Stanley's conjectures on the Stern poset

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Abstract. The Stern poset $\mathcal{S}$ is a graded infinite poset naturally associated with Stern's triangle, which was defined by Stanley, in analogy with Pascal's triangle. Stanley noted that every interval in $\mathcal{S}$ is a distributive lattice. Let P_n denote the unique (up to isomorphism) poset for which the set of its order ideals, ordered by inclusion, is isomorphic to the interval from the unique element of row 0 of Stern's triangle to the n-th element of row r for sufficiently large r. For $n \geq 1$ let

$$L_n(q) = 2 \cdot \left(\sum_{k=1}^{2^n-1} A_{P_k}(q) \right) + A_{P_{2^n}}(q),$$

where $A_P(q)$ represents the corresponding P-Eulerian polynomial. For all $n \geq 1$ Stanley conjectured that $L_n(q)$ has only real zeros and $L_{4n+1}(q)$ is divisible by $L_{2n}(q)$. In this paper we obtain a simple recurrence relation satisfied by $L_n(q)$ and affirmatively solve Stanley's conjectures.

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1 Introduction

Stanley [9] introduced a sequence of polynomials $\{b_n(q)\}_{n\geq 1}$ by defining $b_1(q) = 1$ and

$$b_{2n}(q) = b_n(q), \tag{1.1}$$

$$b_{4n+1}(q) = qb_{2n}(q) + b_{2n+1}(q), \tag{1.2}$$

$$b_{4n+3}(q) = b_{2n+1}(q) + qb_{2n+2}(q). \tag{1.3}$$

So, we have

$$\begin{aligned}
&b_1(q) = b_2(q) = b_4(q) = b_8(q) = 1,\\
&b_3(q) = b_6(q) = q + 1,\\
&b_5(q) = b_{10}(q) = 2q + 1,\\
&b_7(q) = 2q + 1,\\
&b_9(q) = 3q + 1.
\end{aligned}$$

For $n \geq 1$, let

$$L_n(q) = 2 \cdot \left(\sum_{k=1}^{2^n-1} b_k(q) \right) + b_{2^n}(q). \tag{1.4}$$

The following is a list of the few values of $L_n(q)$:

$$\begin{aligned}
L_1(q) &= 3,\\
L_2(q) &= 2q + 7,\\
L_3(q) &= 12q + 15,\\
L_4(q) &= 4q^2 + 46q + 31,\\
L_5(q) &= 36q^2 + 144q + 63,\\
L_6(q) &= 8q^3 + 192q^2 + 402q + 127,\\
L_7(q) &= 96q^3 + 792q^2 + 1044q + 255,\\
L_8(q) &= 16q^4 + 656q^3 + 2796q^2 + 2582q + 511,\\
L_9(q) &= 240q^4 + 3360q^3 + 8892q^2 + 6168q + 1023.
\end{aligned}$$

The main objective of this paper is to prove Stanley's conjectures on the real zeros and divisibility of $L_n(q)$.

Let us first review some background. Stanley's conjectures considered here arose in the study of Stern's triangle S, which is an array of numbers similar to Pascal's triangle. We follow Stanley [8] to give a description of Stern's triangle. We number the rows of Stern's triangle by consecutive natural numbers beginning with 0. Row 0 consists of a single 1, row 1 consists of three 1's, and for $r \geq 2$ row r is obtained from row $r-1$ by inserting, between two consecutive elements c and d, their sum $c+d$, and then placing a 1 at the beginning and end. The first five rows of Stern's triangle look like

$$\begin{array}{c}
1\\
1 \quad 1 \quad 1\\
1 \quad 1 \quad 2 \quad 1 \quad 2 \quad 1 \quad 1\\
1 \; 1 \; 2 \; 1 \; 3 \; 2 \; 3 \; 1 \; 3 \; 2 \; 3 \; 1 \; 2 \; 1 \; 1\\
1\,1\,2\,1\,3\,2\,3\,1\,4\,3\,5\,2\,5\,3\,4\,1\,4\,3\,5\,2\,5\,3\,4\,1\,3\,2\,3\,1\,2\,1\,1
\end{array}.$$

It is clear that row r consists of $2^{r+1}-1$ terms. We number the elements of row r from 0 to $2^{r+1}-2$ and use $\left\langle {r \atop n} \right\rangle$ to denote the $(n+1)$-th element of row r. Thus, we have the recurrence relations

$$\left\langle {r \atop 2n+1} \right\rangle = \left\langle {r-1 \atop n} \right\rangle, \qquad \left\langle {r \atop 2n} \right\rangle = \left\langle {r-1 \atop n-1} \right\rangle + \left\langle {r-1 \atop n} \right\rangle, \tag{1.5}$$

where we set $\left\langle {r \atop n} \right\rangle = 0$ for $n < 0$ or $n > 2^{r+1}-2$ for convenience. For any $r \geq 1$ Stanley showed that

$$\sum_{n\geq 0} \left\langle {r \atop n} \right\rangle x^n = \prod_{i=0}^{r-1} (1 + x^{2^i} + x^{2\cdot 2^i}). \tag{1.6}$$

Letting $r \to \infty$ in (1.6), we get

$$\prod_{i\geq 0} (1 + x^{2^i} + x^{2\cdot 2^i}) = \sum_{n\geq 1} b_n x^{n-1},$$

where the sequence $\{b_n\}_{n\geq 0}$ with $b_0 = 0$ is Stern's well-known diatomic sequence [11]. For more information on Stern's diatomic sequence, see Northshield [4]. It is known that $\{b_n\}_{n\geq 0}$ satisfies the following recurrence relations

$$b_{2n} = b_n, \qquad b_{2n+1} = b_n + b_{n+1}. \tag{1.7}$$

Comparing the above recurrence relations with (1.1), (1.2) and (1.3), we see that $\{b_n(q)\}_{n\geq 1}$ is a polynomial analogue of Stern's diatomic sequence.

Stanley [9] showed that the polynomials $b_n(q)$ also arise as P-Eulerian polynomials of certain posets P naturally associated with Stern's triangle S. Let P be a naturally labelled poset, and let $\mathcal{L}(P)$ denote the set of linear extensions of P, regarded as permutations of the labels of P. The P-Eulerian polynomial, denoted $A_P(q)$, is defined by

$$A_P(q) = \sum_{\sigma\in\mathcal{L}(P)} q^{\mathrm{des}\,(\sigma)},$$

where $\mathrm{des}\,(\sigma)$ denotes the number of descents of the permutation σ, namely, $\mathrm{des}\,(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$. For other related definitions on posets, see Stanley [7, Chapter 3]. To take S as a poset, we will consider $\left\langle {r \atop n} \right\rangle$ as a symbol instead of a number. According to (1.5), we may impose a partial order $\preceq_{\mathcal{S}}$ on S by letting

$$\left\langle {r-1 \atop n} \right\rangle \preceq_{\mathcal{S}} \left\langle {r \atop 2n+1} \right\rangle, \quad \left\langle {r-1 \atop n} \right\rangle \preceq_{\mathcal{S}} \left\langle {r \atop 2n} \right\rangle, \quad \left\langle {r-1 \atop n} \right\rangle \preceq_{\mathcal{S}} \left\langle {r \atop 2n+2} \right\rangle$$

for $0 \leq n \leq 2^r - 2$ and then taking the transitive closure. Following Stanley [9] we call $(S, \preceq_{\mathcal{S}})$ the Stern poset, denoted by $\mathcal{S}$, which is a special case of the upper homogeneous posets studied in [10]. See Figure 1.1 for the first four levels of the Stern poset.

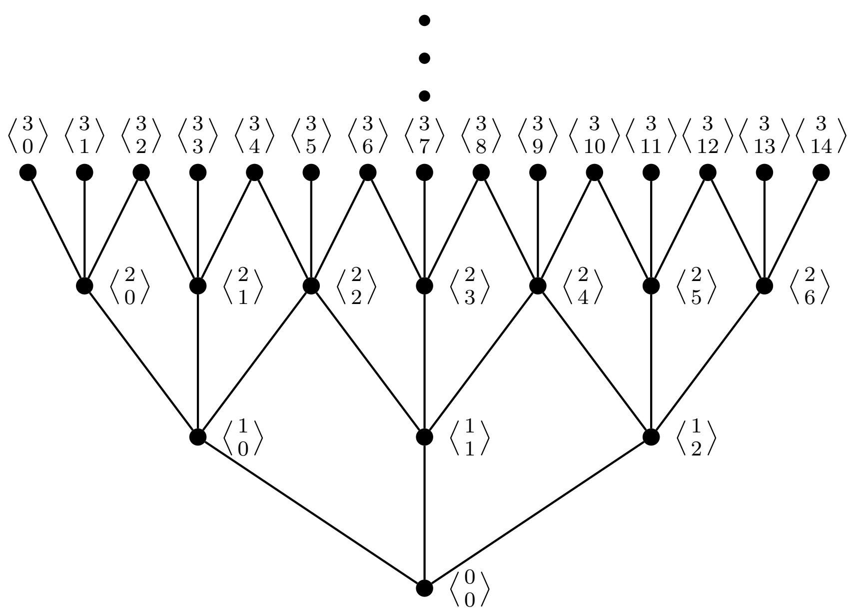

Figure 1.1: The Stern poset $\mathcal{S}$

Fixing a positive integer n, suppose that

$$2^k \leq n-1 < 2^{k+1}$$

for some $k \geq 0$. A little thought shows that for $r \geq k+1$ the interval $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{r \atop n-1}\right\rangle\right]$ of $\mathcal{S}$ is the ordinal sum of the chain $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{r-k-1 \atop 0}\right\rangle\right]$ and the interval $\left[\left\langle{r-k-1 \atop 0}\right\rangle, \left\langle{r \atop n-1}\right\rangle\right]$, the latter being isomorphic to the interval $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{k+1 \atop n-1}\right\rangle\right]$ of $\mathcal{S}$ for any $r \geq k+1$. Thus for sufficiently large r we may associate with the n-th element of row r in Stern's triangle the poset $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{k+1 \atop n-1}\right\rangle\right]$. As pointed out by Stanley [9], every interval in $\mathcal{S}$ is a distributive lattice. This fact can be seen by inspection of its cover relations. Thus there exists a unique (up to isomorphism) poset, denoted by P_n, for which $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{k+1 \atop n-1}\right\rangle\right] \cong J(P_n)$, the set of order ideals of P_n ordered by inclusion. The following result was stated without proof in [9]. For the sake of completeness, we include a proof here.

Theorem 1.1 ([9]) *For any $n \geq 1$ let $b_n(q)$ be defined as in (1.1), (1.2) and (1.3). Then $b_n(q)$ is equal to the P_n-Eulerian polynomial, namely*

$$b_n(q) = \sum_{\sigma \in \mathcal{L}(P_n)} q^{\operatorname{des}(\sigma)}, \tag{1.8}$$

where $\mathcal{L}(P_n)$ denotes the set of linear extensions of P_n, provided that P_n is naturally labeled.

Proof. Given $n \geq 1$, let k be the unique integer such that $2^k \leq n-1 < 2^{k+1}$. By [7, Proposition 3.4.2] we can take P_n to be the subposet of the interval $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{k+1 \atop n-1}\right\rangle\right]$ in $\mathcal{S}$ induced by its join-irreducibles. Let $c_n(q)$ denote the right-hand side of (1.8). Note that the poset P_1 is just a single-element poset and thus $c_1(q) = 1 = b_1(q)$. To prove (1.8), it suffices to show that $c_n(q)$ satisfies the same recurrence as $b_n(q)$. Now the proof breaks into three cases.

Case 1: $n = 2m$. Since the poset P_2 is just a chain of length one, we have $c_2(q) = 1 = c_1(q)$. We proceed to consider the case of $m \geq 2$ for which $2^{k-1} \leq m-1 < 2^k$. Observe that the interval $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{k+1 \atop 2m-1}\right\rangle\right]$ is equal to the ordinal sum of $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{k \atop m-1}\right\rangle\right]$ and $\left[\left\langle{k \atop m-1}\right\rangle, \left\langle{k+1 \atop 2m-1}\right\rangle\right]$, while the cardinality of the latter interval is 2 by the cover relations. This fact implies that $\left\langle{k+1 \atop 2m-1}\right\rangle$ is join-irreducible and P_{2m} is equal to the ordinal sum P_m and the single-element poset $\left\langle{k+1 \atop 2m-1}\right\rangle$. Since $\left\langle{k+1 \atop 2m-1}\right\rangle$ is the maximum element of P_{2m}, we have $c_{2m}(q) = c_m(q)$.

Case 2: $n = 4m+3$. Note that P_3 consists of two incomparable elements, and hence $c_3(q) = 1 + q = c_1(q) + qc_2(q)$. Thus we may assume that $m \geq 1$, thereby $2^{k-1} \leq 2m < 2m+1 < 2^k$. By the cover relations, one can see that $\left\langle{k \atop 2m+1}\right\rangle$ is join-irreducible in the interval $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{k+1 \atop 4m+2}\right\rangle\right]$, while $\left\langle{k \atop 2m}\right\rangle$ and $\left\langle{k+1 \atop 4m+2}\right\rangle$ are not. For ease of notation, let $y = \left\langle{k \atop 2m+1}\right\rangle$. Since each level of $\left[\left\langle{0 \atop 0}\right\rangle, \left\langle{k+1 \atop 4m+2}\right\rangle\right]$ contains at most two elements, the poset P_{4m+3} contains exactly two maximal elements, one of which is y, and the other is denoted by z. In view of the cover relations $\left\langle{k \atop 2m+1}\right\rangle \preceq_{\mathcal{S}} \left\langle{k+1 \atop 4m+2}\right\rangle$ and $\left\langle{k \atop 2m}\right\rangle \preceq_{\mathcal{S}} \left\langle{k+1 \atop 4m+2}\right\rangle$, the poset P_{2m+1} can be obtained from P_{4m+3} by removing y, and P_{2m+2} can be obtained from P_{4m+3} by removing z. In fact, y is the maximum element of P_{2m+2} and $w \preceq_{\mathcal{S}} y$, provided $w \in P_{2m+1}$ and $w \neq z$. Consider a natural labeling ω of P_{4m+3} such that it assigns to y the largest label, and to z the second largest label. Now those linear extensions in $\mathcal{L}(P_{4m+3})$ ending with $\omega(y)$ are in one-to-one correspondence with $\mathcal{L}(P_{2m+1})$, thus contributing $c_{2m+1}(q)$ to $c_{4m+3}(q)$. Other linear extensions in $\mathcal{L}(P_{4m+3})$ must end with $\omega(y)\omega(z)$, and thus are in one-to-one correspondence with $\mathcal{L}(P_{2m+2})$, contributing $qc_{2m+2}(q)$ to $c_{4m+3}(q)$, where the factor q records the descent generated by the pair $(\omega(y), \omega(z))$. To summarize, we have $c_{4m+3}(q) = c_{2m+1}(q) + qc_{2m+2}(q)$.

Case 3: $n = 4m + 1$. In this case we have $c_{4m+1}(q) = qc_{2m}(q) + c_{2m+1}(q)$. The proof is similar to that of Case 2, and so is omitted.

Combining the above three cases leads to the desired result. This completes the proof. ∎

Therefore, the polynomials $L_n(q)$ defined in (1.4) can be considered as a q-analog of the row sums of Stern's triangle. To see this, by induction on n one notices that (1.5) and (1.7) imply $\left\langle {n \atop k} \right\rangle = b_{k+1}$ for $0 \leq k \leq 2^n - 1$, and thus (1.6) implies that

$$L_n(1) = 2 \cdot \left(\sum_{k=1}^{2^n-1} b_k \right) + b_{2^n} = \sum_{k=0}^{2^{n+1}-2} \left\langle {n \atop k} \right\rangle = 3^n.$$

Stanley [8] showed that for any $m \geq 1$ the summation

$$\sum_k \left\langle {n \atop k} \right\rangle^m$$

obeys a homogeneous linear recurrence with constant coefficients, and conjectured the least order of a homogeneous linear recurrence. (Speyer [5] proved that the above sum satisfies such a recurrence of the conjectured minimal order.) For the polynomials $L_n(q)$, Stanley [9] proposed the following interesting conjectures.

Conjecture 1.2 ([9]) *For all $n \geq 1$ the polynomial $L_n(q)$ has only real zeros.*

Conjecture 1.3 ([9]) *For all $n \geq 1$ the polynomial $L_{4n+1}(q)$ is divisible by $L_{2n}(q)$.*

In the next section we derive a recurrence relation satisfied by $L_n(q)$, and then prove the above two conjectures.

2 Proofs of Conjectures 1.2 and 1.3

As will be shown below, both conjectures are immediate consequences of the recurrence relation (2.2). In the following we will show how (2.2) was found.

Note that various techniques have been developed for showing a univariate polynomial has only real zeros; see excellent surveys on this topic by Brändén [1], Brenti [2] and Stanley [6]. Given a sequence $\{F_n(q)\}_{n\geq 0}$ of polynomials, one basic method for proving that it has real zeros is to find a simple recurrence relation for these polynomials and then to use induction to show that they form a generalized Sturm sequence, whose definition will be given below.

Recall that a real polynomial is said to be standard if either it is identically zero or its leading coefficient is positive. Let RZ denote the set of real polynomials in q with only real zeros. Given two polynomials $f(q), g(q) \in \mathrm{RZ}$, let $\{u_i\}$ and $\{v_j\}$ be all zeros of $f(q)$ and $g(q)$ in weakly decreasing order respectively. Following Liu and Wang [3], we say that $g(q)$ alternates left of $f(q)$ if $\deg f(q) = \deg g(q) = n$ and

$$v_n \le u_n \le v_{n-1} \le \cdots \le v_2 \le u_2 \le v_1 \le u_1,$$

and we say that $g(q)$ interlaces $f(q)$ if $\deg f(q) = \deg g(q) + 1 = n$ and

$$u_n \le v_{n-1} \le \cdots \le v_2 \le u_2 \le v_1 \le u_1.$$

Let $g(q) \preceq f(q)$ denote the statement that either $g(q)$ alternates left of $f(q)$ or $g(q)$ interlaces $f(q)$. For notational convenience, let $a \preceq bq + c$ for any real numbers a, b, c and also let $f(q) \preceq 0$ and $0 \preceq f(q)$ for any $f(q) \in \mathrm{RZ}$. We say that a sequence $\{F_n(q)\}_{n\ge 0}$ of standard polynomials is a generalized Sturm sequence if $F_n(q) \in \mathrm{RZ}$ and $F_n(q) \preceq F_{n+1}(q)$ for all $n \ge 0$. Liu and Wang [3] gave the following sufficient condition for a polynomial sequence $\{F_n(q)\}_{n\ge 0}$ to form a generalized Sturm sequence.

Theorem 2.1 ([3, Corollary 2.4]) *Suppose that $\{F_n(q)\}_{n\ge 0}$ is a sequence of standard polynomials with nonnegative coefficients with $\deg F_n(q) = \deg F_{n-1}(q)$ or $\deg F_{n-1}(q) + 1$, which satisfy the following recurrence relation:*

$$F_{n+1}(q) = A_n(q)F_n(q) + B_n(q)F_n'(q) + C_n(q)F_{n-1}(q) \tag{2.1}$$

for some real polynomials $A_n(q), B_n(q)$ and $C_n(q)$. If $F_0(q) \preceq F_1(q)$ and for each n, either $B_n(q) < 0$ or $C_n(q) < 0$ whenever $q \le 0$, then $\{F_n(q)\}_{n\ge 0}$ is a generalized Sturm sequence.

By definition all the polynomials in a generalized Sturm sequence have only real zeros, so part of the conclusion of Theorem 2.1 is that the polynomials $F_n(q)$ have real zeros. It is obvious that each $L_n(q)$ is a polynomial with nonnegative coefficients by (1.4). Keeping Theorem 2.1 in mind motivated us to consider whether the polynomials $L_n(q)$ satisfy a recurrence of the form (2.1) as those $F_n(q)$. Computer experiments suggest the following recurrence relation satisfied by $L_n(q)$ by assuming that the degrees of $A_n(q), B_n(q)$ and $C_n(q)$ are independent of n.

Theorem 2.2 *For any $n \ge 2$ we have*

$$L_{n+1}(q) = 3L_n(q) + 2(q-1)L_{n-1}(q). \tag{2.2}$$

Proof. From (1.4) it follows that

$$L_{n+1}(q) = 2 \cdot \left(\sum_{k=1}^{2^{n+1}-1} b_k(q) \right) + b_{2^{n+1}}(q).$$

By partitioning indices k into those are even, congruent to 1 modulo 4 and conguent to 3 modulo 4, and then applying (1.1), (1.2) and (1.3) respectively, we find that

$$L_{n+1}(q) = \left(2 \sum_{\substack{k=1 \\ k\equiv 0 \,(\mathrm{mod}\, 2)}}^{2^{n+1}-1} b_k(q) + b_{2^{n+1}}(q) \right) + 2 \sum_{\substack{k=1 \\ k\equiv 1 \,(\mathrm{mod}\, 4)}}^{2^{n+1}-1} b_k(q) + 2 \sum_{\substack{k=1 \\ k\equiv 3 \,(\mathrm{mod}\, 4)}}^{2^{n+1}-1} b_k(q)$$

$$
\begin{aligned}
&= \left(2\sum_{k=1}^{2^n-1} b_k(q) + b_{2^n}(q)\right) + 2\left(1 + \sum_{k=1}^{2^{n-1}-1} (qb_{2k}(q) + b_{2k+1}(q))\right)\\
&\quad + 2\sum_{k=0}^{2^{n-1}-1} (b_{2k+1}(q) + qb_{2k+2}(q))\\
&= L_n(q) + 2 + 2\sum_{k=1}^{2^{n-1}-1} (qb_{2k}(q) + b_{2k+1}(q)) + 2\sum_{k=0}^{2^{n-1}-1} (b_{2k+1}(q) + qb_{2k+2}(q))\\
&= L_n(q) + 2 + 2\sum_{k=1}^{2^{n-1}-1} (b_{2k}(q) + b_{2k+1}(q)) + 2\sum_{k=0}^{2^{n-1}-1} (b_{2k+1}(q) + b_{2k+2}(q))\\
&\quad + (q-1)\left(2\sum_{k=1}^{2^{n-1}-1} b_{2k}(q) + 2\sum_{k=0}^{2^{n-1}-1} b_{2k+2}(q)\right)\\
&= L_n(q) + 2 + (L_n(q) - 3) + (L_n(q) + 1)\\
&\quad + 2(q-1)\left(\sum_{k=1}^{2^{n-1}-1} b_k(q) + \sum_{k=0}^{2^{n-1}-1} b_{k+1}(q)\right)\\
&= 3L_n(q) + 2(q-1)L_{n-1}(q),
\end{aligned}
$$

as desired. This completes the proof. ∎

Combining Theorems 2.1 and 2.2, we immediately obtain the following result, which gives an affirmative answer to Conjecture 1.2.

Corollary 2.3 *For all $n \geq 1$ the polynomial $L_n(q)$ has only real zeros.*

Proof. Since $b_n(q)$ is a polynomial with nonnegative coefficients by (1.1), (1.2) and (1.3), so is $L_n(q)$ by (1.4). For convenience let $L_0(q) = 1$. By induction on n, we can deduce $\deg L_n(q) = \lfloor \frac{n}{2} \rfloor$ from (2.2). If $\{F_n(q)\}_{n\geq 0}$ in (2.1) is taken to be the sequence $\{L_n(q)\}_{n\geq 0}$, one can verify that the conditions of Theorem 2.1 are satisfied. Thus $\{L_n(q)\}_{n\geq 0}$ is a generalized Sturm sequence. In particular, $L_n(q)$ has only real zeros for all $n \geq 1$. This completes the proof. ∎

Actually, not only all zeros of $L_n(q)$ are real, but also can be computed explicitly. By solving (2.2), we obtain Binet's formula for $L_n(q)$.

Corollary 2.4 *For any $n \geq 1$, we have*

$$
L_n(q) = \frac{(r(q))^{n+1} - (s(q))^{n+1}}{\sqrt{8q+1}}, \tag{2.3}
$$

where

$$r(q) = \frac{3+\sqrt{8q+1}}{2} \quad \textit{and} \quad s(q) = \frac{3-\sqrt{8q+1}}{2}.$$

Moreover, the zeros of $L_n(q)$ *are* $-\frac{1}{8} - \frac{9}{8}\tan^2\frac{\pi k}{n+1}$, *where* $k = 1, 2, \ldots, \lfloor n/2 \rfloor$.

Proof. Note that the characteristic equation of (2.2) is

$$x^2 - 3x - 2(q-1) = 0$$

with roots $r(q)$ and $s(q)$, where $r(q)+s(q) = 3$ and $r(q)s(q) = -2(q-1)$. So the general solution of (2.2) is

$$L_n(q) = C(r(q))^{n-1} + D(s(q))^{n-1},$$

where the coefficients C and D are to be determined.

The initial conditions $L_1(q) = 3$ and $L_2(q) = 7+2q = 3r(q) + 3s(q) - r(q)s(q)$ yield the following system:

$$C + D = 3,$$

$$Cr(q) + Ds(q) = 2q + 7.$$

Solving this, we get

$$C = \frac{(r(q))^2}{r(q)-s(q)} \quad \text{and} \quad D = -\frac{(s(q))^2}{r(q)-s(q)}.$$

Thus

$$L_n(q) = \frac{(r(q))^{n+1} - (s(q))^{n+1}}{r(q)-s(q)},$$

as desired.

Next we determine the zeros of $L_n(q)$. In view of (2.3), q is a zero of $L_n(q)$ if and only if $\frac{3+\sqrt{8q+1}}{3-\sqrt{8q+1}}$ is a non-trivial root of unity of degree $n+1$ (that is, not equal to 1), or equivalently, $(8q+1)/9 = -\tan^2\frac{\pi k}{n+1}$ for $1 \le k \le \lfloor n/2 \rfloor$. This completes the proof. ∎

We proceed to prove Conjecture 1.3. It is well known that $(x^k - y^k)$ divides $(x^n - y^n)$ whenever k divides n. In view of (2.3), the polynomial $L_k(q)$ divides $L_m(q)$ whenever $k+1$ divides $m+1$. Taking $k = n$ and $m = 2n+1$ immediately leads to the following result, which is stronger than Conjecture 1.3.

Corollary 2.5 *For any* $n \ge 1$, *the polynomial* $L_{2n+1}(q)$ *is divisible by* $L_n(q)$.

Remark 2.6 Recall that the n*-th Chebyshev polynomial* $U_n(q)$ *of the second kind has the following explicit expression*

$$U_n(q) = \frac{(q+\sqrt{q^2-1})^{n+1} - (q-\sqrt{q^2-1})^{n+1}}{2\sqrt{q^2-1}}.$$

Comparing this with (2.3) *leads to*

$$U_n(q) = \frac{2^n q^n}{3^n} \cdot L_n\left(1 - \frac{9}{8q^2}\right).$$

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