The effect on the largest eigenvalue of degree-based weighted adjacency matrix of a graph perturbed by vertex contraction or edge subdivision*

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Abstract

Let G be a connected graph. Denote by d_i the degree of a vertex v_i in G. Let $f(x,y)>0$ be a real symmetric function. Consider an edge-weighted graph in such a way that for each edge v_iv_j of G, the weight of v_iv_j is equal to the value $f(d_i,d_j)$. Therefore, we have a degree-based weighted adjacency matrix $A_f(G)$ of G, in which the (i,j)-entry is equal to $f(d_i,d_j)$ if v_iv_j is an edge of G and is equal to zero otherwise. Let $\mathbf{x}$ be the eigenvector corresponding to the largest eigenvalue $\lambda_1(A_f(G))$ of the weighted adjacency matrix $A_f(G)$. In this paper, we firstly consider the unimodality of the eigenvector $\mathbf{x}$ on an induced path of G. Secondly, if $f(x,y)$ is increasing in the variable x, then we investigate how the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ changes when G is perturbed by vertex contraction or edge subdivision. The aim of this paper is to unify the study of spectral properties for the degree-based weighted adjacency matrices of graphs.

Keywords: degree-based edge-weight; weighted adjacency matrix (eigenvalue, eigenvector); topological function-index; graph operation

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1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. For notation and terminology not defined here, we refer to [2]. Let $G = (V(G), E(G))$ be a graph of order n with vertex set $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E(G)$. If a pair of vertices v_i and v_j are adjacent, then we denote $v_iv_j \in E(G)$. For a vertex $v_i \in V(G)$, let $N_G(v_i)$ be the set of neighbours of v_i in G. The degree of the vertex v_i, denoted by d_i, is equal to $|N_G(v_i)|$. The closed neighborhood of v_i in G is the set $N_G[v_i] = N_G(v_i) \cup \{v_i\}$. If $d_i = 1$, then the vertex v_i of G is said to be a pendant vertex. The distance between two vertices v_i and v_j in a graph G is the length of a shortest v_iv_j-path in G. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a subgraph of G. Futhermore, if H is a subgraph of G and H contains all the edges $v_iv_j \in E(G)$ for any $v_i, v_j \in V(H)$, then H is an induced subgraph of G. We denote by $K_{1,n-1}$, P_n and C_n, respectively, the star, the path and the cycle of order n.

In chemical graph theory, graphical or topological indices are applied to represent chemical structures of molecular graphs and reflect molecular properties. The general form of degree-based topological indices is

$$TI(G) = \sum_{v_iv_j \in E(G)} f(d_i, d_j),$$

where the edge-weight function $f(x, y)$ is a real symmetric function with variables $x > 0$ and $y > 0$, and the value $f(d_i, d_j)$ is the edge-weight of the edge v_iv_j of G. In fact, each index is obtained by summing up the edge-weights of all edges in a molecular graph with edge-weights defined by the function $f(x, y)$, and it maps a molecular graph into a single number. For a symmetric function $f(x, y)$, if the first partial derivative $f'_x(x, y) \geq 0$, then $f(x, y)$ is said to be increasing in variable x. There are many important and well-studied indices collected by Gutman [4], as in Table 1. It is not difficult for us to find that the first fourteen edge-weight functions $f(x, y)$ in Table 1 are increasing in variable x. This means that increasing property is very important to studying topological indices.

In spectral graph theory, a matrix associated with graph G is a critical tool. In 2015, one of the authors Li in [8] proposed that if we use a matrix to represent the structure of a molecular graph with edge-weights separately on its edges, it would keep much more structural information than a topological index. Subsequently, matrices defined by topological indices from algebraic viewpoint were studied separately, including the first(second) Zagreb matrix [13], Nirmala matrix [6], Sombor matrix [5] and inverse sum indeg matrix [1].

In 2018, Das et al. first published in [3] the following definition of the weighted

Edge-weight function f(x,y)	**The corresponding index**
$x+y$	first Zagreb index
xy	second Zagreb index
$(x+y)^2$	first hyper-Zagreb index
$(xy)^2$	second hyper-Zagreb index
$\sqrt{x+y}$	reciprocal sum-connectivity index
$\sqrt{xy}$	reciprocal Randić index
$x+y+xy$	first Gourava index
$(x+y)xy$	second Gourava index
$(x+y+xy)^2$	first hyper-Gourava index
$((x+y)xy)^2$	second hyper-Gourava index
$\sqrt{(x+y)xy}$	product-connectivity Gourava index
x^2+y^2	forgotten index
$\sqrt{x^2+y^2}$	Sombor index
$xy/(x+y)$	inverse sum index
$x^{-2}+y^{-2}$	inverse degree
$x^{-3}+y^{-3}$	modified first Zagreb index
$1/\sqrt{xy}$	Randić index
$1/\sqrt{x+y}$	sum-connectivity index
$2/(x+y)$	harmonic index
$1/\sqrt{x+y+xy}$	sum-connectivity Gourava index
$\|x-y\|$	Albertson index
$(x-y)^2$	sigma index
$(x/y+y/x)/2$	extended index
$\sqrt{(x+y-2)/(xy)}$	atom-bond-connectivity (ABC) index
$(xy/(x+y-2))^3$	augmented Zagreb index
$2\sqrt{xy}/(x+y)$	geometric-arithmetic (GA) index
$(x+y)/(2\sqrt{xy})$	arithmetic-geometric (AG) index

Table 1: Some well-studied chemical indices

adjacency matrix for a graph with degree-based edge-weights.

Definition 1.1 *Let G be a graph of order n and $f(x, y)$ be a real symmetric function. The weighted adjacency matrix $A_f(G)$ is defined as*

$$(A_f(G))_{ij} = \begin{cases} f(d_i, d_j), & v_iv_j \in E(G), \\ 0, & \textit{otherwise}. \end{cases}$$

We name the eigenvalues of the $n \times n$ matrix $A_f(G)$ as the *weighted adjacency eigenvalues* of a graph G with edge-weight function $f(x, y)$. Because $f(x, y)$ is a real symmetric function, then $A_f(G)$ is a real symmetric matrix, and therefore its eigenvalues are all real. Then the weighted adjacency eigenvalues can be ordered as

$$\lambda_1(A_f(G)) \geq \lambda_2(A_f(G)) \geq \cdots \geq \lambda_n(A_f(G)),$$

which are always arranged in a non-increasing order and repeated according to their multiplicity. $\lambda_1(A_f(G))$ is the largest weighted adjacency eigenvalue. If we let $\mathbf{x} = (x_0, x_1, ..., x_{n-1})^T$ be the eigenvector corresponding to $\lambda_1(A_f(G))$, then $A_f(G)\mathbf{x} = \lambda_1(A_f(G))\mathbf{x}$. Moreover, the vector $\mathbf{x}$ can be regarded as a function on $V(G)$. For any vertex v_i, the entry of $\mathbf{x}$ corresponding to v_i is denoted by x_i.

Up to now, there have been a few articles studying the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$. Let us list some known results. In 2021, Li and Wang [9] first attempted to study the extremal tree with the largest value of $\lambda_1(A_f(G))$, which is a star or a double star when the symmetric real function $f(x, y)$ is increasing and convex in variable x, and with the smallest value of $\lambda_1(A_f(G))$, which is a path when $f(x, y)$ is a symmetric polynomial with nonnegative coefficients and zero constant term. In 2022, Zheng et al. [14] added a restriction P^* to $f(x, y)$ and they confirmed that star is the unique tree with the largest value of $\lambda_1(A_f(G))$ among all trees of order n. They also obtained the extremal unicyclic graphs with the largest and smallest value of $\lambda_1(A_f(G))$, respectively. Recently, Li and Yang [12] gave some lower and upper bounds for $\lambda_1(A_f(G))$ and characterized the corresponding extremal graphs. In 2022, Li and Yang [10, 11] got uniform interlacing inequalities for the weighted adjacency eigenvalues under some kinds of graph operations, including edge deletion, edge subdivision, vertex deletion and vertex contraction, and examples were given to show that the interlacing inequalities are the best possible for their type when $f(x, y)$ is increasing in variable x. Although, we can get some upper and lower bounds for $\lambda_1(A_f(G))$ from the interlacing inequalities, but it cannot be reflected directly that how the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ changes when one graph is transformed to another graph. In this paper, we are interested in the impact on the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ under two graph perturbations. So we first give the definitions of graph operations.

Definition 1.2 *(Vertex contraction) The contraction of a pair of vertices $u, v \in V(G)$ produces a new graph $G_{\{u,v\}}$, where $V(G_{\{u,v\}}) = (V(G) \setminus \{u,v\}) \cup \{s\}$, s is a new vertex with $N_{G_{\{u,v\}}}(s) = (N_G(u) \cup N_G(v)) \setminus \{u,v\}$, and $E(G_{\{u,v\}}) = (E(G) \setminus (\{uz : z \in N_G(u)\} \cup \{vz : z \in N_G(v)\})) \cup \{sz : z \in N_{G_{\{u,v\}}}(s)\}$.*

Definition 1.3 *(Edge subdivision) The subdivision of an edge $e = v_iv_j \in E(G)$ produces a new graph G_e, where $V(G_e) = V(G) \cup \{v_n\}$, such that $v_n \notin V(G)$, and $E(G_e) = (E(G) \setminus e) \cup \{v_iv_n, v_jv_n\}$.*

The structure of this paper is as follows. In Section 2, we present some known results that will be used in the subsequent sections. In Section 3, since the eigenvector $\mathbf{x}$ corresponding to the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ plays an important role in the investigation of $\lambda_1(A_f(G))$, we first study the property of $\mathbf{x}$ on an induced path of G. Then, the effects on the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ perturbed by the vertex contraction and edge subdivision of G are described, respectively, when $f(x, y) > 0$ is a real symmetric function and increasing in variable x.

2 Prelimininaries

At the very beginning, we state some fundamental results on matrix theory, which will be used in the sequel. Let $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{n\times m}$ be two matrices. If $a_{ij} \le b_{ij}$ for all i and j, then we say that $A \le B$. If $A \le B$ and $A \ne B$, then we say that $A < B$.

Lemma 2.1 *[7] Let A, B be $n \times n$ nonnegative symmetric matrices. If $A \le B$, then*

$$\lambda_1(A) \le \lambda_1(B).$$

Futhermore, if B is irreducible and $A < B$, then $\lambda_1(A) < \lambda_1(B)$.

The next result plays a very important role in the proof of our main results.

Lemma 2.2 *[7] Let A be an $n \times n$ nonnegative matrix and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})^T$ be a positive vector. If $\alpha, \beta \ge 0$, such that $\alpha\mathbf{y} \le A\mathbf{y} \le \beta\mathbf{y}$, then*

$$\alpha \le \lambda_1(A) \le \beta.$$

If $\alpha\mathbf{y} < A\mathbf{y}$, then $\alpha < \lambda_1(A)$; if $A\mathbf{y} < \beta\mathbf{y}$, then $\lambda_1(A) < \beta$.

Finally we state the famous Perron–Frobenius Theorem.

Lemma 2.3 *[2] Let A be an irreducible symmetric matrix with nonnegative entries. Then the largest eigenvalue $\lambda_1(A)$ of A is simple, with a corresponding eigenvector whose entries are all positive.*

3 Main results

In this section, we first study the property of the eigenvector $\mathbf{x}$ corresponding to the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$. For a connected graph G, if $f(x, y) > 0$ is a real symmetric function, then $A_f(G)$ is an irreducible symmetric matrix with nonnegative entries. From Lemma 2.3, we have a positive eigenvector $\mathbf{x}$ corresponding to the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$. The following result says the unimodality of $\mathbf{x}$ on an induced path of G.

Theorem 3.1 *For a connected graph G of order n and a real symmetric function $f(x, y) > 0$, let $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})^T$ be a positive eigenvector corresponding to the eigenvalue $\lambda_1(A_f(G))$ and $P_k = v_1v_2 \ldots v_k$ be an induced path of G such that $d_i = 2$ for $1 \le i \le k$. If $\lambda_1(A_f(G)) > 2f(2, 2)$, then the following statements hold.*

(1) If $x_1 = x_k$, then

$$x_1 > x_2 > \cdots > x_{\lfloor \frac{k+1}{2} \rfloor} = x_{\lceil \frac{k+1}{2} \rceil} < \cdots < x_{k-1} < x_k$$

and $x_i = x_{k+1-i}$ for $2 \le i \le k - 1$.

(2) If $x_1 < x_k$, then there is an integer $1 \le j \le \lfloor \frac{k+1}{2} \rfloor$ such that

$$x_1 > x_2 > \cdots > x_j > x_{j+1} < \cdots < x_{k-1} < x_k$$

or

$$x_1 > x_2 > \cdots > x_j \le x_{j+1} < \cdots < x_{k-1} < x_k.$$

Moreover, $x_i < x_{k+1-i}$ for $2 \le i \le \lceil \frac{k+1}{2} \rceil - 1$.

Proof. Since $\mathbf{x}$ is a positive eigenvector corresponding to $\lambda_1(A_f(G))$, we have $x_i \ge 0$ for $0 \le i \le n - 1$. Recall that $P_k = v_1v_2 \ldots v_k$ is an induced path of G such that $d_i = 2$ for $1 \le i \le k$. Hence, it is not difficult to get the following relation:

$$\lambda_1(A_f(G))x_i = f(2, 2)x_{i-1} + f(2, 2)x_{i+1},$$

where $2 \le i \le k-1$. This means that

$$\frac{\lambda_1(A_f(G))}{f(2,2)} x_i = x_{i-1} + x_{i+1},$$

where $2 \le i \le k-1$. Clearly, this is a recurrence relation and the characteristic equation is

$$t^2 - \frac{\lambda_1(A_f(G))}{f(2,2)} t + 1 = 0.$$

Since $\lambda_1(A_f(G)) > 2f(2,2)$, we can deduce that this characteristic equation has two unequal real roots t_1 and t_2, such that $t_1 t_2 = 1, t_1 + t_2 > 2$. Without loss of generality, we assume that $t_2 > 1 > t_1 > 0$. The solution of this linear homogeneous recurrence relation with constant coefficients is given by

$$x_i = A t_1^i + B t_2^i.$$

Let x_1 and x_k be the initial conditions. We can determine constants A and B from the initial conditions:

$$\begin{cases} A t_1 + B t_2 = x_1, \\ A t_1^k + B t_2^k = x_k. \end{cases}$$

This implies that

$$A = \frac{x_1 t_2^{k+1} - x_k t_2^2}{t_2^k - t_1^{k-2}}, \quad B = \frac{x_k - x_1 t_1^{k-1}}{t_2^k - t_1^{k-2}}.$$

Because $t_2 > 1 > t_1 > 0$ and $k \ge 2$, it follows that $t_2^k - t_1^{k-2} > 0$. We then have

$$x_i = \frac{1}{t_2^k - t_1^{k-2}}((x_1 t_2^{k+1} - x_k t_2^2) t_1^i + (x_k - x_1 t_1^{k-1}) t_2^i)$$

for $1 \le i \le k$.

(1) Since $x_1 = x_k$, we have

$$\begin{aligned} x_i &= \frac{x_1}{t_2^k - t_1^{k-2}}((t_2^{k+1} - t_2^2) t_1^i + (1 - t_1^{k-1}) t_2^i) \\ &= \frac{x_1}{t_2^k - t_1^{k-2}}(t_2^{k+1-i} - t_1^{i-2} + t_2^i - t_1^{k-1-i}). \end{aligned}$$

Furthermore, we can get

$$x_{k+1-i} = \frac{x_1}{t_2^k - t_1^{k-2}}(t_2^i - t_1^{k-1-i} + t_2^{k+1-i} - t_1^{i-2}).$$

Hence for $1 \leq i \leq k$, we have $x_i = x_{k+1-i}$. Now we let

$$\begin{aligned} f(i) &= t_2^{k+1-i} - t_1^{i-2} + t_2^i - t_1^{k-1-i} \\ &= t_2^{k+1-i} - \frac{1}{t_2^{i-2}} + t_2^i - \frac{1}{t_2^{k-1-i}} \\ &= \frac{t_2^{k-3+i} + t_2^{2k-2-i} - t_2^{k-1-i} - t_2^{i-2}}{t_2^{k-3}}. \end{aligned}$$

Since $t_2^k - t_1^{k-2} > 0$ and $x_1 > 0$, it follows that the monotonicity of $f(i)$ is the same as the monotonicity of x_i. Suppose that

$$g(i) = t_2^{k-3+i} + t_2^{2k-2-i} - t_2^{k-1-i} - t_2^{i-2}.$$

Because $t_2^{k-3} > 0$, the monotonicity of $g(i)$ is the same as the monotonicity of $f(i)$.

We now consider the monotonicity of $g(i)$. By the first derivative of $g(i)$, it follows that

$$\begin{aligned} g'(i) &= t_2^{k-3+i} \ln t_2 - t_2^{2k-2-i} \ln t_2 + t_2^{k-1-i} \ln t_2 - t_2^{i-2} \ln t_2 \\ &= t_2^{k-1}(t_2^{i-2} - t_2^{k-1-i}) \ln t_2 + (t_2^{k-1-i} - t_2^{i-2}) \ln t_2 \\ &= (t_2^{k-1} - 1)(t_2^{i-2} - t_2^{k-1-i}) \ln t_2. \end{aligned}$$

Recalling that $t_2 > 1$, it follows that $t_2^{k-1} - 1 > 0$ and $\ln t_2 > 0$. If $i - 2 > k - 1 - i$, that is, $i > \frac{k+1}{2}$, then we have $t_2^{i-2} > t_2^{k-1-i}$, hence $g'(i) > 0$. This means that x_i is monotonically increasing for $i > \frac{k+1}{2}$. If $i - 2 < k - 1 - i$, that is, $i < \frac{k+1}{2}$, then we obtain $t_2^{i-2} < t_2^{k-1-i}$, hence $g'(i) < 0$. This means that x_i is monotonically decreasing for $i < \frac{k+1}{2}$. Thus we conclude that $x_1 > x_2 > \cdots > x_{\lfloor \frac{k+1}{2} \rfloor} = x_{\lceil \frac{k+1}{2} \rceil} < \cdots < x_{k-1} < x_k$. The proof of (1) is complete.

(2) We assume that $x_1 < x_k$. Otherwise, we can relabel the vertices on P_k. Note that

$$x_i = \frac{1}{t_2^k - t_1^{k-2}}((x_1 t_2^{k+1} - x_k t_2^2) t_1^i + (x_k - x_1 t_1^{k-1}) t_2^i).$$

For $2 \leq i \leq \lceil \frac{k+1}{2} \rceil - 1$, we have

$$\begin{aligned} &x_{k+1-i} - x_i \\ &= \frac{1}{t_2^k - t_1^{k-2}} \left((x_1 t_2^{k+1} - x_k t_2^2) t_1^{k+1-i} + (x_k - x_1 t_1^{k-1}) t_2^{k+1-i} - ((x_1 t_2^{k+1} - x_k t_2^2) t_1^i + (x_k - x_1 t_1^{k-1}) t_2^i)\right) \\ &= \frac{1}{t_2^k - t_1^{k-2}} \left(x_1 t_2^i - x_k t_1^{k-i-1} + x_k t_2^{k+1-i} - x_1 t_1^{i-2} - (x_1 t_2^{k+1-i} - x_k t_1^{i-2} + x_k t_2^i - x_1 t_1^{k-1-i})\right) \\ &= \frac{1}{t_2^k - t_1^{k-2}} (x_k - x_1)(t_2^{k+1-i} - t_2^i + t_1^{i-2} - t_1^{k-i-1}). \end{aligned}$$

Since $2 \leq i \leq \lceil \frac{k+1}{2} \rceil - 1$, it follows that $t_2^{k+1-i} > t_2^i$ and $t_1^{i-2} > t_1^{k-i-1}$. Recalling that $x_k > x_1$, we obtain $x_{k-i+1} > x_i$ for $2 \leq i \leq \lceil \frac{k+1}{2} \rceil - 1$.

Now let us consider a function

$$\begin{aligned} h(i) &= (x_1 t_2^{k+1} - x_k t_2^2) t_1^i + (x_k - x_1 t_1^{k-1}) t_2^i \\ &= x_1 t_2^{k+1-i} - x_k t_2^{2-i} + x_k t_2^i - x_1 t_2^{i-k+1}. \end{aligned}$$

Because $t_2^k - t_1^{k-2} > 0$, it follows that the monotonicity of $h(i)$ is the same as the monotonicity of x_i. By the first derivative of $h(i)$, we obtain

$$\begin{aligned} h'(i) &= -x_1 t_2^{k+1-i} \ln t_2 + x_k t_2^{2-i} \ln t_2 + x_k t_2^i \ln t_2 - x_1 t_2^{i-k+1} \ln t_2 \\ &= (x_k (t_2^i + t_2^{2-i}) - x_1 (t_2^{k+1-i} + t_2^{i-k+1})) \ln t_2. \end{aligned}$$

Then we consider the following two cases.

Case 1. $i > k+1-i$, that is, $i \geq \lfloor \frac{k+1}{2} \rfloor + 1$.

We consider the function $l(i) = t_2^i + t_2^{2-i}$. Since $l'(i) = t_2^i (1 - t_2^{2(1-i)}) \ln t_2$, the function $l(i)$ is monotonically increasing for $i > 1$. This means that $t_2^i + t_2^{2-i} > t_2^{k+1-i} + t_2^{i-k+1}$. Because $x_k > x_1$, it follows that $h'(i) > 0$ for $i \geq \lfloor \frac{k+1}{2} \rfloor + 1$. Hence, x_i is monotonically increasing for $i \geq \lfloor \frac{k+1}{2} \rfloor + 1$. We then have $x_{\lfloor \frac{k+1}{2} \rfloor + 1} < \cdots < x_{k-1} < x_k$.

Case 2. $i \leq k+1-i$, that is, $i \leq \lfloor \frac{k+1}{2} \rfloor$.

Now we consider the function

$$w(i) = \frac{t_2^{k+1-i} + t_2^{i-k+1}}{t_2^i + t_2^{2-i}}.$$

By the first derivative of $w(i)$, we have

$$\begin{aligned} w'(i) &= \frac{((-t_2^{k+1-i} + t_2^{i-k+1})(t_2^i + t_2^{2-i}) - (t_2^i - t_2^{2-i})(t_2^{k+1-i} + t_2^{i-k+1})) \ln t_2}{(t_2^i + t_2^{2-i})^2} \\ &= \frac{2(t_2^{3-k} - t_2^{k+1}) \ln t_2}{(t_2^i + t_2^{2-i})^2}. \end{aligned}$$

It is clear that $w'(i) < 0$ with $k > 1$. There are two possibilities.

Subcase 2.1. $\frac{x_k}{x_1} > w(i)$ for $1 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$.

Since $w(i)$ is monotonically decreasing, we have

$$h'(i) = (x_k (t_2^i + t_2^{2-i}) - x_1 (t_2^{k+1-i} + t_2^{i-k+1})) \ln t_2 > 0$$

for $1 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$. This means that x_i is monotonically increasing for $1 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$. Together with Case 1, it follows that $x_1 < x_2 < \cdots < x_k$.

Subcase 2.2. There exits an integer $1 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$ such that $\frac{x_k}{x_1} \leq w(i)$.

Since $w(i)$ is monotonically decreasing, there is only an integer $1 \leq j \leq \lfloor \frac{k+1}{2} \rfloor$ such that $w(j) \geq \frac{x_k}{x_1}$ and $w(j+1) < \frac{x_k}{x_1}$. Thus we can say that

$$h'(i) = (x_k(t_2^i + t_2^{2-i}) - x_1(t_2^{k+1-i} + t_2^{i-k+1})) \ln t_2 \leq 0$$

for $1 \leq i \leq j$. This means that x_i is monotonically decreasing for $1 \leq i \leq j$. Furthermore,

$$h'(i) = (x_k(t_2^i + t_2^{2-i}) - x_1(t_2^{k+1-i} + t_2^{i-k+1})) \ln t_2 > 0$$

for $j+1 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$. This means that x_i is monotonically increasing for $j+1 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$. Together with Case 1, we can conclude that $x_1 > x_2 > \cdots > x_j > x_{j+1} < \cdots < x_{k-1} < x_k$ or $x_1 > x_2 > \cdots > x_j \leq x_{j+1} < \cdots < x_{k-1} < x_k$. This proof is thus complete. $\square$

In fact, there are many graphs satisfying $\lambda_1(A_f(G)) > 2f(2,2)$. Here we give a result as follows.

Theorem 3.2 *Let $G \neq C_n$ be a connected graph of order n. Assume that $f(x,y) > 0$ is a real symmetric function and increasing in variable x. If G contains a cycle, then $\lambda_1(A_f(G)) > 2f(2,2)$.*

Proof. Without loss of generality, we suppose that G has an induced cycle $C_{k+1} = v_0v_1 \ldots v_kv_0$. Let B be a $(k+1) \times (k+1)$ matrix, which is obtained by choosing the rows and columns associated with $v_0, v_1, \ldots, v_k$ from $A_f(G)$. Since $f(x,y) > 0$ is an increasing function in variable x, it is clear that $B \geq A_f(C_{k+1})$. From Lemma 2.1, we have $\lambda_1(B) \geq \lambda_1(A_f(C_{k+1})) = 2f(2,2)$.

Adding $n-k-1$ zero rows and zero columns to B, we have an $n \times n$ matrix C. Clearly, $\lambda_1(B) = \lambda_1(C)$. Because $G \neq C_n$ and $f(x,y) > 0$, it follows that $A_f(G) > C$. Recalling that G is a connected graph, by using Lemma 2.1 again we have $\lambda_1(A_f(G)) > \lambda_1(C) = \lambda_1(B)$. Hence, $\lambda_1(A_f(G)) > 2f(2,2)$. The required result is obtained. $\square$

In fact, for a subgraph H of a connected graph G, if $G \neq H$, using a similar argument in Theorem 3.2, then we can prove that $\lambda_1(A_f(G)) > \lambda_1(A_f(H))$. Next, when $f(x,y) > 0$ is a real symmetric function and increasing in variable x, we consider the effect on the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ by vertex contraction.

Theorem 3.3 *Let G be a connected graph of order n and $H = G_{\{u,v\}}$, where u and v are two distinct vertices of G such that the distance between u and v is at least 3.*

Assume that $f(x, y) > 0$ is a real symmetric function and increasing in variable x. Then

$$\lambda_1(A_f(G)) < \lambda_1(A_f(H)).$$

Proof. Since the distance between u and v is at least 3, we have $uv \notin E(G)$ and $N_G(u) \cap N_G(v) = \emptyset$. Without loss of generality, we suppose that $N_G(u) = \{u_1, u_2, \ldots, u_p\}$ and $N_G(v) = \{v_1, v_2, \ldots, v_q\}$, where $p, q \geq 1$. Contracting the two vertices u and v, we obtain a new vertex s. For the weighted adjacency matrix $A_f(H)$, we have a positive eigenvector $\mathbf{x}$ corresponding to $\lambda_1(A_f(H))$. That is, $A_f(H)\mathbf{x} = \lambda_1(A_f(H))\mathbf{x}$. Let n-dimensional vector $\mathbf{y}$ be an assignment of G satisfying

$$\begin{cases} y_u = y_v = x_s, \\ y_w = x_w, \qquad\qquad w \in V(G) \setminus \{u, v\}, \end{cases}$$

where x_z and y_z are the entries of $\mathbf{x}$ and $\mathbf{y}$ corresponding to vertex z, respectively. Obviously, the vector $\mathbf{y}$ is positive. Next we prove $A_f(G)\mathbf{y} < \lambda_1(A_f(H))\mathbf{y}$.

First, we compare the entry $(A_f(G)\mathbf{y})_u$ to $\lambda_1(A_f(H))y_u$. We have

$$\begin{aligned} (A_f(G)\mathbf{y})_u &= \sum_{i=1}^{p} f(p, d_{u_i}) y_{u_i} = \sum_{i=1}^{p} f(p, d_{u_i}) x_{u_i} \\ &\leq \sum_{i=1}^{p} f(p+q, d_{u_i}) x_{u_i} \\ &< \sum_{i=1}^{p} f(p+q, d_{u_i}) x_{u_i} + \sum_{i=1}^{q} f(p+q, d_{v_i}) x_{v_i} \\ &= \lambda_1(A_f(H)) x_s = \lambda_1(A_f(H)) y_u. \end{aligned}$$

Since $f(p+q, d_{v_i}) > 0$ and $x_{v_i} > 0$, it follows that $\sum_{i=1}^{q} f(p+q, d_{v_i}) x_{v_i} > 0$. Thus, the above inequality is strict. Similarly, we have

$$\begin{aligned} (A_f(G)\mathbf{y})_v &= \sum_{i=1}^{q} f(q, d_{v_i}) y_{v_i} = \sum_{i=1}^{q} f(q, d_{v_i}) x_{v_i} \\ &\leq \sum_{i=1}^{q} f(p+q, d_{v_i}) x_{v_i} \\ &< \sum_{i=1}^{q} f(p+q, d_{v_i}) x_{v_i} + \sum_{i=1}^{p} f(p+q, d_{u_i}) x_{u_i} \\ &= \lambda_1(A_f(H)) x_s = \lambda_1(A_f(H)) y_v. \end{aligned}$$

Because $\sum_{i=1}^{p} f(p+q, d_{u_i})x_{u_i} > 0$, this inequality is also strict. For any vertex $w \in V(G) \setminus (N[u] \cup N[v])$, we have

$$\begin{aligned}(A_f(G)\mathbf{y})_w &= \sum_{z \in N_G(w)} f(d_w, d_z)y_z = \sum_{z \in N_H(w)} f(d_w, d_z)x_z \\ &= \lambda_1(A_f(H))x_w = \lambda_1(A_f(H))y_w.\end{aligned}$$

Moreover, for a vertex u_i, $1 \le i \le p$, we have

$$\begin{aligned}(A_f(G)\mathbf{y})_{u_i} &= \sum_{z \in N_G(u_i) \setminus u} f(d_{u_i}, d_z)y_z + f(d_{u_i}, p)y_u \\ &\le \sum_{z \in N_H(u_i) \setminus s} f(d_{u_i}, d_z)x_z + f(d_{u_i}, p+q)x_s \\ &= \lambda_1(A_f(H))x_{u_i} = \lambda_1(A_f(H))y_{u_i}.\end{aligned}$$

For a vertex v_i, $1 \le i \le q$, we have

$$\begin{aligned}(A_f(G)\mathbf{y})_{v_i} &= \sum_{z \in N_G(v_i) \setminus v} f(d_{v_i}, d_z)y_z + f(d_{v_i}, q)y_v \\ &\le \sum_{z \in N_H(v_i) \setminus s} f(d_{v_i}, d_z)x_z + f(d_{v_i}, p+q)x_s \\ &= \lambda_1(A_f(H))x_{v_i} = \lambda_1(A_f(H))y_{v_i}.\end{aligned}$$

Thus, $A_f(G)\mathbf{y} < \lambda_1(A_f(H))\mathbf{y}$. From Lemma 2.2, we can conclude that $\lambda_1(A_f(G)) < \lambda_1(A_f(H))$. This completes the proof. □

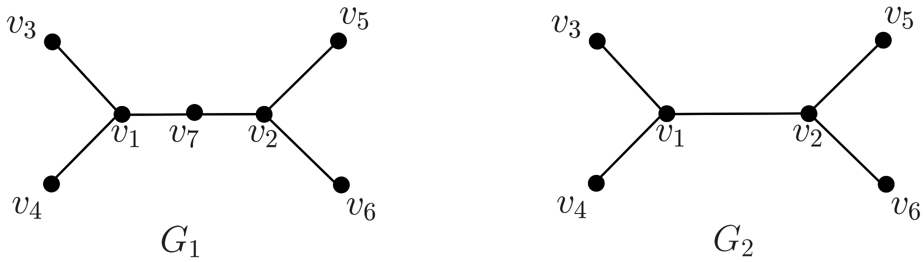

Figure 1: Graphs G_1 and G_2.

Remark 1. In Theorem 3.3, the condition "the distance between u and v is at least 3" is reasonable. Here are some examples to illustrate the situation. First, we consider that the distance between u and v is 1. Suppose that a connected graph G contains a pendent vertex v. Contracting the vertex v and its neighbour, we can get a graph H. Since H is an induced subgraph of G and $f(x, y) > 0$ is increasing in variable x, it follows that $\lambda_1(A_f(G)) > \lambda_1(A_f(H))$. In addition, if $G = G_1$(see Figure 1), contracting v_1 and v_7, then we can get a graph G_2(see Figure 1). Set $f(x, y) = \sqrt{\lg(xy)}$, by calculation it is not difficult to get that $1.5845 \approx \lambda_1(A_f(G_1)) > \lambda_1(A_f(G_2)) \approx 1.5806$. Second, we consider that the distance between u and v is 2. If $G = K_{1,n-1}$, contracting two pendant vertices, then we have $H = K_{1,n-2}$. Because $f(x, y) > 0$ is increasing in variable x, it suffices to prove that $f(1, n-1)\sqrt{n-1} = \lambda_1(A_f(K_{1,n-1})) > \lambda_1(A_f(K_{1,n-2})) = f(1, n-2)\sqrt{n-2}$. Moreover, if $G = G_1$(see Figure 1), contracting v_1 and v_2, then we can obtain $K_{1,5}$. When $f(x, y) = (xy)^3$, a short calculation reveals that $307.8474 \approx \lambda_1(A_f(G_1)) > \lambda_1(A_f(K_{1,5})) \approx 279.5085$.

Finally, we establish the relation between the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ and $\lambda_1(A_f(G_e))$, where $f(x, y) > 0$ is a real symmetric function and increasing in variable x. Now we introduce the definition of an internal path of a graph G in the first place.

Definition 3.4 *Let G be a connected graph of order n. The walk $v_0v_1 \dots v_{k+1}$ is an internal path of G if one of the following holds:*

(i) $k \geq 2$, the vertices $v_0, v_1, \dots, v_k$ are distinct, $v_{k+1} = v_0$, $v_iv_{i+1} \in E(G)$ where $0 \leq i \leq k$, $d_0 \geq 3$ and $d_i = 2$ where $1 \leq i \leq k$;

(ii) $k \geq 0$, the vertices $v_0, v_1, \dots, v_{k+1}$ are distinct, $v_iv_{i+1} \in E(G)$ where $0 \leq i \leq k$, $d_0 \geq 3$, $d_{k+1} \geq 3$ and $d_i = 2$ where $1 \leq i \leq k$.

Theorem 3.5 *Let G be a connected graph of order n and $H = G_e$. Assume that $f(x, y) > 0$ is a real symmetric function and increasing in variable x. Let $\mathbf{x} = \{x_0, x_1, \dots, x_{n-1}\}^T$ be a positive eigenvector corresponding to $\lambda_1(A_f(G))$ and $P_{k+2} = v_0v_1 \dots v_{k+1}$ be an internal path of G such that $x_0 \leq x_{k+1}$. Then the following statements hold.*

(1) If $G \neq C_n$ and e does not belong to an internal path of G, then

$$\lambda_1(A_f(H)) > \lambda_1(A_f(G)).$$

(2) If for any vertex $v_i \in N_G(v_0)$, $d_i \geq 2$ and e belongs to an internal path of G, then

$$\lambda_1(A_f(H)) < \lambda_1(A_f(G)).$$

Proof. (1) If $G \neq C_n$ and e does not belong to an internal path of G, then we can get G by deleting a pendent vertex v from H. This means that H has a proper subgraph $H' = H \backslash v$ isomorphic to G. Now deleting the row and column associated with v from

$A_f(H)$, we get a matrix B. Then, adding a zero row and a zero column to B, we have an $(n+1)\times(n+1)$ matrix C. It is not difficult to verify that $\lambda_1(B) = \lambda_1(C)$. The matrix $A_f(H) - C$ has two same nonnegative nonzero entries $f(2,1)$ in symmetric place, and all other entries of $A_f(H) - C$ are zero. Since G is connected, $A_f(H)$ is irreducible. Now using Lemma 2.1, we have $\lambda_1(A_f(H)) > \lambda_1(C) = \lambda_1(B)$.

Note that the matrix $B - A_f(G)$ has two same nonnegative entries $f(2,2) - f(2,1)$ in symmetric place, and all other entries of $B - A_f(G)$ are zero. Since $f(x,y) > 0$ is an increasing function in variable x, we have $B \geq A_f(G)$. Using Lemma 2.1 again, we get $\lambda_1(B) \geq \lambda_1(A_f(G))$. Until now, we can obtain $\lambda_1(A_f(H)) > \lambda_1(A_f(G))$.

(2) For convenience, we suppose that v_n is the addition vertex which appears in subdividing edge e. Next, we prove the result by discussing the type of the internal path with the edge e.

Case 1. e belongs to an internal path $P_{k+2} = v_0v_1\ldots v_{k+1}$ of type (i).

Let x_i be the entry of $\mathbf{x}$ corresponding to the vertex v_i of G where $i = 0, 1, \ldots, k$. Since $v_0 = v_{k+1}$ and $v_1v_2\ldots v_k$ is an induced path of G, by Theorem 3.1 we have $x_i = x_{k+1-i}$ for $1 \leq i \leq k$. Next we consider the following two cases.

Case 1.1. k is even.

We have $x_{\frac{k}{2}} = x_{\frac{k}{2}+1}$. Without loss of generality, we take $e = v_{\frac{k}{2}}v_{\frac{k}{2}+1}$. Let $\mathbf{y}$ be an $(n+1)$-dimensional vector obtained from $\mathbf{x}$ by inserting the addition entry $y_n = x_{\frac{k}{2}} = x_{\frac{k}{2}+1}$. That is,

$$\begin{cases} y_n = x_{\frac{k}{2}}, \\ y_i = x_i, \qquad v_i \in V(G). \end{cases}$$

Hence, the vector $\mathbf{y}$ is positive. Then we have

$$\begin{aligned} (A_f(H)\mathbf{y})_n &= f(2,2)x_{\frac{k}{2}} + f(2,2)x_{\frac{k}{2}+1} \\ &= 2f(2,2)x_{\frac{k}{2}} \\ &< \lambda_1(A_f(G))x_{\frac{k}{2}} = \lambda_1(A_f(G))y_n. \end{aligned}$$

Since e belongs to an internal path of type (i), G has a cycle as an induced subgraph. From Theorem 3.2, we know that the above inequality is strict. For any vertex $v_i \neq v_n$, we can easily obtain $(A_f(H)\mathbf{y})_i = \lambda_1(A_f(G))y_i$.

Hence, $A_f(H)\mathbf{y} < \lambda_1(A_f(G))\mathbf{y}$. Using Lemma 2.2, we get $\lambda_1(A_f(H)) < \lambda_1(A_f(G))$.

Case 1.2. k is odd.

We have $x_{\frac{k-1}{2}} = x_{\frac{k+3}{2}}$. It is not difficult to see that

$$\lambda_1(A_f(G))x_{\frac{k+1}{2}} = f(2,2)x_{\frac{k-1}{2}} + f(2,2)x_{\frac{k+3}{2}} = 2f(2,2)x_{\frac{k-1}{2}}.$$

Since $\lambda_1(A_f(G)) > 2f(2,2)$, we obtain $x_{\frac{k+1}{2}} < x_{\frac{k-1}{2}}$. Without loss of generality, we take $e = v_{\frac{k-1}{2}}v_{\frac{k+1}{2}}$. Let vector $\mathbf{y}$ be obtained from $\mathbf{x}$ by inserting the addition entry $y_n = x_{\frac{k+1}{2}}$. That is,

$$\begin{cases} y_n = x_{\frac{k+1}{2}}, \\ y_i = x_i, \qquad v_i \in V(G). \end{cases}$$

Hence, vector $A_f(H)\mathbf{y}$ differs from $\lambda_1(A_f(G))\mathbf{y}$ only in the $\frac{k+1}{2}$-th and n-th entries. Comparing the two corresponding entries in $A_f(H)\mathbf{y}$ and $\lambda_1(A_f(G))\mathbf{y}$, respectively, we get

$$\begin{aligned} (A_f(H)\mathbf{y})_{\frac{k+1}{2}} &= f(2,2)y_n + f(2,2)x_{\frac{k+3}{2}} \\ &= f(2,2)x_{\frac{k+1}{2}} + f(2,2)x_{\frac{k+3}{2}} \\ &< f(2,2)x_{\frac{k-1}{2}} + f(2,2)x_{\frac{k+3}{2}} \\ &= \lambda_1(A_f(G))x_{\frac{k+1}{2}} = \lambda_1(A_f(G))y_{\frac{k+1}{2}}, \end{aligned}$$

and

$$\begin{aligned} (A_f(H)\mathbf{y})_n &= f(2,2)x_{\frac{k-1}{2}} + f(2,2)x_{\frac{k+1}{2}} \\ &< f(2,2)x_{\frac{k-1}{2}} + f(2,2)x_{\frac{k-1}{2}} \\ &= f(2,2)x_{\frac{k-1}{2}} + f(2,2)x_{\frac{k+3}{2}} \\ &= \lambda_1(A_f(G))x_{\frac{k+1}{2}} = \lambda_1(A_f(G))y_n. \end{aligned}$$

It follows that $A_f(H)\mathbf{y} < \lambda_1(A_f(G))\mathbf{y}$. From Lemma 2.2, we get $\lambda_1(A_f(H)) < \lambda_1(A_f(G))$.

Case 2. e belongs to an internal path $P_{k+2} = v_0v_1 \dots v_{k+1}$ of type (ii).

Let x_i be the entry of $\mathbf{x}$ corresponding to the vertex v_i of G where $i = 0, 1, \dots, k+1$. Let t be the smallest index such that $x_t = \min\{x_0, x_1, \dots, x_{k+1}\}$. Because $x_0 \le x_{k+1}$, we have $0 \le t < k+1$. Without loss of generality, we take $e = v_tv_{t+1}$. Here we still discuss by distinguishing two cases.

Case 2.1. $t > 0$.

Since $v_1v_2 \dots v_k$ is an induced path, by Theorem 3.1, it follows that $0 < t \le \lfloor \frac{k+1}{2} \rfloor$. Furthermore, we have $x_i > x_t$ for $0 \le i < t$, and $x_t \le x_{t+1} < x_i$ for $t+1 < i \le k+1$. Let $\mathbf{y}$ be obtained from $\mathbf{x}$ by inserting the addition entry $y_n = x_t$. That is,

$$\begin{cases} y_n = x_t, \\ y_i = x_i, \quad v_i \in V(G). \end{cases}$$

We can deduce that $A_f(H)\mathbf{y}$ differs from $\lambda_1(A_f(G))\mathbf{y}$ only in the t-th and n-th entries. It is not difficult to get that

$$\begin{aligned}(A_f(H)\mathbf{y})_t &= f(2,d_{t-1})x_{t-1} + f(2,2)y_n \\ &= f(2,d_{t-1})x_{t-1} + f(2,2)x_t \\ &\le f(2,d_{t-1})x_{t-1} + f(2,d_{t+1})x_{t+1} \\ &= \lambda_1(A_f(G))x_t = \lambda_1(A_f(G))y_t,\end{aligned}$$

and

$$\begin{aligned}(A_f(H)\mathbf{y})_n &= f(2,2)x_t + f(2,d_{t+1})x_{t+1} \\ &< f(2,d_{t-1})x_{t-1} + f(2,d_{t+1})x_{t+1} \\ &= \lambda_1(A_f(G))x_t = \lambda_1(A_f(G))y_n.\end{aligned}$$

Thus, $A_f(H)\mathbf{y} < \lambda_1(A_f(G))\mathbf{y}$. Using Lemma 2.2, we have $\lambda_1(A_f(H)) < \lambda_1(A_f(G))$.

Case 2.2. $t = 0$.

We take $e = v_0v_1$. According to the choice of t, it follows that $x_0 \le x_i$ for $1 \le i \le k+1$. For convenience, let S be the set of neighbours of v_0 other than v_1 in G, and $s = \sum\limits_{v_j \in S} f(d_0, d_j)x_j$, and let R be the set of neighbours of v_1 other than v_0 in G, and $r = \sum\limits_{v_j \in R} f(d_1, d_j)x_j$.

Subcase 2.2.1. $f(d_0,2)x_0 + f(d_1,2)x_1 < \lambda_1(A_f(G))x_0$.

Let $\mathbf{y}$ be obtained from $\mathbf{x}$ by inserting the addition entry $y_n = x_0$, that is,

$$\begin{cases} y_n = x_0, \\ y_i = x_i, \quad v_i \in V(G). \end{cases}$$

It is easy to show that vector $A_f(H)\mathbf{y}$ differs from $\lambda_1(A_f(G))\mathbf{y}$ in at most three entries: 0-th, 1-th and n-th. For the vertex v_0, we have

$$\begin{aligned}(A_f(H)\mathbf{y})_0 &= f(d_0,2)y_n + \sum_{v_j \in S} f(d_0,d_j)x_j \\ &= f(d_0,2)x_0 + s \\ &\le f(d_0,d_1)x_1 + s \\ &= \lambda_1(A_f(G))x_0 = \lambda_1(A_f(G))y_0.\end{aligned}$$

For the vertex v_1, we have

$$\begin{aligned}(A_f(H)\mathbf{y})_1 &= f(d_1,2)y_n + \sum_{v_j\in R} f(d_1,d_j)x_j \\ &= f(d_1,2)x_0 + r \\ &\le f(d_1,d_0)x_0 + r \\ &= \lambda_1(A_f(G))x_1 = \lambda_1(A_f(G))y_1.\end{aligned}$$

For the vertex v_n, we have

$$\begin{aligned}(A_f(H)\mathbf{y})_n &= f(d_0,2)x_0 + f(d_1,2)x_1 \\ &< \lambda_1(A_f(G))x_0 = \lambda_1(A_f(G))y_n.\end{aligned}$$

It follows that $A_f(H)\mathbf{y} < \lambda_1(A_f(G))\mathbf{y}$. From Lemma 2.2, we get $\lambda_1(A_f(H)) < \lambda_1(A_f(G))$.

Subcase 2.2.2. $f(d_0,2)x_0 + f(d_1,2)x_1 \ge \lambda_1(A_f(G))x_0$.

Since $\lambda_1(A_f(G))x_0 = s + f(d_0,d_1)x_1$, we obtain $s \le f(d_0,2)x_0 + f(d_1,2)x_1 - f(d_0,d_1)x_1$. Recall that $f(x,y) > 0$ is an increasing function. Then, $f(d_1,2) - f(d_0,d_1) \le 0$ and $f(d_0,2) > 0$. Because each entry of $\mathbf{x}$ is positive, we can get $0 < s = \sum_{v_j\in S} f(d_0,d_j)x_j \le f(d_0,2)x_0$. Hence, $\frac{s}{f(d_0,2)} \le x_0$. Now let vector $\mathbf{y}$ be an assignment of the vertices of G satisfying that

$$\begin{cases} y_0 = \frac{\lambda_1(A_f(G))x_0 - f(d_0,d_1)x_1}{f(d_0,2)}, \\ y_n = x_0, \\ y_i = x_i, & v_i \in V(G)\setminus\{v_0\}. \end{cases}$$

Since $s = \lambda_1(A_f(G))x_0 - f(d_0,d_1)x_1$, we have $y_0 = \frac{s}{f(d_0,2)}$, and then $0 < y_0 \le x_0$. In addition, $\mathbf{x}$ is a positive vector, and hence $\mathbf{y}$ is also a positive vector. Next we prove that the $(n+1)$-dimensional vector $\mathbf{y}$ satisfies $A_f(H)\mathbf{y} < \lambda_1(A_f(G))\mathbf{y}$. The vector $A_f(H)\mathbf{y}$ differs from $\lambda_1(A_f(G))\mathbf{y}$ in at most the following entries. For the vertex v_0, we establish that

$$\begin{aligned}
(A_f(H)\mathbf{y})_0 &= f(d_0,2)y_n + \sum_{v_j\in S} f(d_0,d_j)x_j \\
&= f(d_0,2)x_0 + s \\
&\le 2f(d_0,2)x_0 \\
&\le (d_0-1)f(d_0,2)x_0 \\
&= \sum_{v_j\in S} f(d_0,2)x_0 \\
&\le \sum_{v_j\in S} f(d_0,d_j)x_0 \\
&\le \sum_{v_j\in S} \frac{f(d_0,d_j)}{f(d_0,2)}\cdot f(d_0,d_j)x_0 \\
&< \sum_{v_j\in S} \frac{f(d_0,d_j)}{f(d_0,2)}\cdot \lambda_1(A_f(G))x_j \\
&= \frac{\lambda_1(A_f(G))}{f(d_0,2)}\cdot s = \lambda_1(A_f(G))y_0.
\end{aligned}$$

Because the degrees of the neighbours of v_0 are at least 2 and $f(x,y)>0$ is increasing, we get $f(d_0,2)\le f(d_0,d_j)$. It follows that the third and fourth inequalities hold. If $v_j\in S$, we have

$$f(d_0,d_j)x_0 < \sum_{v_k\in N_G(v_j)\setminus v_0} f(d_k,d_j)x_k + f(d_0,d_j)x_0 = \lambda_1(A_f(G))x_j.$$

Hence, the last inequality is strict. For the vertex v_1, we have

$$\begin{aligned}
(A_f(H)\mathbf{y})_1 &= f(d_1,2)y_n + \sum_{v_j\in R} f(d_1,d_j)x_j \\
&= f(d_1,2)x_0 + r \\
&\le f(d_1,d_0)x_0 + r \\
&= \lambda_1(A_f(G))x_1 = \lambda_1(A_f(G))y_1.
\end{aligned}$$

For the vertex $v_j\in S$, we have

$$\begin{aligned}
(A_f(H)\mathbf{y})_j &= \sum_{v_k\in N_H(v_j)\setminus v_0} f(d_k,d_j)x_k + f(d_j,d_0)y_0 \\
&\le \sum_{v_k\in N_G(v_j)\setminus v_0} f(d_k,d_j)x_k + f(d_j,d_0)x_0 \\
&= \lambda_1(A_f(G))x_j = \lambda_1(A_f(G))y_j.
\end{aligned}$$

For the vertex v_n, we have

$$\begin{aligned}(A_f(H)\mathbf{y})_n &= f(d_0,2)y_0 + f(d_1,2)y_1\\ &= s + f(d_1,2)x_1\\ &\leq s + f(d_1,d_0)x_1\\ &= \lambda_1(A_f(G))x_0 = \lambda_1(A_f(G))y_n.\end{aligned}$$

Thus, $A_f(H)\mathbf{y} < \lambda_1(A_f(G))\mathbf{y}$. By Lemma 2.2, it follows that $\lambda_1(A_f(H)) < \lambda_1(A_f(G))$. The proof is now complete. $\square$

Remark 2. In Theorem 3.5 (2), the condition "for any vertex $v_i \in N_G(v_0)$, $d_i \geq 2$" is necessary. Otherwise, there are graphs G and H which do not satisfy $\lambda_1(A_f(H)) < \lambda_1(A_f(G))$. For example, set $f(x,y) = \sqrt{\lg(xy)}$, considering $G = G_2$ and $H = G_1$ in Figure 1, then we can have $1.5845 \approx \lambda_1(A_f(G_1)) > \lambda_1(A_f(G_2)) \approx 1.5806$.

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