On the saturation spectrum of families of cycle subdivisions

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Abstract

Given a family of graphs $\mathcal{F}$, a graph G is $\mathcal{F}$-saturated if G contains no member of $\mathcal{F}$ as a subgraph, but $G+e$ contains a member of $\mathcal{F}$ as a subgraph for each edge e in the complement of G. The edge spectrum of $\mathcal{F}$ is the set of all possible sizes of $\mathcal{F}$-saturated graphs on n vertices. A G-subdivision is a graph derived from G by replacing each edge of G with a path of arbitrary length. Let $\mathcal{C}_{\geq k}$ denote the family of C_k-subdivisions, where C_k is a cycle of length k with $k \geq 3$. Determining the minimum or maximum number of edges in n-vertex $\mathcal{F}$-saturated graphs are two of the most important problems in the study of extremal graph theory. This is also a very important optimization problem in graph theory. The study of this problem is closely related to the development of other branches of mathematics, computer science, network, modern information science and technology. In this paper, we determine the edge spectrum of $\mathcal{C}_{\geq k}$ for each $k \in \{3,4,5,6\}$.

Keywords: edge spectrum; extremal numbers; saturation numbers; cycle subdivisions

1 Introduction

All graphs considered in this paper are finite and simple. We follow [10] for undefined notation and terminology. Let k be a positive integer and $[k] = \{1,2,3,...,k\}$. Let $G = G(V(G), E(G))$ be a graph, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. We denote $e(G) = |E(G)|$ and call it the *size* of G. Denote by $\overline{G}$ the complement of G. Let K_k denote the complete graph on k vertices. A *G-subdivision* is a graph derived from G by replacing each edge of G with a path of arbitrary length. Let $\mathcal{C}_{\geq k}$ denote the family

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of C_k-subdivisions, where C_k is a cycle of length k with $k \geq 3$. Note that $C_k \in \mathcal{C}_{\geq k}$ and $\mathcal{C}_{\geq k}$ is a family of cycles that each of them has length at least k. A maximal connected subgraph of G that has no cut-vertex is called a *block* of G. For a graph H, the H-block of G is a block of G isomorphic to H.

Given a family of graphs $\mathcal{F}$, a graph G is $\mathcal{F}$*-saturated* if G contains no member of $\mathcal{F}$ as a subgraph, but $G+e$ contains a member of $\mathcal{F}$ as a subgraph for each $e \in E(\overline{G})$. Let $ex(n, \mathcal{F}) = \max\{e(G) : G$ is $\mathcal{F}$-saturated and $|V(G)| = n\}$ and $sat(n, \mathcal{F}) = \min\{e(G) : G$ is $\mathcal{F}$-saturated and $|V(G)| = n\}$. We shall refer to $ex(n, \mathcal{F})$ as the extremal number of $\mathcal{F}$ and $sat(n, \mathcal{F})$ as the saturation number of $\mathcal{F}$. By the definitions above, if an $\mathcal{F}$-saturated graph of order n has m edges, then $sat(n, \mathcal{F}) \leq m \leq ex(n, \mathcal{F})$. It is natural to consider the converse that whether there exists an $\mathcal{F}$-saturated graph of order n and size m for any integer m with $sat(n, \mathcal{F}) \leq m \leq ex(n, \mathcal{F})$. In order to study this problem, the concept of the edge spectrum was introduced.

The edge spectrum of $\mathcal{F}$, denoted by $ES(n, \mathcal{F})$, is the set of all possible sizes of $\mathcal{F}$-saturated graphs on n vertices. That is, $ES(n, \mathcal{F}) = \{e(G) : G$ is $\mathcal{F}$-saturated and $|V(G)| = n\}$. When $\mathcal{F} = \{F\}$, we simply write F-saturated for $\mathcal{F}$-saturated and replace $ES(n, \mathcal{F})$ with $ES(n, F)$. For some special graph classes $\mathcal{F}$, we may have $ES(n, \mathcal{F}) = \{m : m$ is an integer with $sat(n, \mathcal{F}) \leq m \leq ex(n, \mathcal{F})\}$. For instance, in 2018, Balister and Dogan [3] proved that $ES(n, K_{1,t})$ consists of all integers in the interval $[sat(n, K_{1,t}), ex(n, K_{1,t})]$, where t is a positive integer with $t \leq n-1$. However, it fails in general. Some gaps of edge spectrum have been found for graphs K_t [1, 2, 4], $K_4 - e$ [7], P_5 and P_6 [8], and so on. It is natural to consider which graph has gapless edge spectrum and which graph has gaps in its edge spectrum. In this paper, we consider the edge spectrum for $\mathcal{C}_{\geq k}$.

Erdős and Gallai [5], and Woodall [11] provided the extremal number of $\mathcal{C}_{\geq k}$ for each $3 \leq k \leq n$. They also provided an extremal graph consisting of $\lfloor \frac{n-1}{k-2} \rfloor$ copies of K_{k-1} and one copy of K_{t+1}, where $n-1 \equiv t \ (mod\ (k-2))$, $0 \leq t \leq k-3$, and all copies share exactly one vertex in common. That is $ex(n, \mathcal{C}_{\geq k}) = \lfloor \frac{n-1}{k-2} \rfloor \binom{k-1}{2} + \binom{t+1}{2}$. In this paper, we refer the case $k \in \{3, 4, 5, 6\}$, so we present their results as follows.

Theorem 1.1 *([5, 11]) Let n be an integer.*
(1) For $n \geq 3$, $ex(n, \mathcal{C}_{\geq 3}) = n - 1$.
(2) For $n \geq 4$, $ex(n, \mathcal{C}_{\geq 4}) = \lfloor \frac{3n-3}{2} \rfloor$.
(3) For $n \geq 5$, $ex(n, \mathcal{C}_{\geq 5}) = \begin{cases} 2n-2, & \text{if } n \equiv 1 \ (mod\ 3); \\ 2n-3, & \text{if } n \equiv 0 \ (mod\ 3) \text{ or } n \equiv 2 \ (mod\ 3). \end{cases}$
(4) For $n \geq 6$, $ex(n, \mathcal{C}_{\geq 6}) = \begin{cases} \lfloor \frac{5n-5}{2} \rfloor, & \text{if } n \equiv 1 \ (mod\ 4); \\ \lfloor \frac{5n-8}{2} \rfloor, & \text{if } n \equiv 0 \ (mod\ 4),\ n \equiv 2 \ (mod\ 4) \text{ or } n \equiv 3 \ (mod\ 4). \end{cases}$

It is trivial that $sat(n, \mathcal{C}_{\geq 3}) = ex(n, \mathcal{C}_{\geq 3}) = n - 1$. In [6], the authors provided the value $sat(n, \mathcal{C}_{\geq i})$ for each $i \in \{4, 5\}$. Moreover, Ma, Hou, Hei, and Gao [9] determined $sat(n, \mathcal{C}_{\geq 6})$.

Theorem 1.2 *([6, 9]) Let n be an integer.*
(1) For $n \geq 4$, $sat(n, \mathcal{C}_{\geq 4}) = n + \lfloor \frac{n-3}{4} \rfloor$.
(2) For $n \geq 5$, $sat(n, \mathcal{C}_{\geq 5}) = \lceil \frac{10(n-1)}{7} \rceil$.
(3) For $n \geq 6$,
$$sat(n, \mathcal{C}_{\geq 6}) = \begin{cases} 9, & \text{if } n = 6; \\ 11, & \text{if } n = 7; \\ 12, & \text{if } n = 8; \\ 13, & \text{if } n = 9; \\ \lceil \frac{3(n-1)}{2} \rceil, & \text{if } n \geq 10. \end{cases}$$

Combining Theorems 1.1 and 1.2, we prove that there is no gap in $ES(n, \mathcal{C}_{\geq r})$ for each $r \in \{3, 4, 5\}$ and there is a gap in $ES(n, \mathcal{C}_{\geq 6})$ when $n \equiv 0 \ (mod\ 4)$ or $n \equiv 1 \ (mod\ 4)$.

Theorem 1.3 *Let m, n, r be three integers with $n \geq r$. For each $r \in \{3, 4, 5\}$ and each m with $sat(n, \mathcal{C}_{\geq r}) \leq m \leq ex(n, \mathcal{C}_{\geq r})$, there is a $\mathcal{C}_{\geq r}$-saturated graph on n vertices and m edges.*

Theorem 1.4 *Let n and m be two integers with $n \geq 6$ and $sat(n, \mathcal{C}_{\geq 6}) \leq m \leq ex(n, \mathcal{C}_{\geq 6})$. There is a $\mathcal{C}_{\geq 6}$-saturated graph on n vertices and m edges if and only if*

$$m \notin \begin{cases} \{ex(n, \mathcal{C}_{\geq 6}) - 1\}, & if\ n \equiv 0 \ (mod\ 4); \\ \{ex(n, \mathcal{C}_{\geq 6}) - 1, ex(n, \mathcal{C}_{\geq 6}) - 2\}, & if\ n \equiv 1 \ (mod\ 4). \end{cases}$$

The rest of the paper is organized as follows. In Section 2, we introduce some known properties of $\mathcal{C}_{\geq r}$-saturated graphs for each $r \in \{4, 5, 6\}$, which will be used to verify the graphs we constructed are $\mathcal{C}_{\geq r}$-saturated. In Section 3, we prove Theorems 1.3 and 1.4 by giving a complete characterization for the edge spectrum of $\mathcal{C}_{\geq r}$.

2 Structural properties of $\mathcal{C}_{\geq r}$-saturated graphs for each $r \in \{4, 5, 6\}$

In [6], Ferrara, Jacobson, Milans, Tennenhouse, and Wenger characterized some properties of $\mathcal{C}_{\geq 4}$-saturated graphs and $\mathcal{C}_{\geq 5}$-saturated graphs, respectively.

Proposition 2.1 *([6]) A connected graph G with at least two vertices is $\mathcal{C}_{\geq 4}$-saturated if and only if every block of G is isomorphic to either K_2 or K_3, and no two K_2-blocks of G share a vertex.*

The graph $B_t = K_2 \vee \overline{K}_t$ is called a *book*, which is obtained from $K_2 \cup \overline{K}_t$ by joining each vertex of K_2 to each vertex of $\overline{K}_t$. Every vertex of $\overline{K}_t$ is called a *page* of the book B_t.

Proposition 2.2 *([6]) A graph G is $\mathcal{C}_{\geq 5}$-saturated if and only if*
(1) every block of G is isomorphic to either a complete graph of order at most 4 or a book with at least three pages, and
(2) for any K_2-block B of G and block $B' \neq B$ with $B \cap B' \neq \emptyset$, either B' is a K_4-block or B' is a B_t-block with $t \geq 3$ such that $B \cap B'$ is a page of B'.

Let r, s and t be three integers with $r, s \geq 2$ and $t \geq 6$. As shown in Figure 1(1), the graph denoted by $D(r, s)$ has $r+s+3$ vertices and each vertex in $\{v_1, v_2, ..., v_r, u_1, u_2, ..., u_s\}$ has degree 2. As depicted in Figure 1(2), the graph denoted by $H(t, 6, 2)$ has t vertices and each vertex in $\{u_1, u_2, ..., u_{t-4}\}$ has degree 2. In both of $D(r, s)$ and $H(t, 6, 2)$, the white vertices are called *centers* of them. Clearly, every $\mathcal{C}_{\geq 6}$-saturated graph of order at most 5 is a clique.

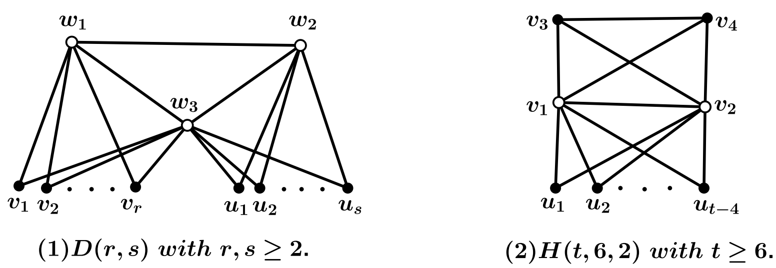


Figure 1: Two $\mathcal{C}_{\geq 6}$-saturated graphs $D(r, s)$ and $H(t, 6, 2)$.

To avoid using more definitions, we use the following proposition from [9], which is stated slightly different, but can be derived from the original statement.

Proposition 2.3 *([9]) A graph G is $\mathcal{C}_{\geq 6}$-saturated if and only if*
(1) G is connected and each block of G is isomorphic to one of $\{K_t : 1 \leq t \leq 5\} \cup \{D(r, s) : s, r \geq 2\} \cup \{H(t, 6, 2) : t \geq 6\}$, and
(2) no two K_3-blocks of G share a vertex, and
(3) for any K_2-block B of G and block $B' \neq B$ with $B \cap B' \neq \emptyset$, we have $B' \cong K_5$, $B' \cong D(r, s)$ or $B' \cong H(t, 6, 2)$ for integers $r, s \geq 2$ and $t \geq 6$ such that $B \cap B'$ is not a center of B' when $B' \cong D(r, s)$ or $B' \cong H(t, 6, 2)$.

3 Proofs of Theorems 1.3 and 1.4

For any graph, we choose one vertex of it as its root vertex. Denote by $G \cdot H$ the graph obtained from two disjoint graphs G and H by identifying the root vertex of G and the root vertex of H, where the identifying vertex is the root vertex of the graph $G \cdot H$. That is, $|V(G \cdot H)| = |V(G)| + |V(H)| - 1$ and $E(G \cdot H) = E(G) \cup E(H)$. Let $\prod\limits^{0} G = K_1$, $\prod\limits^{1} G = G$ and $\prod\limits^{k} G = (\prod\limits^{k-1} G) \cdot G$ for an integer k with $k \geq 2$.

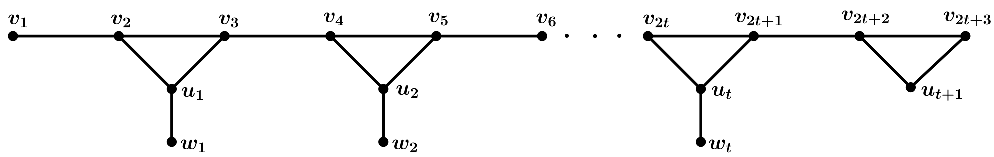


Figure 2: A $\mathcal{C}_{\geq 4}$-saturated graph G.

Proof of Theorem 1.3. Theorem 1.3 holds naturally if $r = 3$, since $ex(n, \mathcal{C}_{\geq 3}) = sat(n, \mathcal{C}_{\geq 3}) = n - 1$. So we may assume $r \geq 4$. The rest proof is split into two cases: $r = 4$ and $r = 5$.

Case 1: $r = 4$.

As shown in Figure 2, the graph G has $4t+4$ vertices, each vertex in $\{u_1, u_2, ..., u_t, v_2, v_3, ..., v_{2t+2}\}$ has degree 3, each vertex in $\{v_1, w_1, w_2, ..., w_t\}$ has degree 1, and each vertex in $\{u_{t+1}, v_{2t+3}\}$ has degree 2. By Proposition 2.1, the graph G is a $\mathcal{C}_{\geq 4}$-saturated graph. For each integer $n \geq 4$, let

$$G_n^0 = \begin{cases} G, & \text{if } n = 4t+4; \\ G - \{v_1\}, & \text{if } n = 4t+3; \\ G - \{v_{2t+3}, u_{t+1}\}, & \text{if } n = 4t+2; \\ G - \{v_{2t+2}, v_{2t+3}, u_{t+1}\}, & \text{if } n = 4t+1. \end{cases}$$

Clearly, we have

$$e(G_n^0) = \begin{cases} 5t+4 = n + \lfloor \frac{n-3}{4} \rfloor, & \text{if } n = 4t+4; \\ 5t+3 = n + \lfloor \frac{n-3}{4} \rfloor, & \text{if } n = 4t+3; \\ 5t+1 = n + \lfloor \frac{n-3}{4} \rfloor, & \text{if } n = 4t+2; \\ 5t = n + \lfloor \frac{n-3}{4} \rfloor, & \text{if } n = 4t+1. \end{cases}$$

By Theorem 1.2(1), $e(G_n^0) = sat(n, \mathcal{C}_{\geq 4})$. Let i be an integer. For each $1 \leq i \leq t$, let

$$G_n^i = \begin{cases} (E(G_n^{i-1}) \setminus \{u_i w_i\}) \cup \{v_{2i-1} w_i, v_{2i} w_i\}, & \text{if } n = 4t+1; \\ (E(G_n^{i-1}) \setminus \{u_i w_i\}) \cup \{v_{2i+1} w_i, v_{2i+2} w_i\}, & \text{if } n \in \{4t+2, 4t+3, 4t+4\}. \end{cases}$$

For each $0 \leq j \leq t$, each block of G_n^j is isomorphic to either K_2 or K_3 and there are no two K_2-blocks of G_n^j sharing a vertex. By Proposition 2.1, the graph G_n^j is $\mathcal{C}_{\geq 4}$-saturated.

Clearly, $e(G_n^i) = e(G_n^{i-1}) + 1$ for $1 \leq i \leq t$ and

$$e(G_n^t) = e(G_n^0) + t = \begin{cases} 6t+4 = \lfloor \frac{3n-3}{2} \rfloor, & \text{if } n = 4t+4; \\ 6t+3 = \lfloor \frac{3n-3}{2} \rfloor, & \text{if } n = 4t+3; \\ 6t+1 = \lfloor \frac{3n-3}{2} \rfloor, & \text{if } n = 4t+2; \\ 6t = \lfloor \frac{3n-3}{2} \rfloor, & \text{if } n = 4t+1. \end{cases}$$

Then we have $e(G_n^t) = ex(n, \mathcal{C}_{\geq 4})$ by Theorem 1.1(2). Hence there is a $\mathcal{C}_{\geq 4}$-saturated graph on $n \geq 4$ vertices and m edges for each integer m with $sat(n, \mathcal{C}_{\geq 4}) \leq m \leq ex(n, \mathcal{C}_{\geq 4})$.

Case 2: $r = 5$.

By Theorems 1.1 and 1.2, we have $sat(5, \mathcal{C}_{\geq 5}) = 6$, $ex(5, \mathcal{C}_{\geq 5}) = 7$, $sat(6, \mathcal{C}_{\geq 5}) = 8$ and $ex(6, \mathcal{C}_{\geq 5}) = 9$, which implies that Theorem 1.3 holds for $5 \leq n \leq 6$. Thus we may assume $n \geq 7$. Inspired by the constructions of Ferrara et al. [6], we construct the graph H_i and H_i has exactly one root vertex v for each $i \in \{0, 1, 2, 3, 4, 5, 6\}$. For each graph in Figure 3, we denote the vertex v as the root vertex. Let $H_2 = H_0 \cdot K_3$, $H_3 = H_1 \cdot K_3$, $H_4 = H_2 \cdot K_3 = H_0 \cdot \prod^{2} K_3$, $H_5 = H_1 \cdot \prod^{2} K_3$, and $H_6 = M_0 \cdot H_1$. We obtain $|V(H_i)| = 7 + i$ and $e(H_i) = 9 + \lfloor \frac{3i}{2} \rfloor$ for each $i \in \{0, 1, 2, 3, 4, 5, 6\}$.

Let

$$G_n^0 = \begin{cases} H_0 \cdot \prod^{t-1} H_1, & \text{if } n = 7t; \\ \prod^{t} H_1, & \text{if } n = 7t+1; \\ H_2 \cdot \prod^{t-1} H_1, & \text{if } n = 7t+2; \\ H_3 \cdot \prod^{t-1} H_1, & \text{if } n = 7t+3; \\ H_4 \cdot \prod^{t-1} H_1, & \text{if } n = 7t+4; \\ H_5 \cdot \prod^{t-1} H_1, & \text{if } n = 7t+5; \\ H_6 \cdot \prod^{t-1} H_1, & \text{if } n = 7t+6. \end{cases}$$

Then

$$e(G_n^0) = \begin{cases} 10t-1 = \lceil \frac{10(n-1)}{7} \rceil, & \text{if } n = 7t; \\ 10t = \lceil \frac{10(n-1)}{7} \rceil, & \text{if } n = 7t+1; \\ 10t+2 = \lceil \frac{10(n-1)}{7} \rceil, & \text{if } n = 7t+2; \\ 10t+3 = \lceil \frac{10(n-1)}{7} \rceil, & \text{if } n = 7t+3; \\ 10t+5 = \lceil \frac{10(n-1)}{7} \rceil, & \text{if } n = 7t+4; \\ 10t+6 = \lceil \frac{10(n-1)}{7} \rceil, & \text{if } n = 7t+5; \\ 10t+8 = \lceil \frac{10(n-1)}{7} \rceil, & \text{if } n = 7t+6. \end{cases}$$

By Theorem 1.2(2), $e(G_n^0) = sat(n, \mathcal{C}_{\geq 5})$.

Let i be an integer with $i \geq 1$. We try to construct the graph G_n^1 based on G_n^0. By the structure of G_n^0, there is a pair of adjacent vertices u and w such that $d_{G_n^0}(w) = 1$

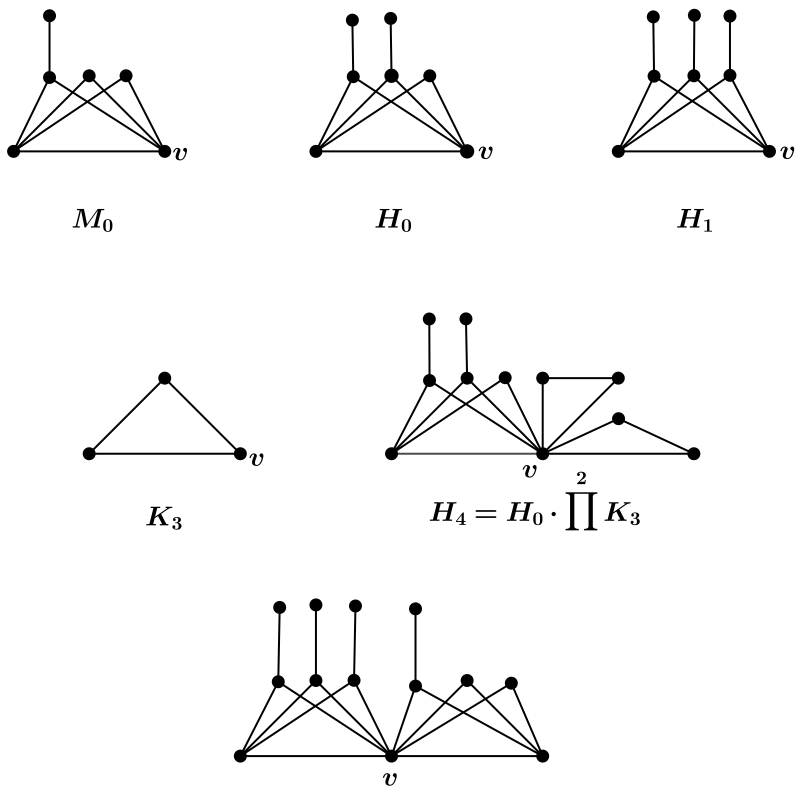


Figure 3: Some $\mathcal{C}_{\geq 5}$-saturated graphs.

and $d_{G_n^0}(u) = 3$, say $N_{G_n^0}(u) = \{u_1, u_2, w\}$. We call such pair as a $(1,3)$-pair. Let $G_n^1 = (G_n^0 \setminus \{wu\}) \cup \{wu_1, wu_2\}$. Observe that there are two $(1,3)$-pairs in H_i for each $i \in \{0, 2, 4\}$, three $(1,3)$-pairs in H_j for each $j \in \{1, 3, 5\}$, and four $(1,3)$-pairs in H_6. Then G_n^0 has $3t - t_0$ such vertex pairs, where

$$t_0 = \begin{cases} 1, & \text{if } n \in \{7t, 7t+2, 7t+4\}; \\ 0, & \text{if } n \in \{7t+1, 7t+3, 7t+5\}; \\ -1, & \text{if } n = 7t+6. \end{cases}$$

Applying this method, we construct the graph G_n^i iteratively for each integer $1 \leq i \leq 3t - t_0$ as follows. Choose a pair of adjacent vertices u and w with $d_{G_n^{i-1}}(w) = 1$, $N_{G_n^{i-1}}(w) = \{u\}$ and $d_{G_n^{i-1}}(u) = 3$. Denote $N_{G_n^{i-1}}(u) = \{u_1, u_2, w\}$. Let $G_n^i = (G_n^{i-1} \setminus \{wu\}) \cup \{wu_1, wu_2\}$. We have $e(G_n^i) = e(G_n^{i-1}) + 1$. Let k and ℓ be two integers. Clearly, for each $1 \leq i \leq 3t - t_0$, each block of G_n^i is isomorphic to either K_ℓ with $2 \leq \ell \leq 3$ or a book B_k with $k \geq 3$. For any K_2-block B of G_n^i and block $B' \neq B$ with $B \cap B' \neq \emptyset$, B' is a B_k-block with $k \geq 3$ and $B \cap B'$ is a page of B'. Proposition 2.2 implies that G_n^i is $\mathcal{C}_{\geq 5}$-saturated. For K_k with $k \in \{2, 3, 4\}$, choose one vertex of K_k as the root vertex, and for the book B_j where j is an integer and $j \geq 3$, choose one vertex of B_j that is not a page as the root vertex. We have

$$G_n^{3t-t_0} = \begin{cases} (\prod^{t-1} B_6) \cdot B_5, & \text{if } n = 7t; \\ \prod^{t} B_6, & \text{if } n = 7t+1; \\ (\prod^{t-1} B_6) \cdot B_5 \cdot K_3, & \text{if } n = 7t+2; \\ (\prod^{t} B_6) \cdot K_3, & \text{if } n = 7t+3; \\ (\prod^{t-1} B_6) \cdot B_5 \cdot \prod^{2} K_3, & \text{if } n = 7t+4; \\ (\prod^{t} B_6) \cdot \prod^{2} K_3, & \text{if } n = 7t+5; \\ (\prod^{t} B_6) \cdot B_4, & \text{if } n = 7t+6. \end{cases}$$

Set

$$H_n^0 = \begin{cases} (\prod^{t-1} B_5) \cdot B_{t+4}, & \text{if } n = 7t; \\ (\prod^{t-1} B_5) \cdot B_{t+5}, & \text{if } n = 7t+1; \\ (\prod^{t-1} B_5) \cdot B_{t+4} \cdot K_3, & \text{if } n = 7t+2; \\ (\prod^{t-1} B_5) \cdot B_{t+5} \cdot K_3, & \text{if } n = 7t+3; \\ (\prod^{t-1} B_5) \cdot B_{t+4} \cdot \prod^{2} K_3, & \text{if } n = 7t+4; \\ (\prod^{t-1} B_5) \cdot B_{t+5} \cdot \prod^{2} K_3, & \text{if } n = 7t+5; \\ (\prod^{t} B_5) \cdot B_{t+4}, & \text{if } n = 7t+6. \end{cases}$$

We see $e(H_n^0) = e(G_n^{3t-t_0})$. Let i be an integer with $0 \leq i \leq t-1$. Let

$$H_n^i = \begin{cases} (\prod^{t-1-i} B_5) \cdot B_{t+4} \cdot \prod^{2i} K_4, & \text{if } n = 7t; \\ (\prod^{t-1-i} B_5) \cdot B_{t+5} \cdot \prod^{2i} K_4, & \text{if } n = 7t+1; \\ (\prod^{t-1-i} B_5) \cdot B_{t+4} \cdot K_3 \cdot \prod^{2i} K_4, & \text{if } n = 7t+2; \\ (\prod^{t-1-i} B_5) \cdot B_{t+5} \cdot K_3 \cdot \prod^{2i} K_4, & \text{if } n = 7t+3; \\ (\prod^{t-1-i} B_5) \cdot B_{t+4} \cdot (\prod^{2} K_3) \cdot \prod^{2i} K_4, & \text{if } n = 7t+4; \\ (\prod^{t-1-i} B_5) \cdot B_{t+5} \cdot (\prod^{2} K_3) \cdot \prod^{2i} K_4, & \text{if } n = 7t+5; \\ (\prod^{t-i} B_5) \cdot B_{t+4} \cdot \prod^{2i} K_4, & \text{if } n = 7t+6, \end{cases}$$

and

$$H_n^t = \begin{cases} H_n^{t-1}, & \text{if } n = 7t; \\ H_n^{t-1}, & \text{if } n = 7t+1; \\ B_{t+6} \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+2; \\ B_{t+7} \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+3; \\ B_{t+6} \cdot K_3 \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+4; \\ B_{t+7} \cdot K_3 \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+5; \\ B_{t+4} \cdot \prod^{2t} K_4, & \text{if } n = 7t+6. \end{cases}$$

Since every block of H_n^i is isomorphic to either K_ℓ with $\ell \in \{3,4\}$ or B_k, where k is an integer and $k \geq 3$, Proposition 2.2 implies that H_n^i is $\mathcal{C}_{\geq 5}$-saturated for each $0 \leq i \leq t$. We may see $e(H_n^i) = e(H_n^{i-1}) + 1$ for each $1 \leq i \leq t-1$ and

$$e(H_n^t) = \begin{cases} e(H_n^{t-1}), & \text{if } n \in \{7t, 7t+1\}; \\ e(H_n^{t-1}) + 1, & \text{if } n \in \{7t+2, 7t+3, 7t+4, 7t+5, 7t+6\}. \end{cases}$$

In addition,

$$e(H_n^t) = \begin{cases} 14t - 3 = 2n - 3, & \text{if } n = 7t; \\ 14t - 1 = 2n - 3, & \text{if } n = 7t+1; \\ 14t + 1 = 2n - 3, & \text{if } n = 7t+2; \\ 14t + 3 = 2n - 3, & \text{if } n = 7t+3; \\ 14t + 4 = 2n - 4, & \text{if } n = 7t+4; \\ 14t + 6 = 2n - 4, & \text{if } n = 7t+5; \\ 14t + 9 = 2n - 3, & \text{if } n = 7t+6. \end{cases}$$

Let $F_n^1 = (\prod^{2(t-1)} K_4) \cdot B_{t+8}$ when $n = 7t+4$, $F_n^1 = (\prod^{2(t-1)} K_4) \cdot B_{t+9}$ when $n = 7t+5$ and $F_n^1 = H_n^t$ when $n \in \{7t, 7t+1, 7t+2, 7t+5, 7t+6\}$. Then $e(F_n^1) = 2n - 3$ for all $n \geq 7$ and

$$F_n^1 = \begin{cases} B_{t+4} \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t; \\ B_{t+5} \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+1; \\ B_{t+6} \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+2; \\ B_{t+7} \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+3; \\ B_{t+8} \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+4; \\ B_{t+9} \cdot \prod^{2(t-1)} K_4, & \text{if } n = 7t+5; \\ B_{t+4} \cdot \prod^{2t} K_4, & \text{if } n = 7t+6. \end{cases}$$

When $t \equiv 0$ $(mod\ 3)$, let

$$F_n^2 = \begin{cases} \prod^{2t+\lfloor \frac{t}{3} \rfloor} K_4, & \text{if } n = 7t+1; \\ \prod^{2t+\lfloor \frac{t}{3} \rfloor+1} K_4, & \text{if } n = 7t+4; \\ F_n^1, & \text{if } n \in \{7t, 7t+2, 7t+3, 7t+5, 7t+6\}. \end{cases}$$

When $t \equiv 1$ $(mod\ 3)$, let

$$F_n^2 = \begin{cases} \prod^{2t+\lfloor \frac{t}{3} \rfloor} K_4, & \text{if } n = 7t; \\ \prod^{2t+\lfloor \frac{t}{3} \rfloor+1} K_4, & \text{if } n = 7t+3; \\ \prod^{2t+\lfloor \frac{t}{3} \rfloor+2} K_4, & \text{if } n = 7t+6; \\ F_n^1, & \text{if } n \in \{7t+1, 7t+2, 7t+4, 7t+5\}. \end{cases}$$

When $t \equiv 2$ $(mod\ 3)$, let

$$F_n^2 = \begin{cases} \prod^{2t+\lfloor \frac{t}{3} \rfloor+1} K_4, & \text{if } n = 7t+2; \\ \prod^{2t+\lfloor \frac{t}{3} \rfloor+2} K_4, & \text{if } n = 7t+5; \\ F_n^1, & \text{if } n \in \{7t, 7t+1, 7t+3, 7t+4, 7t+6\}. \end{cases}$$

For all cases above, we have $e(F_n^2) - e(F_n^1) \leq 1$. When $t \equiv 0$ $(mod\ 3)$,

$$e(F_n^2) = \begin{cases} 2n-2, & \text{if } n \equiv 1 \ (mod\ 3); \\ 2n-3, & \text{if } n \equiv 0 \ (mod\ 3) \text{ or } n \equiv 2 \ (mod\ 3). \end{cases}$$

When $t \equiv 1$ $(mod\ 3)$,

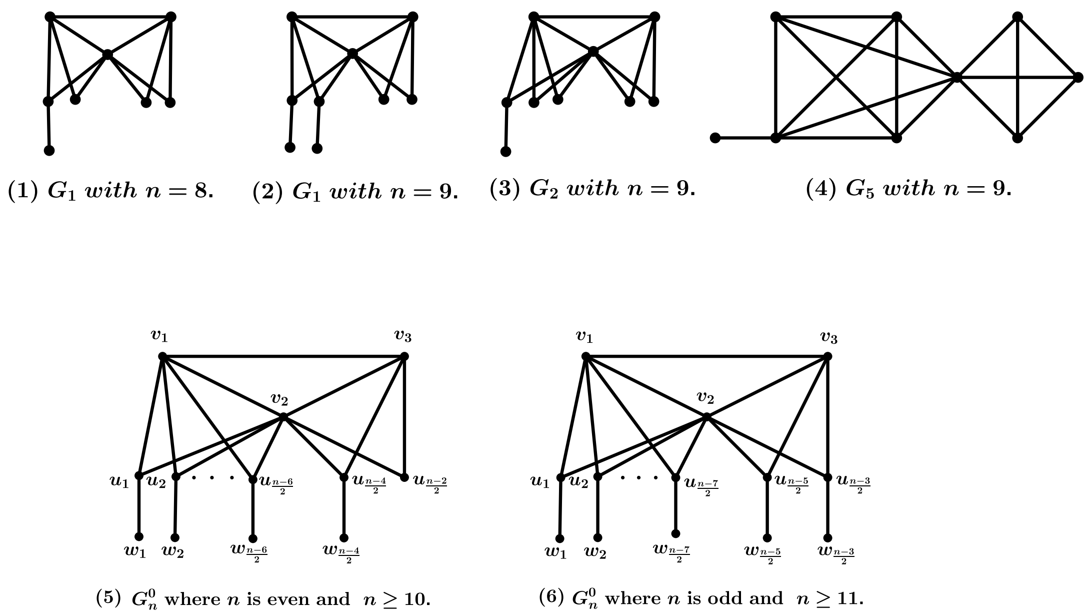
$$e(F_n^2) = \begin{cases} 2n-2, & \text{if } n \equiv 1 \ (mod\ 3); \\ 2n-3, & \text{if } n \equiv 0 \ (mod\ 3) \text{ or } n \equiv 2 \ (mod\ 3). \end{cases}$$

When $t \equiv 2$ $(mod\ 3)$,

$$e(F_n^2) = \begin{cases} 2n-2, & \text{if } n \equiv 1 \ (mod\ 3); \\ 2n-3, & \text{if } n \equiv 0 \ (mod\ 3) \text{ or } n \equiv 2 \ (mod\ 3). \end{cases}$$

In all cases, $e(F_n^2) = ex(n, \mathcal{C}_{\geq 5})$ by Theorem 1.1(3). Therefore, given an integer n with $n \geq 5$, for any integer m with $sat(n, \mathcal{C}_{\geq 5}) \leq m \leq ex(n, \mathcal{C}_{\geq 5})$, we have constructed a $\mathcal{C}_{\geq 5}$-saturated graph on n vertices and m edges. ■

Proof of Theorem 1.4. Firstly, let us construct a series of $\mathcal{C}_{\geq 6}$-saturated graphs.



(1) G_1 *with* $n = 8$. (2) G_1 *with* $n = 9$. (3) G_2 *with* $n = 9$. (4) G_5 *with* $n = 9$.

(5) G_n^0 where n is even and $n \geq 10$. (6) G_n^0 where n is odd and $n \geq 11$.

Figure 4: Some $\mathcal{C}_{\geq 6}$-saturated graphs.

Claim 1 *Let n and m be two integers such that $n \geq 6$, $sat(n, \mathcal{C}_{\geq 6}) \leq m \leq ex(n, \mathcal{C}_{\geq 6})$ and*

$$m \notin \begin{cases} \{ex(n, \mathcal{C}_{\geq 6}) - 1\}, & \text{if } n \equiv 0 \ (mod\ 4); \\ \{ex(n, \mathcal{C}_{\geq 6}) - 2, ex(n, \mathcal{C}_{\geq 6}) - 1\}, & \text{if } n \equiv 1 \ (mod\ 4). \end{cases}$$

Then there exists a $\mathcal{C}_{\geq 6}$-saturated graph on n vertices with size m.

Proof. Firstly, we consider the case $6 \leq n \leq 9$. For K_t with $2 \leq t \leq 5$, choose one vertex of K_t as the root vertex. When $n = 6$, let $G_1 = K_4 \cdot K_3$, $G_2 = H(6, 6, 2)$ and $G_3 = K_5 \cdot K_2$. When $n = 7$, let $G_1 = D(2, 2)$, $G_2 = K_4 \cdot K_4$ and $G_3 = K_5 \cdot K_3$. Clearly, for both cases, G_i is $\mathcal{C}_{\geq 6}$-saturated for each $i \in [3]$. By Theorem 1.2(3),

$$e(G_1) = sat(n, \mathcal{C}_{\geq 6}) = \begin{cases} 9, & \text{if } n = 6; \\ 11, & \text{if } n = 7. \end{cases}$$

By Theorem 1.1(4),

$$e(G_3) = e(G_2) + 1 = e(G_1) + 2 = \left\lfloor \frac{5n - 8}{2} \right\rfloor = ex(n, \mathcal{C}_{\geq 6}).$$

When $n = 8$, let G_1 be the graph shown in Figure 4(1). We have $e(G_1) = 12 = sat(8, \mathcal{C}_{\geq 6})$.

Let $G_2 = D(2,3)$, $G_3 = H(8,6,2)$ and $G_4 = K_5 \cdot K_4$. Then

$$e(G_3) = e(G_2) + 1 = e(G_1) + 2 = 14 = ex(n, \mathcal{C}_{\geq 6}) - 2$$

and $e(G_4) = 16 = \lfloor \frac{5n-8}{2} \rfloor = ex(n, \mathcal{C}_{\geq 6})$ by Theorem 1.1(4). When $n = 9$, the graphs G_1, G_2 and G_5 are shown in Figure 4(2), Figure 4(3) and Figure 4(4), respectively. Let $G_3 = D(4,2)$, $G_4 = H(9,6,2)$ and $G_6 = K_5 \cdot K_5$. We obtain $e(G_1) = 13 = sat(9, \mathcal{C}_{\geq 6})$, $e(G_{i+1}) = e(G_i) + 1$ for each $i \in [4]$, $e(G_5) = 17 = ex(n, \mathcal{C}_{\geq 6}) - 3$ and $e(G_6) = 20 = \lfloor \frac{5n-5}{2} \rfloor = ex(n, \mathcal{C}_{\geq 6})$. So Claim 1 holds for $6 \leq n \leq 9$.

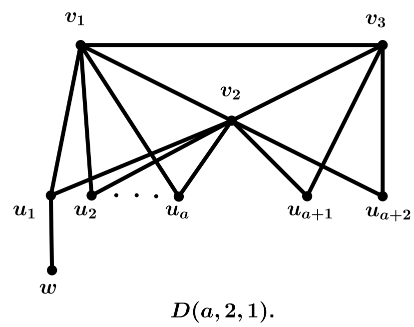


Figure 5: A $\mathcal{C}_{\geq 6}$-saturated graph $D(a,2,1)$.

Next we consider the case $n \geq 10$. The graphs shown in Figure 4(5) and Figure 4(6) are constructed by Ma et al. [9]. These graphs are $\mathcal{C}_{\geq 6}$-saturated and satisfy $e(G_n^0) = sat(n, \mathcal{C}_{\geq 6})$. Let i be an integer with $1 \leq i \leq \frac{n-3}{2}$ and $G_n^i = \{G_n^{i-1} \setminus w_i u_i\} \cup \{w_i v_1, w_i v_2\}$. So $e(G_n^i) = e(G_n^{i-1}) + 1$. For each $1 \leq i \leq \frac{n-3}{2}$, each block of G_n^i is isomorphic to K_2 or $D(r,s)$ with $r, s \geq 2$, and for any K_2-block of G_n^i and block B' with $B \cap B' \neq \emptyset$, B' is isomorphic to $D(r,s)$ and $B \cap B'$ is not a center of B'. By Proposition 2.3, G_n^i is $\mathcal{C}_{\geq 6}$-saturated. Let v_1 be the root vertex of G_n^i for each $1 \leq i \leq \frac{n-3}{2}$ and choose one vertex of K_5 as the root vertex. We see $G_n^{\lfloor \frac{n-3}{2} \rfloor} = D(n-5, 2)$. Let $H_n^0 = G_n^{\lfloor \frac{n-3}{2} \rfloor}$. When i is odd and $1 \leq i \leq 2\lfloor \frac{n-7}{4} \rfloor$, let $H_n^i = [((H_n^{i-1} - \{u_i, u_{i+1}, w_i, w_{i+1}\}) \setminus \{w_{i+2}v_1, w_{i+2}v_2\}) \cup \{w_{i+2}w_{i+3}\}] \cdot K_5$. When i is even and $1 \leq i \leq 2\lfloor \frac{n-7}{4} \rfloor$, let $H_n^i = (H_n^{i-1} \setminus \{w_{i+1}w_{i+2}\}) \cup \{w_{i+1}v_1, w_{i+1}v_2\}$. That is

$$H_n^i = \begin{cases} D(n-2i-8, 2, 1) \cdot \prod^{\frac{i+1}{2}} K_5, & \text{if } i \equiv 1 \ (mod\ 2); \\ D(n-2i-5, 2) \cdot \prod^{\frac{i}{2}} K_5, & \text{if } i \equiv 0 \ (mod\ 2). \end{cases}$$

The graph $D(a,2,1)$ is defined in Figure 5 for some integer a on $a+6$ vertices, where each vertex in $\{u_1, u_2, \ldots, u_{a+2}\}$ has degree 2. By Proposition 2.3, H_n^i is $\mathcal{C}_{\geq 6}$-saturated for each $1 \leq i \leq 2\lfloor \frac{n-7}{4} \rfloor$. We see $e(H_n^i) = e(H_n^{i-1}) + 1$. Let r, s be two integers with $r, s \geq 2$. Let one of the centers of $D(r,s)$ be the root vertex of it. Set

$$F_n^0 = H_n^{2\lfloor\frac{n-7}{4}\rfloor} = \begin{cases} D(3,2)\cdot \prod^{\lfloor\frac{n-7}{4}\rfloor} K_5, & \text{if } n\equiv 0\ (mod\ 4);\\ D(4,2)\cdot \prod^{\lfloor\frac{n-7}{4}\rfloor} K_5, & \text{if } n\equiv 1\ (mod\ 4);\\ D(5,2)\cdot \prod^{\lfloor\frac{n-7}{4}\rfloor} K_5, & \text{if } n\equiv 2\ (mod\ 4);\\ D(2,2)\cdot \prod^{\lfloor\frac{n-7}{4}\rfloor} K_5, & \text{if } n\equiv 3\ (mod\ 4).\end{cases}$$

Let one of the centers in $H(t,6,2)$ be the root vertex for each integer $t\geq 6$ and one vertex of K_5 be the root vertex. Denote by K_5^+ the graph obtained from identifying a vertex of K_5 and a vertex of K_2, that is $e(K_5^+) = 11$ and $|V(K_5^+)| = 6$. Let one vertex of K_5^+ of degree 4 be the root vertex. When $n\equiv 3\ (mod\ 4)$, let $F_n^1 = (\prod^{\lfloor\frac{n-7}{4}\rfloor} K_5)\cdot H(7,6,2)$ and $F_n^2 = (\prod^{\lfloor\frac{n-3}{4}\rfloor} K_5)\cdot K_3$. In [11], Woodall proved that the graph $(\prod^{\lfloor\frac{n-1}{4}\rfloor} K_5)\cdot K_{n-4\lfloor\frac{n-1}{4}\rfloor}$ is $\mathcal{C}_{\geq 6}$-saturated and has $ex(n,\mathcal{C}_{\geq 6})$ edges. It follows that $e(F_n^2) = ex(n,\mathcal{C}_{\geq 6})$ and

$$e(F_n^0)+2 = e(F_n^1)+1 = e(F_n^2) = ex(n,\mathcal{C}_{\geq 6}).$$

When $n\equiv 2\ (mod\ 4)$, let

$$F_n^1 = (\prod^{\lfloor\frac{n-11}{4}\rfloor} K_5)\cdot \prod^{2} K_5^+\cdot K_4,\ F_n^2 = (\prod^{\lfloor\frac{n-3}{4}\rfloor} K_5)\cdot K_4\cdot K_3,$$

$$F_n^3 = (\prod^{\lfloor\frac{n-3}{4}\rfloor} K_5)\cdot H(6,6,2),\ \text{and}\ F_n^4 = (\prod^{\lfloor\frac{n+1}{4}\rfloor} K_5)\cdot K_2 = (\prod^{\lfloor\frac{n-1}{4}\rfloor} K_5)\cdot K_2.$$

We have $e(F_n^i) = e(F_n^{i-1})+1$ for each $i\in[4]$ and $e(F_n^4) = ex(n,\mathcal{C}_{\geq 6})$.
When $n\equiv 0\ (mod\ 4)$, let $F_n^1 = H(8,6,2)\cdot \prod^{\lfloor\frac{n-7}{4}\rfloor} K_5$, and $F_n^2 = K_4\cdot \prod^{\lfloor\frac{n-3}{4}\rfloor} K_5$, we see

$$e(F_n^1) = e(F_n^0)+1 = e(F_n^2)-2 = ex(n,\mathcal{C}_{\geq 6})-2.$$

When $n\equiv 1\ (mod\ 4)$, let

$$F_n^1 = (\prod^{\lfloor\frac{n-7}{4}\rfloor} K_5)\cdot H(9,6,2),\ F_n^2 = (\prod^{\lfloor\frac{n-7}{4}\rfloor} K_5)\cdot K_5^+\cdot K_4,\ \text{and}\ F_n^3 = \prod^{\lfloor\frac{n+1}{4}\rfloor} K_5.$$

We see $e(F_n^2) = e(F_n^1)+1 = e(F_n^3)-3 = ex(n,\mathcal{C}_{\geq 6})-3$. By Proposition 2.3, we can verify that F_n^i is $\mathcal{C}_{\geq 6}$-saturated for each $i\in[4]$. ∎

Next we prove that above necessary condition for the existence of $\mathcal{C}_{\geq 6}$-saturated graphs of order n and size m is also sufficient.

Claim 2 *When $n \equiv 0$ $(mod\ 4)$ or $n \equiv 1$ $(mod\ 4)$, there is no $\mathcal{C}_{\geq 6}$-saturated graph G with n vertices such that $e(G) = ex(n, \mathcal{C}_{\geq 6}) - 1$ and when $n \equiv 1$ $(mod\ 4)$, there is no $\mathcal{C}_{\geq 6}$-saturated graph G with n vertices such that $e(G) = ex(n, \mathcal{C}_{\geq 6}) - 2$.*

Proof. By contradiction, suppose G is $\mathcal{C}_{\geq 6}$-saturated and

$$e(G) \in \begin{cases} \{ex(n, \mathcal{C}_{\geq 6}) - 1\}, & \text{if } n \equiv 0\ (mod\ 4); \\ \{ex(n, \mathcal{C}_{\geq 6}) - 2, ex(n, \mathcal{C}_{\geq 6}) - 1\}, & \text{if } n \equiv 1\ (mod\ 4). \end{cases}$$

Firstly, we conclude that G has at least two blocks. Otherwise, suppose G has only one block, by Proposition 2.3 and $n \geq 6$, then $G = D(r, s)$ where $r, s \geq 2$ and $r + s + 3 = n$, or $G = H(n, 6, 2)$. When $G = H(n, 6, 2)$, we have $e(G) = 2n - 2$, and when $G = D(r, s)$, we have $e(G) = 2n - 3$. If $n \equiv 1$ $(mod\ 4)$ and $e(G) = ex(n, \mathcal{C}_{\geq 6}) - 2$, by Theorem 1.1(4), then $e(G) = \lfloor \frac{5n-5}{2} \rfloor - 2$ and $\lfloor \frac{5n-5}{2} \rfloor - 2 \notin \{2n - 2, 2n - 3\}$ for $n \geq 6$. Thus we may assume that $e(G) = ex(n, \mathcal{C}_{\geq 6}) - 1$. If $G = H(n, 6, 2)$, then $e(G) = 2n - 2$, but for $n \geq 6$,

$$2n - 2 \notin \begin{cases} \{\lfloor \frac{5n-8}{2} \rfloor - 1\}, & \text{if } n \equiv 0\ (mod\ 4); \\ \{\lfloor \frac{5n-5}{2} \rfloor - 1\}, & \text{if } n \equiv 1\ (mod\ 4). \end{cases}$$

This implies $G = D(r, s)$. But for $n \geq 6$,

$$2n - 3 \notin \begin{cases} \{\lfloor \frac{5n-8}{2} \rfloor - 1\}, & \text{if } n \equiv 0\ (mod\ 4); \\ \{\lfloor \frac{5n-5}{2} \rfloor - 1\}, & \text{if } n \equiv 1\ (mod\ 4), \end{cases}$$

a contradiction to the assumption of G. Therefore G has at least two blocks. By Proposition 2.3, each block B of G satisfies $B \cong D(r, s)$ or $B \cong H(t, 6, 2)$ or $B \cong K_k$ where $r, s \geq 2$, $t \geq 6$ and $2 \leq k \leq 5$. We contract a block B of G to a vertex and denote the resulting graph by $G_1 = G \diagup B$. We first consider the case $e(G) = ex(n, \mathcal{C}_{\geq 6}) - 1$ with $n \equiv 0$ $(mod\ 4)$ or $n \equiv 1$ $(mod\ 4)$. If $n \equiv 0$ $(mod\ 4)$, then $e(G) = \lfloor \frac{5n-8}{2} \rfloor - 1 = \frac{5n-10}{2}$ and $e(G_1) = e(G) - e(B) \leq ex(n - |B| + 1, \mathcal{C}_{\geq 6})$, which follows that

$$e(B) \geq \begin{cases} \frac{5n-10}{2} - \lfloor \frac{5(n-|B|+1)-5}{2} \rfloor = \frac{5n-10}{2} - \frac{5n-5|B|}{2} = \frac{5|B|-10}{2}, & |B| \equiv 0\ (mod\ 4); \\ \frac{5n-10}{2} - \lfloor \frac{5(n-|B|+1)-8}{2} \rfloor = \frac{5n-10}{2} - \frac{5(n-|B|+1)-9}{2} = \frac{5|B|-6}{2}, & |B| \equiv 2\ (mod\ 4); \\ \frac{5n-10}{2} - \lfloor \frac{5(n-|B|+1)-8}{2} \rfloor = \frac{5n-10}{2} - \frac{5(n-|B|+1)-8}{2} = \frac{5|B|-7}{2}, & |B| \equiv 1 \text{ or } 3\ (mod\ 4). \end{cases}$$

Thus we have $B \ncong K_k$ for any $k \in \{2, 3\}$, $B \ncong D(r, s)$ for any $r, s \geq 2$, and $B \ncong H(t, 6, 2)$ for any $t \geq 6$. Therefore every block of G is isomorphic to K_4 or K_5. We may assume that $G = (\prod^{x} K_4) \cdot (\prod^{y} K_5)$ with $n = 3x + 4y + 1$ and $e(G) = 6x + 10y = \frac{5n-10}{2}$, yielding $3x = 5$, which contradicts the fact that x is an integer. Therefore, there is no $\mathcal{C}_{\geq 6}$-saturated graph G with $e(G) = ex(n, \mathcal{C}_{\geq 6}) - 1$ when $n \equiv 0$ $(mod\ 4)$. If $n \equiv 1$ $(mod\ 4)$, then $e(G) = \frac{5n-7}{2}$ and $e(G_1) = e(G) - e(B) \leq ex(n - |B| + 1, \mathcal{C}_{\geq 6})$, which follows that

$$e(B) \geq \begin{cases} \frac{5n-7}{2} - \lfloor \frac{5(n-|B|+1)-5}{2} \rfloor = \frac{5n-7}{2} - \frac{5(n-|B|+1)-5}{2} = \frac{5|B|-7}{2}, & |B| \equiv 1\ (mod\ 4); \\ \frac{5n-7}{2} - \lfloor \frac{5(n-|B|+1)-8}{2} \rfloor = \frac{5n-7}{2} - \frac{5(n-|B|+1)-9}{2} = \frac{5|B|-3}{2}, & |B| \equiv 3\ (mod\ 4); \\ \frac{5n-7}{2} - \lfloor \frac{5(n-|B|+1)-8}{2} \rfloor = \frac{5n-7}{2} - \frac{5(n-|B|+1)-8}{2} = \frac{5|B|-4}{2}, & |B| \equiv 0 \text{ or } 2\ (mod\ 4). \end{cases}$$

We have $B \ncong K_k$ for any $2 \le k \le 4$, $B \ncong D(r,s)$ for any $r,s \ge 2$ and $B \ncong H(t,6,2)$ for any $t \ge 6$. Therefore every block of G is isomorphic to K_5, impling that $G = \prod^{x} K_5$ with $n = 4x+1$ and $e(G) = 10x \neq \frac{5n-7}{2}$, a contradiction.

Next we consider the case $e(G) = ex(n, \mathcal{C}_{\ge 6}) - 2 = \lfloor \frac{5n-5}{2} \rfloor - 2 = \frac{5n-9}{2}$ with $n \equiv 1$ $(mod\ 4)$. In this case $e(G_1) = \frac{5n-9}{2} - e(B) \le ex(n - |B| + 1, \mathcal{C}_{\ge 6})$, yielding

$$e(B) \ge \begin{cases} \frac{5n-9}{2} - \lfloor \frac{5(n-|B|+1)-5}{2} \rfloor = \frac{5n-9}{2} - \frac{5(n-|B|+1)-5}{2} = \frac{5|B|-9}{2}, & |B| \equiv 1\ (mod\ 4); \\ \frac{5n-9}{2} - \lfloor \frac{5(n-|B|+1)-8}{2} \rfloor = \frac{5n-9}{2} - \frac{5(n-|B|+1)-9}{2} = \frac{5|B|-5}{2}, & |B| \equiv 3\ (mod\ 4); \\ \frac{5n-9}{2} - \lfloor \frac{5(n-|B|+1)-8}{2} \rfloor = \frac{5n-9}{2} - \frac{5(n-|B|+1)-8}{2} = \frac{5|B|-6}{2}, & |B| \equiv 0 \text{ or } 2\ (mod\ 4). \end{cases}$$

Thus, we have that B is not isomorphic to any one of $\{K_2, K_3, K_4, D(r,s), H(t,6,2)\}$ where $r,s \ge 2$ and $t \ge 6$. That is $G = \prod^{x} K_5$ with $n = 4x+1$ and $e(G) = 10x \neq \frac{5n-9}{2}$, a contradiction. Therefore, there is no $\mathcal{C}_{\ge 6}$-saturated graph with $e(G) = ex(n, \mathcal{C}_{\ge 6}) - 1$ when $n \equiv 0$ or 1 $(mod\ 4)$, or $e(G) = ex(n, \mathcal{C}_{\ge 6}) - 2$ when $n \equiv 1$ $(mod\ 4)$. ■

This completes the proof of Theorem 1.4. ■

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