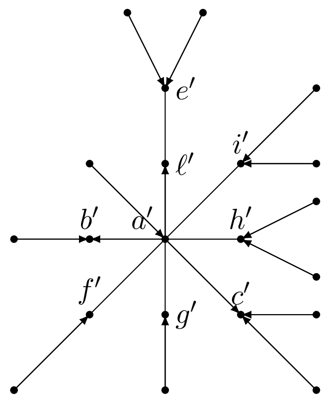

Figure 2: T_1

Proof. We prove this lemma by contradiction. Suppose that there is a proper 3-biorientation D of T_3 and D has the minimum possible number of arcs among such biorientations. By Lemma 1, D has at least one non-single arc and, by Observation 1, all non-single arcs have endpoints with in-degree 3 and 2, respectively. Assume first that $xx', x'x \in A(D)$ are the only non-single arcs. Then modify D by deleting xx' and adding a new vertex x'' and two new arcs xx'' and $x''x'$. Note that the new digraph is a proper 3-orientation of the undirected underlying graph. However, this contradicts Lemma 1. So we may assume one of the copies of R_3 has a non-single arc.

Without loss of generality, we assume that the copy of R_3 with root x contains a non-single arc yz. Since $d^-(y), d^-(z) \in \{2, 3\}$ and by symmetry, it suffices to consider the following cases.

Case 1: cx is a non-single arc. Then delete arc xc and replace any arc(s) between c and h by arc ch. The new proper 3-biorientation has fewer arcs than D, a contradiction.

Case 2: ℓg is a non-single arc. Then delete ℓg and replace any arc(s) between g and m by arc gm. The new proper 3-biorientation has fewer arcs than D, a contradiction.

Case 3: $x\ell$ is a non-single arc. Then replace all arcs incident to ℓ by a (single) arc pointing away from d, replace any arc between g and m by mg, and replace any arc between n and p by pn. The new proper 3-biorientation has fewer arcs than D, a contradiction. □

It is natural to ask if there exists a tree T such that $\text{pbon}(T) < \text{pon}(T)$, and below we give a positive answer to this question. Let us construct a graph T^* from the tree T_1 depicted in Figure 2 as follows. Let us identify each leaf vertex of T_1 with the root of a copy of R_3 such that the other vertices of the copies are not identified with any vertex of T_1.

Theorem 3. *We have* $\text{pon}(T^*) = 4$ *and* $\text{pbon}(T^*) = 3$.

We prove this theorem by showing the following three lemmas.

Lemma 3. *There is a proper 3-orientation D of R_3 such that $d^-_D(x) = 0$.*

Proof. We use the labelling of vertices of R_3 in Fig. 1. Orient from x all edges incident with x. Orient all edges not incident with x but incident with b or c towards b or c. Orient the subtree of R_3 induced by $\{\ell, q, g, m, n, p\}$ using the following arcs: $\ell q, g\ell, mg, nd, pn$. Finally, orient similarly the subtrees with roots at e and f. It is not hard to verify that we have obtained a required orientation. □

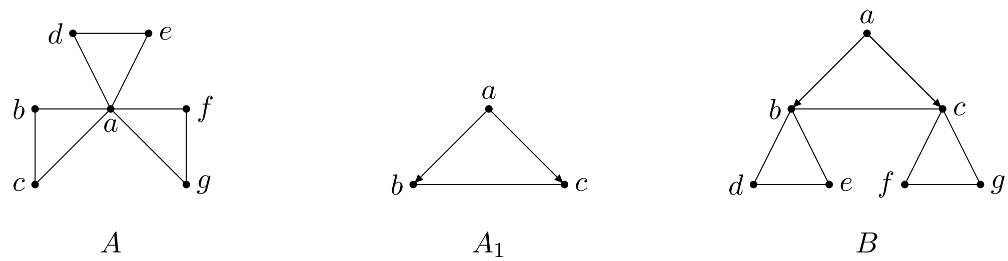

Figure 3: Graphs A, A_1 and B

Lemma 4. $\text{pon}(T^*) = 4$.

Proof. By Theorem 1, it suffices to prove that the $\text{pon}(T^*) \geq 4$. Suppose that there is a proper 3-orientation D of T^*, then by Lemma 1 for every leaf u of T_1 and its neighbor v, we have $uv \in A(D)$. Observe that there is an arc from a' to at least one vertex of each set $\{b', f', g'\}$ and $\{c', h', i'\}$ respectively, say b' and c', since $d^-(a') \leq 3$. Now we have $d^-(b') = 2$ and $d^-(c') = 3$, so $d^-(a') = 1$ and $a'l' \in A(D)$. Then no matter how $l'e'$ is oriented in D we have either $d^-(a') = d^-(l') = 1$ or $d^-(l') = d^-(e') = 2$ which contradicts the assumptions that D is a proper orientation. Thus, we conclude that $\text{pon}(T^*) = 4$. □

Lemma 5. $\text{pbon}(T^*) = 3$.

Proof. In Fig. 2, orient from a' all un-oriented edges incident with a' and replace edge $e'l'$ by two mutually opposite arcs. Use a proper 3-orientation for every copy of R_3 such that the root has in-degree zero (it is possible by Lemma 3). This proves that T^* has a proper 3-biorientation. Now we can show that $\text{pbon}(T^*) = 3$ using Observation 1 (as in the proof of Lemma 2). □

Theorem 4. *There is an infinite number of trees T_n with* $\text{pbon}(T_n) < \text{pon}(T_n)$.

Proof. For every $n \geq 0$, we construct a tree T_n^* from T^* by adding n paths of length 2 which share only vertex a' with each other and T^* such that a' is not a central vertex of the paths. Consider the proper 3-biorientation of T^* described in the proof of Lemma 5. Note that $d^-(a') = 1$. Orient all edges of the n paths towards the central vertices. Observe that the biorientation of T_n^* is a proper 3-biorientation. As in Lemma 5, we can see that $\text{pbon}(T_n^*) = 3$. From the proof of Lemma 4, it follows that $\text{pon}(T_n) = 4$. □

2.2 pbon(G) vs pon(G) for cacti G

A graph G is a *cactus* if G is connected and every two cycles of G have at most one vertex in common. Araujo et al. [5] prove the following bound for cacti.

Theorem 5. *[5] If G is a cactus, then* $\text{pon}(G) \leq 7$ *and this bound is tight.*

To prove the tightness of the bound in Theorem 5, Araujo et al. [5] constructed a graph G_1 such that $\text{pon}(G_1) = 7$. Let us describe G_1. Let F be a graph which is the union of sixteen triangles:

(i) K with vertex set $\{v_1, v_2, v_3\}$;
(ii) K_i^j with vertex set $\{v_i, y_i^j, z_i^j\}$ for $i \in \{1, 2, 3\}$ and $j \in \{1, \ldots, 5\}$.

Let G_1 be the graph obtained from F by adding a copy of A (see Fig. 3) and five copies of B (see Fig. 3) at every vertex $v \in F$, identifying the vertex a of both A and B with each vertex of F. We will use a proof similar to that in [5] to show that $\text{pbon}(G_1) = 7$. We first prove two lemmas.

Lemma 6. *Let G be a graph which contains A as an a-pendant subgraph ($V(G) \neq V(A)$). For every proper biorientation D of G, we have $d_D^-(a) \notin \{1, 2\}$.*

Proof. We prove this lemma by contradiction. Assume that G has a proper biorientation such that $d^-(a) \in \{1, 2\}$. Since A has three triangles, one of them must be oriented as A_1 (see Fig. 3). Then if bc is replaced by a single arc then either $d^-(b) = 1$ and $d^-(c) = 2$ or $d^-(b) = 2$ and $d^-(c) = 1$, a contradiction for both cases. If bc is replaced by a pair of mutually opposite arcs, then $d^-(b) = d^-(c) = 2$, a contradiction. □

Lemma 7. *Let G be a graph which contains B as an a-pendant subgraph ($V(G) \neq V(B)$). Let D be a proper biorientation of G such that ab and ac are single arcs in D. Then $d_D^-(b) \notin \{1, 2\}$ and $d_D^-(c) \notin \{1, 2\}$ implying that $d_D^-(b), d_D^-(c) \in \{3, 4\}$.*

Proof. We prove this lemma by contradiction. Suppose that $d^-(b) = 1$ or $d^-(b) = 2$. If $d^-(b) = 1$, then bd and be are single arcs in D and no matter how we replace edge de by a single arc or two mutually opposite arcs, we arrive at a contradiction. If $d^-(b) = 2$, then without loss of generality $d^-(d) = 1$ and $d^-(e) = 0$ since $d^-(d) \leq 2$ and $d^-(e) \leq 2$. Hence eb, ed and db are single arcs of D implying that $d^-(b) \geq 3$, a contradiction. Therefore, $d^-(b) \notin \{1, 2\}$. Similarly, we can prove that $d^-(c) \notin \{1, 2\}$. By the restrictions on $d^-(b)$ and $d^-(c)$ and the fact that $d^-(a) = 0$, we conclude that $d_D^-(b), d_D^-(c) \in \{3, 4\}$. □

Theorem 6. *We have* $\text{pbon}(G_1) \geq 7$.

Proof. We prove this theorem by contradiction. Suppose that there is a proper 6-biorientation D of G_1. If there is a vertex u of F with $d_D^-(u) \in \{3, 4\}$, then in the proper 6-biorientation of one of the five copies of B corresponding to u we have two single arcs directed from $u = a$ (as B in Fig. 3). However, this contradicts Lemma 7. Then by Lemma 6, $d^-(u) \in \{0, 5, 6\}$ for all vertices $u \in V(F)$.

Since K is a triangle and D is a proper 6-biorientation, without loss of generality, we may assume that $d^-(v_1) = 5$ and $d^-(v_2) = 0$. Then v_2v_1 is a single arc in D. Since each K_1^j ($j \in \{1, \ldots, 5\}$) is a triangle, each of them has a vertex of in-degree zero. This implies that $d^-(v_1) = 6$, a contradiction. □

By Theorems 5 and 6 and inequality (1), we obtain the following:

Theorem 7. *If G is a cactus, then* $\text{pbon}(G) \leq 7$*, and this bound is tight.*

2.3 pbon(G) vs pon(G) for arbitrary graph G

Theorem 8. *Let G be a graph, if* $\text{pon}(G) \leq 3$*, then* $\text{pbon}(G) = \text{pon}(G)$.

Proof. We prove this theorem by contradiction. Suppose that there is a graph G with $\text{pon}(G) \leq 3$, but $\text{pbon}(G) < \text{pon}(G)$. If $\text{pbon}(G) = 1$, then the corresponding biorientation must be an orientation, so $\text{pon}(G) = 1$, a contradiction. If $\text{pbon}(G) = 2$, then there is a proper 2-biorientation D of G and all mutually opposite arcs xy, yx of D satisfy $d^-(x) = 2$ and $d^-(y) = 1$. By Observation 1, we can delete xy to obtain a proper 2-biorientation and therefore a proper 2-orientation which contradicts our assumption. □

3 spon(G) vs pbon(G) for arbitrary graphs G

In this section, we will first prove that $\mathrm{spon}(G) \leq \mathrm{pbon}(G)$ for every graph G and then obtain a characterization of graphs G for which $\mathrm{spon}(G) = \mathrm{pbon}(G)$.

Theorem 9. *For every graph G, $\mathrm{spon}(G) \leq \mathrm{pbon}(G)$. Moreover, for every proper $\mathrm{pbon}(G)$-biorientation D of G, there is a semi-proper orientation D' of G such that the in-weight of every vertex x in D' is no more than its in-degree of x in D.*

Proof. Let D be a proper $\mathrm{pbon}(G)$-biorientation of G. A vertex $v \in V(G)$ is called of *the first type* if there are arcs into v but they are all non-single, and of *the second type*, otherwise. Let v be a vertex of the first type. Then delete all (non-single) arcs into v. Note that in the new D, the in-degree of v equals zero and the in-degree of each of its neightbors in G is positive and has not changed. Thus, D remains a proper biorientation of G and the in-degree of every vertex has not increased. Note that v is now a vertex of the second type. If the new D has a vertex of the first type, continue as above.

Now we may assume that all vertices of D are of the second type. We will perform the following procedure. For every vertex u incident with $2p_u(>0)$ non-single arcs in D (we count pairs of arcs of the form uv, vu), delete every non-single arc into u and set the weight of some single arc into u to $p_u + 1$. Set the weight of every non-weighted arc to 1. Note that when the procedure ends we get a semi-proper orientation D' of G in which the in-weight of every vertex is no more than its in-degree in the initial proper $\mathrm{pbon}(G)$-biorientation of G. Thus, we are done. □

Theorem 10. *For a graph G and integer k, we have $\mathrm{spon}(G) = \mathrm{pbon}(G) = k$ if and only if $\mathrm{spon}(G) = k$ and there is a semi-proper k-orientation such that the in-weight of each vertex is no more than its degree.*

Proof. If $\mathrm{spon}(G) = \mathrm{pbon}(G) = k$, then clearly there is a proper k-biorientation of G. By Theorem 9, we can obtain a semi-proper k-orientation of G from a proper k-biorientation of G such that the in-weight of each vertex in the semi-proper orientation is no more than its in-degree in the biorientation. We are done.

Conversely, assume that $\mathrm{spon}(G) = k$ and there is a semi-proper k-orientation D' of G such that the in-weight of each vertex in D' is no more than its degree in G. Then we can obtain a proper k-biorientation D of G in the following way. Since the in-weight of each vertex in D' is no more than its degree in G, for every vertex v of G add some arcs opposite to existing single arcs to make the number of arcs into each vertex equal to its in-weight. Now we can set the weight of every edge to 1 to obtain a proper k-biorientation. Since $k = \mathrm{spon}(G) \leq \mathrm{pbon}(G)$, we conclude that $\mathrm{pbon}(G) = k$. □

There is an infinite number of graphs G with $\mathrm{spon}(G) < \mathrm{pbon}(G)$. Indeed, Dehghan and Havet [9] observed that for every tree T, $\mathrm{spon}(T) \leq 2$ due to the following semi-proper 2-orientation. Choose a vertex v of T and for an edge xy of T call x the v-closer vertex of xy if the path from v to y includes x. Orient every edge xy from its v-closer vertex to the other vertex and assign weight 1 (2, respectively) to every edge xy with v-closer vertex x such that the distance from v to y is odd (even, respectively). However, the trees T_n constructed in the proof of Theorem 4 are of the proper biorientation number 3.

4 Open Problems

We have provided only a sufficient condition for $\mathrm{pbon}(G) = \mathrm{pon}(G)$. It would be interesting to establish a full characterization. All graphs G which we studied satisfy $\mathrm{pon}(G) - \mathrm{pbon}(G) \leq 1$. Is this true in general? If not, is there a constant c such that $\mathrm{pon}(G) - \mathrm{pbon}(G) \leq c$ for every graph G?

There is a large number of open problems on the proper orientation number of graphs listed in [1, 2, 3, 4, 5]. It would be interesting to investigate biorientation analogs of these problems. The most studied proper orientation number problems are the following two posed in [5, 6]: Is there a constant c such that for every outerplanar (planar, respectively) graph G, $\mathrm{pon}(G) \leq c$. Recently, Chen et al. [8] proved that $pon(G) \leq 14$ for every planar graph G and $pon(G) \leq 10$ if G is outerplanar, which solved the above questions. Planar graph with $pon(G) = 10$ has been constructed by Araujo et al. [5], but no example with $pon(G) > 10$ is known. For outerplanar graphs, there is a lower bound $pon(G) \geq 7$ [5]. It would be interesting to see whether the upper bounds 14 and 10 are tight for the proper orientation number and proper biorientation number for planar graphs and outerplanar graphs, respectively, and if it is not the case then what are the tight upper bounds?

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6 Declarations

Not applicable.

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