Hecke-type double sums and mock theta functions

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Abstract

Mock theta functions were first introduced by Ramanujan in his last letter to Hardy. In the literature, Hecke-type double sums for the classical mock theta functions play a very important role in proving mock theta conjectures. In this paper, with the aid of some q-series identities, we establish the Hecke-type double sums for some new mock theta functions. All the Hecke-type double sums are represented in the form of $f_{a,b,c}(x, y, q)$ which is defined by Hickerson and Mortenson. Meanwhile, we express these functions in terms of Appell–Lerch sums and theta functions.

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1 Introduction

Throughout the paper, we use the standard q-series notation [GR04]. Let q denote a complex number with $|q| < 1$. Then for positive integers n and m,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a_1,a_2,\cdots,a_m;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1,a_2,\cdots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is defined as

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,x\right) := \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r}x^n.$$

Define

$$j(x;q) := (x,q/x,q;q)_\infty.$$

Let a and m be integers with m positive. Then

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m), \quad J_m := J_{m,3m} = \prod_{i=1}^{\infty}(1-q^{mi}).$$

In this paper, we mainly focus on Hecke-type double sums

$$\sum_{(m,n)\in D}(-1)^{H(m,n)}q^{Q(m,n)+L(m,n)},$$

where $H(m,n)$ and $L(m,n)$ are linear forms, $Q(m,n)$ is an indefinite quadratic form, and D is some subset of $\mathbb{Z}\times\mathbb{Z}$ such that $Q(m,n)\geq 0$ for all $(m,n)\in D$. The object of this paper is to find the Hecke-type double sums for some new mock theta functions.

Mock theta functions were first introduced by Ramanujan [R27] in his last letter to Hardy. These functions have certain asymptotic properties as q approaches a root of unity which are similar to theta functions, but they are not really theta functions. Historically, mock theta functions can be represented by Eulerian forms, Hecke-type double sums, Appell–Lerch sums, and Fourier coefficients of meromorphic Jacobi forms. Hecke-type double sums for the classical mock theta functions have received a great deal of attention in the literature, and this kind of representations plays a very important role in the development of mock theta functions.

In 1981, by means of Bailey's lemma, Andrews [A86] established the Hecke-type double sums for the fifth and seventh order mock theta functions. For example, Andrews showed that

$$J_1 f_0(q) = \sum_{n=0}^{\infty}(1-q^{2n+1})q^{\frac{5n^2+n}{2}}\sum_{j=-n}^{n}(-1)^j q^{-j^2}, \tag{1.1}$$

$$J_1 f_1(q) = \sum_{n=0}^{\infty}(1-q^{4n+2})q^{\frac{5n^2+3n}{2}}\sum_{j=-n}^{n}(-1)^j q^{-j^2}, \tag{1.2}$$

where the fifth order mock theta functions $f_0(q)$ and $f_1(q)$ are given by

$$f_0(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n} \quad \text{and} \quad f_1(q) = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n}.$$

Later, Hickerson [H881] introduced the universal mock theta function $g(x,q)$ where

$$g(x,q) = x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(x^{-1}q;q)_n}\right) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(x,x^{-1}q;q)_{n+1}}.$$

Then in view of the Hecke-type identities (1.1) and (1.2), Hickerson [H881] proved

$$f_0(q) = \frac{J_{5,10}J_{2,5}}{J_1} - 2q^2 g(q^2,q^{10}), \tag{1.3}$$

$$f_1(q) = \frac{J_{5,10}J_{4,5}}{J_1} - 2q^3 g(q^4,q^{10}), \tag{1.4}$$

which are called "mock theta conjectures". Customarily, analogous identities of (1.3) and (1.4) involving mock theta functions and theta functions are referred to as "mock theta conjectures". Later, using the same method, Hickerson [H882] proved three mock theta conjectures related to the seventh order mock theta functions.

In 2012, in view of q-orthogonal polynomials, Andrews [A12] provided the Hecke-type double sums related to some third order mock theta functions and found two new mock theta functions. Later, the new Hecke-type double sums for some third order mock theta functions were derived by Mortenson [M13]. In 2018, some mock theta conjectures for the third order mock theta functions were established by McIntosh [M18].

In [GM12], it was shown that the mock theta functions with odd order can be expressed by the universal mock theta function $g(x,q)$. Similarly, the mock theta functions with even order are related to another universal mock theta function $g_2(x,q)$ where

$$g_2(x,q)=\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+1)/2}}{(x,x^{-1}q;q)_{n+1}}.$$

In [AH91], Andrews and Hickerson presented the Hecke-type double sums for the sixth order mock theta functions. The Hecke-type double sums for two new sixth order mock theta functions were found by Berndt and Chan [BC07]. Meanwhile, they established some linear relations between these two mock theta functions and Ramanujan's mock theta functions.

In [HM14], Hickerson and Mortenson gave the following definition of Hecke-type double sums.

Definition 1.1. Let $x,y\in\mathbb{C}^*:=\mathbb{C}\backslash\{0\}$ and define $sg(r):=1$ for $r\geq 0$ and $sg(r):=-1$ for $r<0$. Then

$$f_{a,b,c}(x,y,q):=\sum_{sg(r)=sg(s)} sg(r)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}. \tag{1.5}$$

Notice that

$$f_{a,b,a}(x,y,q)=f_{a,b,a}(y,x,q). \tag{1.6}$$

Based on the Hecke-type double sums for some mock theta functions given in [AH91, BC07], we can rewrite these Hecke-type double sums in the form of $f_{a,b,c}(x,y,q)$. For example, for the sixth order mock theta functions,

(1.7)
$$\phi_6(q)=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n^2}}{(-q;q)_{2n}}=\frac{1}{\overline{J}_{1,4}}f_{1,2,1}(q,-q,q),$$

(1.8)
$$\psi_6(q)=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}=\frac{q}{2\overline{J}_{1,4}}f_{1,2,1}(q^2,-q^2,q),$$

(1.9)
$$\overline{\phi}_6(q)=\sum_{n=1}^{\infty}\frac{(-q;q)_{2n-1}q^n}{(q;q^2)_n}=\frac{1}{J_{1,2}}\left(f_{1,5,1}(q,q^3,q^2)-q^4f_{1,5,1}(q^7,q^9,q^2)\right)-1,$$

(1.10)
$$\overline{\psi}_6(q)=\sum_{n=1}^{\infty}\frac{(-q;q)_{2n-2}q^n}{(q;q^2)_n}=\frac{q}{J_{1,2}}f_{1,5,1}(q^3,q^5,q^2),$$

where (1.10) appeared in [CW20].

In 2000, Gordon and McIntosh [GM00] found eight eighth order mock theta functions. Later, McIntosh [M07] studied three second order mock theta functions. In [CGH19], the Hecke-type double sums for these mock theta functions were provided. For example, the following eighth order mock theta functions have the Hecke-type double sums below.

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q^2)_n} = \frac{1}{J_{1,4}} f_{1,3,1}(-q,-q^2,-q), \tag{1.11}$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)^2}}{(q;q^2)_{n+1}} = \frac{q}{2J_{1,4}} f_{1,3,1}(-q^2,-q^3,-q). \tag{1.12}$$

For the tenth order mock theta functions appearing in Ramanujan's lost notebook [R88], Choi [C99, C00, C02, C07] established the Hecke-type double sums for these functions, and proved eight linear relation-s given by Ramanujan. For more on mock theta functions, one can see [A81, AB18, AG89, BHL11, G15, GM03, M12].

In [HM14], Hickerson and Mortenson provided the following definition of Appell–Lerch sums.

Definition 1.2. Let $x, z \in \mathbb{C}^*$ with neither z nor xz an integral power of q. Then

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1 - q^{r-1}xz}.$$

Then they expressed all the classical mock theta function in terms of Appell–Lerch sums. For example,

$$\phi_6(q) = 2m(q,q^3,-1), \tag{1.13}$$

$$\psi_6(q) = m(1,q^3,-q), \tag{1.14}$$

$$\overline{\phi}_6(q) = -m(q,q^3,q) - q\frac{\overline{J}_{3,12}^3}{J_1\overline{J}_{1,4}}, \tag{1.15}$$

$$\overline{\psi}_6(q) = -\frac{1}{2}m(1,q^3,q) + q\frac{J_6^3}{2J_1J_2}, \tag{1.16}$$

$$V_0(q) = -2q^{-1}m(1,q^8,q) - \frac{\overline{J}_{1,4}^2}{J_{2,8}}, \tag{1.17}$$

$$V_1(q) = -m(q^2,q^8,q). \tag{1.18}$$

In this paper, with the aid of some q-series identities given by Liu [L131, L132] and Chen and Wang [CW20], we find the Hecke-type double sums for some new mock theta functions. The main theorems are stated as follows.

Theorem 1.3. *We have*

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_nq^{n^2-n}}{(q^2;q^4)_n}=\frac{1}{J_{2,4}}\left(f_{1,2,1}(q^2,q^4,q^4)+f_{1,2,1}(q^4,q^6,q^4)\right). \tag{1.19}$$

Theorem 1.4. *We have*

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_nq^{n^2+n}}{(q^2;q^4)_n}=\frac{1}{(1+q^2)J_{2,4}}\left(f_{1,2,1}(q^2,q^4,q^4)+q^2f_{1,2,1}(q^4,q^6,q^4)\right). \tag{1.20}$$

Theorem 1.5. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{n^2+2n}}{(-q;q)_{2n}}=&-\frac{1}{(1-q)\overline{J}_{1,4}}\left(f_{1,2,1}(q,-q,q)\right.\\&\left.+\,qf_{1,2,1}(q^2,-q^2,q)\right)+\frac{2}{1-q}.\end{aligned} \tag{1.21}$$

Theorem 1.6. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(q;q^2)_n}=&\frac{1}{J_{1,2}}\left(2qf_{1,5,1}(q^3,q^5,q^2)-f_{1,5,1}(q,q^3,q^2)\right.\\&\left.+q^4f_{1,5,1}(q^7,q^9,q^2)\right)+1.\end{aligned} \tag{1.22}$$

Theorem 1.7. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+2}q^{n^2}}{(q;q^2)_{n+1}}=&\frac{2q}{J_{1,4}}f_{1,3,1}(q^5,q^7,q^4)+\frac{1+q^2}{2J_{1,4}}f_{1,3,1}(-q,-q^2,-q)\\&+\frac{1-q^2}{2}.\end{aligned} \tag{1.23}$$

Let $M_1(q)$, $M_2(q)$, $M_3(q)$, $M_4(q)$, and $M_5(q)$ be the left-hand sides of (1.19)-(1.23), respectively. Then based on the above results, we obtain the following corollary.

Corollary 1.8. *We have*

$$(1-q)M_3(q)=-\left(\phi_6(q)+2\psi_6(q)\right)+2, \tag{1.24}$$

$$M_4(q)=2\overline{\psi}_6(q)-\overline{\phi}_6(q), \tag{1.25}$$

$$M_5(q)=\frac{1}{2}(1+q^2)V_0(q)+2V_1(q)+\frac{1}{2}(1-q^2). \tag{1.26}$$

Finally, we express these five functions in terms of Appell–Lerch sums and theta functions.

Corollary 1.9. *We have*

$$M_1(q) = m(q^8, q^{12}, -1) - q^{-2} m(1, q^{12}, -1) + \frac{J_4^2 \overline{J}_{6,12}}{J_{2,4} \overline{J}_{0,4}} + q^{-2} \frac{\overline{J}_{2,4} J_{6,12}^2}{2 J_{2,4} \overline{J}_{0,12}}, \tag{1.27}$$

$$(1+q^2) M_2(q) = m(q^8, q^{12}, -1) - m(1, q^{12}, -1) + \frac{J_4^2 \overline{J}_{6,12}}{J_{2,4} \overline{J}_{0,4}} + \frac{\overline{J}_{2,4} J_{6,12}^2}{2 J_{2,4} \overline{J}_{0,12}}, \tag{1.28}$$

$$(1-q) M_3(q) = -2m(q, q^3, -1) - 2m(1, q^3, -q) + 2, \tag{1.29}$$

$$M_4(q) = m(q, q^3, q) - m(1, q^3, q) + q \frac{\overline{J}_{3,12}^3}{J_1 \overline{J}_{1,4}} + q \frac{J_6^3}{J_1 J_2}, \tag{1.30}$$

$$M_5(q) = -q^{-1}(1+q^2) m(1, q^8, q) - 2m(q^2, q^8, q) - (1+q^2) \frac{\overline{J}_{1,4}^2}{2 J_{2,8}} + \frac{1}{2}(1-q^2). \tag{1.31}$$

This paper is organized as follows. In Section 2, we state some preliminary results. In Section 3, we prove Theorems 1.3-1.7 and Corollaries 1.8-1.9.

2 Preliminaries

Racall that Jacobi's triple product identity is given by [GR04, Eq. (1.6.1)]

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (x, q/x, q; q)_\infty. \tag{2.1}$$

Throughout the paper, we use the following identities without mention.

$$\overline{J}_{0,1} = \frac{2J_2^2}{J_1}, \quad J_{1,2} = \frac{J_1^2}{J_2}, \quad \overline{J}_{1,2} = \frac{J_2^5}{J_1^2 J_4^2}, \quad \overline{J}_{1,3} = \frac{J_2 J_3^2}{J_1 J_6},$$
$$J_{1,6} = \frac{J_1 J_6^2}{J_2 J_3}, \quad \overline{J}_{1,6} = \frac{J_2^2 J_3 J_{12}}{J_1 J_4 J_6}.$$

Meanwhile, we need the following identity:

$$j(x; q) = j(q/x; q) = -x j(x^{-1}; q).$$

To prove the main results, we need the following identities given by Liu [L131, L132].

Lemma 2.1 ([L131, Theorem 1.7][L132, p. 2089])**.** *For* $\max\{|uab/q|, |ua|, |ub|, |c|, |d|\} < 1$,

$$\frac{(uq, uab/q; q)_\infty}{(ua, ub; q)_\infty} {}_3\phi_2\left(\begin{matrix} q/a, & q/b, & v \\ & c, & d \end{matrix}; q, \frac{uab}{q}\right) = \sum_{n=0}^{\infty} \frac{(1-uq^{2n})(u, q/a, q/b; q)_n}{(1-u)(q, ua, ub; q)_n}(-uab)^n q^{\frac{n^2-3n}{2}} {}_3\phi_2\left(\begin{matrix} q^{-n}, & uq^n, & v \\ & c, & d \end{matrix}; q, q\right). \tag{2.2}$$

Lemma 2.2 ([L132, Eq. (3.14)])**.** *For any nonnegative integer n,*

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & \alpha q^n, & \beta \\ & c, & d \end{matrix}; q, q\right) = (-c)^n q^{\frac{n^2-n}{2}} \frac{(q\alpha/c; q)_n}{(c; q)_n} \times {}_3\phi_2\left(\begin{matrix} q^{-n}, & \alpha q^n, & d/\beta \\ & d, & q\alpha/c \end{matrix}; q, q\beta/c\right). \tag{2.3}$$

Chen and Wang [CW20] established the following identities.

Lemma 2.3 ([CW20, Lemma 2.2])**.** *For any nonnegative integer n,*

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & \alpha q^{n+1}, & q/c \\ & \alpha d, & q^2/c \end{matrix}; q, d\right) = \left(\frac{q}{c}\right)^n \frac{(\alpha c, q; q)_n}{(q^2/c, \alpha q; q)_n} \times \sum_{j=0}^{n}(-1)^j \frac{(1-\alpha q^{2j})(\alpha, q/c, q/d; q)_j}{(1-\alpha)(q, \alpha c, \alpha d; q)_j} \times \left(\frac{cd}{q}\right)^j q^{\frac{-j^2-j}{2}}. \tag{2.4}$$

Lemma 2.4 ([CW20, Lemma 2.3])**.** *For any nonnegative integer n,*

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & q^n, & q/c \\ & d/q, & q^2/c \end{matrix}; q, d\right) = \left(\frac{q}{c}\right)^n (1-q^n)\frac{(c/q; q)_n}{(q^2/c; q)_n} \times \left(\frac{(c+d)q + cd(q-2) - q^3}{(c-q)(d-q)(1-q)} + \sum_{j=2}^{n}(-1)^j \frac{(1-q^{2j-1})(q/c, q/d; q)_j}{(1-q^j)(1-q^{j-1})(c/q, d/q; q)_j} (cd)^j q^{\frac{-j^2-3j}{2}}\right). \tag{2.5}$$

Next, in view of the above lemmas, we establish the following two lemmas which are the basis in the proofs of the main theorems.

Lemma 2.5. *We have*

$$\frac{(q, ab/q; q)_\infty}{(a, b; q)_\infty} {}_3\phi_2\left(\begin{matrix} q/a, & q/b, & q \\ & c, & d \end{matrix}; q, \frac{ab}{q}\right) \tag{2.6}$$

$$
\begin{aligned}
&= 1+\sum_{n=1}^{\infty}\frac{(1-q^{2n})(q/a,q/b,q/c,q/d;q)_n}{(a,b,c,d;q)_n}(abcd)^n q^{n^2-3n}\\
&\quad\times\Bigg(\frac{q(c+d)+q^2(q-2)-cdq}{(q-c)(q-d)(1-q)}\\
&\quad+\sum_{j=2}^{n}(-1)^j\frac{(1-q^{2j-1})(c/q,d/q;q)_j}{(1-q^{j-1})(1-q^j)(q/c,q/d;q)_j}(cd)^{-j}q^{\frac{-j^2+5j}{2}}\Bigg).
\end{aligned}
$$

Proof. Setting $u=1$ and $v=q$ in (2.2) yields

(2.7)

$$
\begin{aligned}
&\frac{(q,ab/q;q)_\infty}{(a,b;q)_\infty}{}_3\phi_2\left(\begin{matrix} q/a, & q/b, & q\\ & c, & d\end{matrix};q,\frac{ab}{q}\right)\\
&= 1+\sum_{n=1}^{\infty}\frac{(1+q^n)(q/a,q/b;q)_n}{(a,b;q)_n}(-ab)^n q^{\frac{n^2-3n}{2}}{}_3\phi_2\left(\begin{matrix} q^{-n}, & q^n, & q\\ & c, & d\end{matrix};q,q\right)\\
&= 1+\sum_{n=1}^{\infty}\frac{(1+q^n)(q/a,q/b,q/c;q)_n}{(a,b,c;q)_n}(abc)^n q^{n^2-2n}\\
&\quad\times{}_3\phi_2\left(\begin{matrix} q^{-n}, & q^n, & d/q\\ & d, & q/c\end{matrix};q,\frac{q^2}{c}\right),
\end{aligned}
$$

where we use (2.3) with $\alpha=1$ and $\beta=q$ to obtain the second equality.

Then substituting (2.5) with $c=q^2/d$ and $d=q^2/c$ into (2.7), we complete the proof. □

Lemma 2.6. *We have*

$$
\begin{aligned}
&\frac{(uq,uab/q;q)_\infty}{(ua,ub;q)_\infty}{}_3\phi_2\left(\begin{matrix} q/a, & q/b, & q\\ & c, & d\end{matrix};q,\frac{uab}{q}\right)\\
&= \sum_{n=0}^{\infty}\frac{(1-uq^{2n})(q/a,q/b,uq/c,uq/d;q)_n}{(1-u)(ua,ub,c,d;q)_n}(uabcd)^n q^{n^2-3n}\\
&\quad\times\sum_{j=0}^{n}(-1)^j\frac{(1-uq^{2j-1})(uq^{-1},c/q,d/q;q)_j}{(1-uq^{-1})(q,uq/c,uq/d;q)_j}(cd)^{-j}q^{\frac{-j^2+5j}{2}}.
\end{aligned}
\tag{2.8}
$$

Proof. Replacing v by q in (2.2), we arrive at

(2.9)

$$
\begin{aligned}
&\frac{(uq,uab/q;q)_\infty}{(ua,ub;q)_\infty}{}_3\phi_2\left(\begin{matrix} q/a, & q/b, & q\\ & c, & d\end{matrix};q,\frac{uab}{q}\right)\\
&= \sum_{n=0}^{\infty}\frac{(1-uq^{2n})(u,q/a,q/b;q)_n}{(1-u)(q,ua,ub;q)_n}(-uab)^n q^{\frac{n^2-3n}{2}}{}_3\phi_2\left(\begin{matrix} q^{-n}, & uq^n, & q\\ & c, & d\end{matrix};q,q\right)\\
&= \sum_{n=0}^{\infty}\frac{(1-uq^{2n})(u,q/a,q/b,uq/c,;q)_n}{(1-u)(q,ua,ub,c;q)_n}(uabc)^n q^{n^2-2n}
\end{aligned}
$$

$$\times {}_3\phi_2\left(\begin{matrix} q^{-n}, & uq^n, & d/q \\ & d, & uq/c \end{matrix}; q, q^2/c\right),$$

where we obtain the second equality by using (2.3) with $\alpha = u$ and $\beta = q$.

Then utilizing (2.4) with $\alpha = uq^{-1}$, $c = q^2/d$, and $d = q^2/c$ in (2.9), we complete the proof. □

In addition, we need the following properties.

Lemma 2.7 ([HM14]). *For $x, y \in \mathbb{C}^*$,*

$$\begin{aligned} f_{a,b,c}(x,y,q) &= f_{a,b,c}(-x^2q^a, -y^2q^c, q^4) \\ &\quad - xf_{a,b,c}(-x^2q^{3a}, -y^2q^{c+2b}, q^4) \\ &\quad - yf_{a,b,c}(-x^2q^{a+2b}, -y^2q^{3c}, q^4) \\ &\quad + xyq^b f_{a,b,c}(-x^2q^{3a+2b}, -y^2q^{3c+2b}, q^4), \end{aligned} \tag{2.10}$$

$$f_{a,b,c}(x,y,q) = -\frac{q^{a+b+c}}{xy} f_{a,b,c}(q^{2a+b}/x, q^{2c+b}/y, q), \tag{2.11}$$

$$f_{a,b,c}(x,y,q) = -yf_{a,b,c}(q^bx, q^cy, q) + j(x;q^a). \tag{2.12}$$

Recall the following relation between Hecke-type double sums and Appell–Lerch sums obtained by Hickerson and Mortenson [HM14].

Lemma 2.8 ([HM14, Eq. (1.7)]). *For $x, y \in \mathbb{C}^*$,*

$$\begin{aligned} f_{1,2,1}(x,y,q) &= j(y;q)m\left(q^2x/y^2, q^3, -1\right) \\ &\quad + j(x;q)m\left(q^2y/x^2, q^3, -1\right) \\ &\quad - \frac{yJ_3^3 j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3}j(-qy^2/x;q^3)j(-qx^2/y;q^3)}. \end{aligned} \tag{2.13}$$

Lemma 2.9. *We have*

$$f_{1,2,1}(q^2,q^4,q^4) = J_{2,4}m(q^8,q^{12},-1) + \frac{J_4^2\overline{J}_{6,12}}{\overline{J}_{0,4}}, \tag{2.14}$$

$$f_{1,2,1}(q^4,q^6,q^4) = -q^{-2}J_{2,4}m(1,q^{12},-1) + q^{-2}\frac{\overline{J}_{2,4}J_{6,12}^2}{2\overline{J}_{0,12}}. \tag{2.15}$$

Proof. In (2.13), replace q by q^4 and set $x = q^2$ and $y = q^4$. Noticing that $j(q^4;q^4) = 0$, we find

$$\begin{aligned} f_{1,2,1}(q^2,q^4,q^4) &= j(q^4;q^4)m(q^2,q^{12},-1) + j(q^2;q^4)m(q^8,q^{12},-1) \\ &\quad - \frac{q^4J_{12}^3 j(-q^{-2};q^4)j(q^{14};q^{12})}{\overline{J}_{0,12}j(-q^{10};q^{12})j(-q^4;q^{12})} \\ &= J_{2,4}m(q^8,q^{12},-1) + \frac{J_4^3J_{12}^5}{2J_6^2J_8^2J_{24}^2}, \end{aligned}$$

which implies (2.14). Next, in (2.13), replace q by q^4 and set $x = q^4$ and $y = q^6$. Then upon simplification, we deduce (2.15). □

3 Proofs of the main results

In this section, we prove Theorems 1.3-1.7 and Corollaries 1.8-1.9. We claim that $\sum_{n=j}^{k} = 0$ if $k < j$.

Proof of Theorem 1.3. Setting $(a, b, c, d, q) \to (-1, 0, -q, q, q^2)$ in (2.6) and simplifying, we derive

$$\begin{aligned}(3.1)\qquad &\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2-n}}{(q^2;q^4)_n} \\ &= \frac{2}{J_{2,4}} + \frac{1}{J_{2,4}} \sum_{n=1}^{\infty} (1+q^{2n})(1-q^{4n})(-1)^n q^{3n^2-3n} \\ &\quad \times \left(\frac{q^2}{1-q^2} + (1-q^{-2}) \sum_{j=2}^{n} \frac{q^{-j^2+3j}}{(1-q^{2j-2})(1-q^{2j})} \right).\end{aligned}$$

Notice that

$$\begin{aligned}&\frac{q^2}{1-q^2} + (1-q^{-2}) \sum_{j=2}^{n} \frac{q^{-j^2+3j}}{(1-q^{2j-2})(1-q^{2j})} \\ &= \frac{q^2}{1-q^2} - q^{-2} \sum_{j=2}^{n} \left(\frac{q^{-j^2+3j}}{1-q^{2j-2}} - \frac{q^{-j^2+3j+2}}{1-q^{2j}} \right) \\ &= \frac{q^2}{1-q^2} - q^{-2} \left(\sum_{j=1}^{n-1} \frac{q^{-j^2+j+2}}{1-q^{2j}} - \sum_{j=2}^{n} \frac{q^{-j^2+3j+2}}{1-q^{2j}} \right) \\ &= -\sum_{j=1}^{n} q^{-j^2+j} + \frac{q^{-n^2+n}}{1-q^{2n}}.\end{aligned}$$

Since

$$\begin{aligned}(3.2)\qquad \sum_{j=1}^{n} q^{-j^2+j} &= \frac{1}{2} \left(\sum_{j=0}^{n-1} q^{-j^2-j} + \sum_{j=-n}^{-1} q^{-j^2-j} \right) \\ &= \frac{1}{2} \left(\sum_{j=-n}^{n} q^{-j^2-j} - q^{-n^2-n} \right),\end{aligned}$$

we have

$$\begin{aligned}(3.3)\qquad &\frac{q^2}{1-q^2} + (1-q^{-2}) \sum_{j=2}^{n} \frac{q^{-j^2+3j}}{(1-q^{2j-2})(1-q^{2j})} \\ &= -\frac{1}{2} \sum_{j=-n}^{n} q^{-j^2-j} + \frac{(1+q^{2n})q^{-n^2-n}}{2(1-q^{2n})}.\end{aligned}$$

Then substituting (3.3) into (3.1) yields

(3.4)
$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2-n}}{(q^2;q^4)_n}\\
&=\frac{1}{2J_{2,4}}\sum_{n=1}^{\infty}(-1)^n\left(q^{3n^2+n}-q^{3n^2-n}+q^{3n^2+3n}-q^{3n^2-3n}\right)\sum_{j=-n}^{n}q^{-j^2-j}\\
&\quad+\frac{1}{2J_{2,4}}\sum_{n=1}^{\infty}(1+q^{2n})^3(-1)^n q^{2n^2-4n}+\frac{2}{J_{2,4}}\\
&=\frac{1}{2J_{2,4}}\left(\sum_{n=0}^{\infty}(-1)^n q^{3n^2+n}\sum_{j=-n}^{n}q^{-j^2-j}-\sum_{n=1}^{\infty}(-1)^n q^{3n^2-n}\sum_{j=-n+1}^{n-1}q^{-j^2-j}\right)\\
&\quad+\frac{1}{2J_{2,4}}\left(\sum_{n=0}^{\infty}(-1)^n q^{3n^2+3n}\sum_{j=-n}^{n}q^{-j^2-j}-\sum_{n=1}^{\infty}(-1)^n q^{3n^2-3n}\sum_{j=-n+1}^{n-1}q^{-j^2-j}\right)\\
&\quad+\frac{1}{2J_{2,4}}\sum_{n=1}^{\infty}(-1)^n q^{2n^2-2n}(1+q^{4n})+\frac{1}{J_{2,4}}\sum_{n=1}^{\infty}(-1)^n q^{2n^2}+\frac{1}{J_{2,4}}.
\end{aligned}$$

Based on (2.1), we have

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n q^{2n^2-2n}(1+q^{4n})&=\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+2n}-1=-1,\\
\sum_{n=1}^{\infty}(-1)^n q^{2n^2}&=\frac{1}{2}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}-1\right)=\frac{J_{2,4}-1}{2}.
\end{aligned}$$

So, substituting the above two identities into (3.4), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2-n}}{(q^2;q^4)_n}\\
&=\frac{1}{2J_{2,4}}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+n-j^2-j}\\
&\quad+\frac{1}{2J_{2,4}}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+3n-j^2-j}+\frac{1}{2}.
\end{aligned}\tag{3.5}$$

Observe that

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+n-j^2-j}\\
&=\left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}}-\sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^n q^{3n^2+n-j^2-j}.
\end{aligned}\tag{3.6}$$

Then letting $r \equiv s \pmod 2$ and setting $n = (r+s)/2$ and $j = (r-s)/2$ in (3.6), we derive

$$\begin{aligned}(3.7)\quad &\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+n-j^2-j} \\ &= \left(\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}} - \sum_{\substack{r,s<0\\ r\equiv s \pmod 2}}\right)(-1)^{\frac{r+s}{2}} q^{\frac{r^2+s^2+4rs+s}{2}} \\ &= \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r+s}{2}} q^{\frac{r^2+s^2+4rs+s}{2}} \\ &= \sum_{sg(r)=sg(s)} sg(r)(-1)^{r+s}\left(q^{2r^2+2s^2+8rs+2s} - q^{2r^2+2s^2+8rs+6r+8s+4}\right) \\ &= f_{1,2,1}(q^2,q^4,q^4) - q^4 f_{1,2,1}(q^8,q^{10},q^4),\end{aligned}$$

where we obtain the penultimate step by considering the even and odd cases of r and s, and the last step follows from (1.5). From (1.6) and (2.11), it can be seen that

$$f_{1,2,1}(q^2,q^4,q^4) = -q^4 f_{1,2,1}(q^8,q^{10},q^4) + J_{2,4}.$$

Substituting the above identity into (3.7) yields

$$(3.8)\quad \left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+n-j^2-j} = 2f_{1,2,1}(q^2,q^4,q^4) - J_{2,4}.$$

Similarly,

$$\begin{aligned}(3.9)\quad &\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+3n-j^2-j} \\ &= \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r+s}{2}} q^{\frac{r^2+s^2+4rs+2r+4s}{2}} \\ &= f_{1,2,1}(q^4,q^6,q^4) - q^6 f_{1,2,1}(q^{10},q^{12},q^4).\end{aligned}$$

From (1.6) and (2.12), it can be seen that

$$f_{1,2,1}(q^4,q^6,q^4) = -q^6 f_{1,2,1}(q^{10},q^{12},q^4).$$

So, combining the above identity and (3.9), we derive

$$(3.10)\quad \left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+3n-j^2-j} = 2f_{1,2,1}(q^4,q^6,q^4).$$

Finally, substituting (3.8) and (3.10) into (3.5), we complete the proof. □

Proof of Theorem 1.4. Setting $(u,a,b,c,d,q)\to(q^2,-1,0,-q,q,q^2)$ in (2.8) yields

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_nq^{n^2+n}}{(q^2;q^4)_n}&\\ =\frac{1}{(1-q^2)J_{2,4}}&\sum_{n=0}^{\infty}(1-q^{4n+2})^2(-1)^nq^{3n^2-n}\\ \times&\left(1+(1-q^{-2})(1-q^2)\sum_{j=1}^{n}\frac{(1+q^{2j})q^{-j^2+3j}}{(1-q^{4j-2})(1-q^{4j+2})}\right).\end{aligned}\tag{3.11}$$

Next,

$$\begin{aligned}&1+(1-q^{-2})(1-q^2)\sum_{j=1}^{n}\frac{(1+q^{2j})q^{-j^2+3j}}{(1-q^{4j-2})(1-q^{4j+2})}\\ &=1+\frac{1-q^{-2}}{1+q^2}\left(\sum_{j=1}^{n}\frac{(1+q^{2j})q^{-j^2+3j}}{1-q^{4j-2}}-\sum_{j=1}^{n}\frac{(1+q^{2j})q^{-j^2+3j+4}}{1-q^{4j+2}}\right)\\ &=1+\frac{1-q^{-2}}{1+q^2}\left(\sum_{j=1}^{n}\frac{(1+q^{2j})q^{-j^2+3j}}{1-q^{4j-2}}-\sum_{j=2}^{n+1}\frac{(1+q^{2j-2})q^{-j^2+5j}}{1-q^{4j-2}}\right)\\ &=1-\frac{(1-q^2)q^{-2}}{1+q^2}\left(\sum_{j=2}^{n+1}q^{-j^2+3j}+\frac{(1+q^2)q^2}{1-q^2}-\frac{(1+q^{2n+2})q^{-n^2+n+2}}{1-q^{4n+2}}\right)\\ &=-\frac{1-q^2}{1+q^2}\sum_{j=1}^{n}q^{-j^2+j}+\frac{(1-q^2)(1+q^{2n+2})q^{-n^2+n}}{(1+q^2)(1-q^{4n+2})}.\end{aligned}\tag{3.12}$$

Then substituting (3.12) into (3.11), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_nq^{n^2+n}}{(q^2;q^4)_n}\\ &=\frac{1}{(1+q^2)J_{2,4}}\sum_{n=0}^{\infty}(-1)^n\left(-q^{3n^2-n}-q^{3n^2+7n+4}+2q^{3n^2+3n+2}\right)\sum_{j=1}^{n}q^{-j^2+j}\\ &\quad+\frac{1}{(1+q^2)J_{2,4}}\sum_{n=0}^{\infty}(1+q^{2n+2})(1-q^{4n+2})(-1)^nq^{2n^2}\\ &=\frac{1}{(1+q^2)J_{2,4}}\left(\sum_{n=-1}^{\infty}(-1)^nq^{3n^2+5n+2}\sum_{j=1}^{n+1}q^{-j^2+j}-\sum_{n=0}^{\infty}(-1)^nq^{3n^2+7n+4}\right.\end{aligned}\tag{3.13}$$

$$\times \sum_{j=1}^{n} q^{-j^2+j} + 2\sum_{n=0}^{\infty}(-1)^n q^{3n^2+3n+2} \sum_{j=1}^{n} q^{-j^2+j}$$
$$\left. + \sum_{n=0}^{\infty}(1+q^{2n+2})(1-q^{4n+2})(-1)^n q^{2n^2}\right).$$

Notice that

$$\begin{aligned}
(3.14)\qquad &\sum_{n=-1}^{\infty}(-1)^n q^{3n^2+5n+2}\sum_{j=1}^{n+1} q^{-j^2+j} \\
&= \sum_{n=0}^{\infty}(-1)^n q^{3n^2+5n+2}\sum_{j=1}^{n+1} q^{-j^2+j} \\
&= \sum_{n=0}^{\infty}(-1)^n q^{3n^2+5n+2}\sum_{j=1}^{n} q^{-j^2+j} + \sum_{n=0}^{\infty}(-1)^n q^{2n^2+4n+2} \\
&= \sum_{n=0}^{\infty}(-1)^n q^{3n^2+5n+2}\sum_{j=1}^{n} q^{-j^2+j} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2}.
\end{aligned}$$

Then substituting (3.14) into (3.13) yields

(3.15)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+n}}{(q^2;q^4)_n} \\
&= \frac{1}{(1+q^2)J_{2,4}}\sum_{n=0}^{\infty}(1-q^{2n+2})(-1)^n q^{3n^2+5n+2}\sum_{j=1}^{n} q^{-j^2+j} \\
&\quad + \frac{2q^2}{(1+q^2)J_{2,4}}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+3n}\sum_{j=1}^{n} q^{-j^2+j} - \frac{1}{(1+q^2)J_{2,4}}\sum_{n=1}^{\infty}(-1)^n q^{2n^2} \\
&\quad + \frac{1}{(1+q^2)J_{2,4}}\sum_{n=0}^{\infty}(1+q^{2n+2})(1-q^{4n+2})(-1)^n q^{2n^2}.
\end{aligned}$$

Observe that

(3.16)

$$\begin{aligned}
&-\frac{1}{(1+q^2)J_{2,4}}\left(\frac{1}{2}\sum_{n=0}^{\infty}(1-q^{2n+2})(-1)^n q^{2n^2+4n+2} + \sum_{n=0}^{\infty}(-1)^n q^{2n^2+2n+2}\right. \\
&\quad \left. + \sum_{n=1}^{\infty}(-1)^n q^{2n^2} - \sum_{n=0}^{\infty}(1+q^{2n+2})(1-q^{4n+2})(-1)^n q^{2n^2}\right) \\
&= -\frac{1}{(1+q^2)J_{2,4}}\left(\frac{1}{2}\sum_{n=0}^{\infty}(1-q^{2n+2})(-1)^n q^{2n^2+4n+2} - 1\right. \\
&\quad \left. + \sum_{n=0}^{\infty}(1+q^{2n+2})(-1)^n q^{2n^2+4n+2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{(1+q^2)J_{2,4}}\left(\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n q^{2n^2+4n+2}+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{2n^2+6n+4}-1\right)\\
&= \frac{1}{(1+q^2)J_{2,4}}\left(\frac{3}{2}\sum_{n=1}^{\infty}(-1)^n q^{2n^2}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n q^{2n^2+2n}+1\right)\\
&= \frac{1}{(1+q^2)J_{2,4}}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}-\frac{1}{2}\sum_{n=1}^{\infty}(1-q^{2n})(-1)^n q^{2n^2}\right)\\
&= -\frac{1}{2(1+q^2)J_{2,4}}\sum_{n=1}^{\infty}(1-q^{2n})(-1)^n q^{2n^2}+\frac{1}{1+q^2},
\end{aligned}
$$

where we use (2.1) to derive the last step.

Then combining (3.2), (3.15), and (3.16), we derive

$$
\begin{aligned}
(3.17)\qquad &\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+n}}{(q^2;q^4)_n}\\
&= \frac{1}{2(1+q^2)J_{2,4}}\sum_{n=0}^{\infty}(1-q^{2n+2})(-1)^n q^{3n^2+5n+2}\sum_{j=-n}^{n} q^{-j^2-j}\\
&\quad+\frac{q^2}{(1+q^2)J_{2,4}}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+3n}\sum_{j=-n}^{n} q^{-j^2-j}\\
&\quad-\frac{1}{2(1+q^2)J_{2,4}}\sum_{n=1}^{\infty}(1-q^{2n})(-1)^n q^{2n^2}+\frac{1}{1+q^2}.
\end{aligned}
$$

Next, we find that

$$
\begin{aligned}
(3.18)\qquad &\sum_{n=0}^{\infty}(1-q^{2n+2})(-1)^n q^{3n^2+5n+2}\sum_{j=-n}^{n} q^{-j^2-j}\\
&= \sum_{n=1}^{\infty}(-1)^n q^{3n^2+n}\sum_{j=-n+1}^{n-1} q^{-j^2-j}-\sum_{n=1}^{\infty}(-1)^n q^{3n^2-n}\sum_{j=-n+1}^{n-1} q^{-j^2-j}\\
&= \sum_{n=1}^{\infty}(-1)^n q^{3n^2+n}\left(\sum_{j=-n}^{n} q^{-j^2-j}-q^{-n^2-n}-q^{-n^2+n}\right)\\
&\quad-\sum_{n=-\infty}^{-1}(-1)^n q^{3n^2+n}\sum_{j=n+1}^{-n-1} q^{-j^2-j}\\
&= \left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+n-j^2-j}-1-\sum_{n=1}^{\infty}(1+q^{2n})(-1)^n q^{2n^2}\\
&= 2f_{1,2,1}(q^2,q^4,q^4)-J_{2,4}-1-\sum_{n=1}^{\infty}(1+q^{2n})(-1)^n q^{2n^2},
\end{aligned}
$$

where the last step follows from (3.8).

Similarly,

$$\begin{aligned}
(3.19)\qquad &\sum_{n=0}^{\infty}(-1)^n q^{3n^2+3n}\sum_{j=-n}^{n} q^{-j^2-j}\\
&=\frac{1}{2}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{3n^2+3n-j^2-j}\\
&=f_{1,2,1}(q^4,q^6,q^4),
\end{aligned}$$

where the last step follows from (3.10).

Finally, substituting (3.18) and (3.19) into (3.17), and then utilizing (2.1), we complete the proof. □

Proof of Theorem 1.5. Setting $(u,a,b,c,d,q)\to(q^2,q,0,-q,-q^2,q^2)$ in (2.8) yields

$$\begin{aligned}
(3.20)\qquad &\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n^2+2n}}{(-q;q)_{2n}}\\
&=\frac{1}{(1+q)\overline{J}_{1,4}}\sum_{n=0}^{\infty}(1+q^{2n+1})^2(-1)^n q^{3n^2+n}\\
&\quad\times\left(1+2(1+q^{-1})(1+q)\sum_{j=1}^{n}\frac{(-1)^j q^{-j^2+2j}}{(1+q^{2j-1})(1+q^{2j+1})}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&1+2(1+q^{-1})(1+q)\sum_{j=1}^{n}\frac{(-1)^j q^{-j^2+2j}}{(1+q^{2j-1})(1+q^{2j+1})}\\
&=1+2\frac{1+q^{-1}}{1-q}\sum_{j=1}^{n}\left(\frac{(-1)^j q^{-j^2+2j}}{1+q^{2j-1}}-\frac{(-1)^j q^{-j^2+2j+2}}{1+q^{2j+1}}\right)\\
&=1+2\frac{1+q^{-1}}{1-q}\left(\sum_{j=0}^{n-1}\frac{(-1)^{j+1}q^{-j^2+1}}{1+q^{2j+1}}-\sum_{j=1}^{n}\frac{(-1)^j q^{-j^2+2j+2}}{1+q^{2j+1}}\right)\\
&=1+2\frac{1+q^{-1}}{1-q}\left(\sum_{j=1}^{n}(-1)^{j+1}q^{-j^2+1}-\frac{q}{1+q}+\frac{(-1)^n q^{-n^2+1}}{1+q^{2n+1}}\right)\\
&=1+\frac{1+q^{-1}}{1-q}\left(\sum_{j=-n}^{n}(-1)^{j+1}q^{-j^2+1}+q\right)-\frac{2}{1-q}+\frac{2(1+q)(-1)^n q^{-n^2}}{(1-q)(1+q^{2n+1})}\\
&=\frac{1+q}{1-q}\sum_{j=-n}^{n}(-1)^{j+1}q^{-j^2}+\frac{2(1+q)(-1)^n q^{-n^2}}{(1-q)(1+q^{2n+1})}.
\end{aligned}$$

Then substituting the above identity into (3.20), we have

(3.21)
$$
\begin{aligned}
&(1-q)\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{n^2+2n}}{(-q;q)_{2n}}\\
&=\frac{1}{\overline{J}_{1,4}}\sum_{n=0}^{\infty}(1+q^{2n+1})^2(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}(-1)^{j+1}q^{-j^2}+\frac{2}{\overline{J}_{1,4}}\sum_{n=-\infty}^{\infty}q^{2n^2+n}\\
&=\frac{1}{\overline{J}_{1,4}}\sum_{n=0}^{\infty}(1+q^{2n+1})^2(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}(-1)^{j+1}q^{-j^2}+2,
\end{aligned}
$$

where we use (2.1) to obtain the last step.

Observe that

(3.22)
$$
\begin{aligned}
&\sum_{n=0}^{\infty}(1+q^{4n+2})(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}(-1)^jq^{-j^2}\\
&=\sum_{n=0}^{\infty}(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}(-1)^jq^{-j^2}-\sum_{n=-\infty}^{-1}(-1)^nq^{3n^2+n}\sum_{j=n+1}^{-n-1}(-1)^jq^{-j^2}\\
&=\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^rq^{\frac{r^2+s^2+4rs+r+s}{2}}\\
&=f_{1,2,1}(-q^3,-q^3,q^4)-q^4f_{1,2,1}(-q^9,-q^9,q^4).
\end{aligned}
$$

Meanwhile, replacing a, b, c, x, and y by 1, 2, 1, q, and $-q$ in (2.10), respectively, and then using (1.6), we obtain

$$
f_{1,2,1}(q,-q,q)=f_{1,2,1}(-q^3,-q^3,q^4)-q^4f_{1,2,1}(-q^9,-q^9,q^4).
$$

Thus, combining (3.22) and the above identity, we derive

$$
\sum_{n=0}^{\infty}(1+q^{4n+2})(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}(-1)^jq^{-j^2}=f_{1,2,1}(q,-q,q). \tag{3.23}
$$

Similarly, we deduce

(3.24)
$$
\begin{aligned}
&2\sum_{n=0}^{\infty}(-1)^nq^{3n^2+3n}\sum_{j=-n}^{n}(-1)^jq^{-j^2}\\
&=\sum_{n=0}^{\infty}(-1)^nq^{3n^2+3n}\sum_{j=-n}^{n}(-1)^jq^{-j^2}\\
&\quad-\sum_{n=-\infty}^{-1}(-1)^nq^{3n^2+3n}\sum_{j=n+1}^{-n-1}(-1)^jq^{-j^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^r q^{\frac{r^2+s^2+4rs+3r+3s}{2}}\\
&= f_{1,2,1}(-q^5,-q^5,q^4) - q^6 f_{1,2,1}(-q^{11},-q^{11},q^4)\\
&= f_{1,2,1}(q^2,-q^2,q),
\end{aligned}
$$

where we derive the last equality by using (1.6) and (2.10).

Finally, in view of (3.21), (3.23), and (3.24), we complete the proof. □

Proof of Theorem 1.6. Setting $(u,a,b,c,d,q)\to(q^2,-1,-q,0,q,q^2)$ in (2.8), we obtain

(3.25)
$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^n}{(q;q^2)_n} &= \frac{1}{(1-q)J_{1,2}}\sum_{n=0}^{\infty}(1-q^{2n+1})^2(-1)^n q^{3n^2+n}\\
&\quad\times\left(1+(1-q^{-1})(1-q)\sum_{j=1}^{n}\frac{(1+q^{2j})q^{-2j^2+j}}{(1-q^{2j-1})(1-q^{2j+1})}\right).
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&1+(1-q^{-1})(1-q)\sum_{j=1}^{n}\frac{(1+q^{2j})q^{-2j^2+j}}{(1-q^{2j-1})(1-q^{2j+1})}\\
&=1+(1-q^{-1})\sum_{j=1}^{n}\left(\frac{q^{-2j^2+j}}{1-q^{2j-1}}-\frac{q^{-2j^2+j+1}}{1-q^{2j+1}}\right)\\
&=1+(1-q^{-1})\left(\sum_{j=0}^{n-1}\frac{q^{-2j^2-3j-1}}{1-q^{2j+1}}-\sum_{j=1}^{n}\frac{q^{-2j^2+j+1}}{1-q^{2j+1}}\right)\\
&=(1-q^{-1})\sum_{j=0}^{n-1}(1+q^{2j+1})q^{-2j^2-3j-1}+(1-q)\frac{q^{-2n^2+n}}{1-q^{2n+1}}\\
&=(1-q^{-1})\sum_{j=-n}^{n-1}q^{-2j^2-j}+(1-q)\frac{q^{-2n^2+n}}{1-q^{2n+1}}\\
&=(1-q^{-1})\sum_{j=-n}^{n}q^{-2j^2-j}+(1-q)\frac{q^{-2n^2+n}}{1-q^{2n+1}}-(1-q^{-1})q^{-2n^2-n}.
\end{aligned}
$$

Then substituting the above identity into (3.25) yields that

(3.26)
$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(q;q^2)_n}\\
&=-\frac{1}{J_{1,2}}\sum_{n=0}^{\infty}(1-q^{2n+1})^2(-1)^n q^{3n^2+n}\sum_{j=-n}^{n}q^{-2j^2-j}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{q}{J_{1,2}}\sum_{n=0}^{\infty}(1-q^{2n+1})(-1)^nq^{n^2+2n}+\frac{1}{J_{1,2}}\sum_{n=0}^{\infty}(1-q^{2n+1})^2(-1)^nq^{n^2}\\
=&-\frac{1}{J_{1,2}}\sum_{n=0}^{\infty}(1-q^{2n+1})^2(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}q^{-2j^2-j}\\
&+\frac{1}{J_{1,2}}\sum_{n=0}^{\infty}(1-q^{2n+1})(-1)^nq^{n^2}\\
=&-\frac{1}{J_{1,2}}\sum_{n=0}^{\infty}(1+q^{4n+2})(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}q^{-2j^2-j}\\
&+\frac{2q}{J_{1,2}}\sum_{n=0}^{\infty}(-1)^nq^{3n^2+3n}\sum_{j=-n}^{n}q^{-2j^2-j}+1,
\end{aligned}
$$

where we utilize (2.1) to derive the last equality.

Observe that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(1+q^{4n+2})(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}q^{-2j^2-j}\\
&=\sum_{n=0}^{\infty}(-1)^nq^{3n^2+n}\sum_{j=-n}^{n}q^{-2j^2-j}-\sum_{n=-\infty}^{-1}(-1)^nq^{3n^2+n}\sum_{j=n+1}^{-n-1}q^{-2j^2-j}\\
&=\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}}sg(r)(-1)^{\frac{r+s}{2}}q^{\frac{r^2+s^2+10rs+4s}{4}}\\
&=f_{1,5,1}(q,q^3,q^2)-q^4f_{1,5,1}(q^7,q^9,q^2).
\end{aligned}
\tag{3.27}
$$

Meanwhile,

$$
\begin{aligned}
&2\sum_{n=0}^{\infty}(-1)^nq^{3n^2+3n}\sum_{j=-n}^{n}q^{-2j^2-j}\\
&=\sum_{n=0}^{\infty}(-1)^nq^{3n^2+3n}\sum_{j=-n}^{n}q^{-2j^2-j}-\sum_{n=-\infty}^{-1}(-1)^nq^{3n^2+3n}\sum_{j=n+1}^{-n-1}q^{-2j^2-j}\\
&=f_{1,5,1}(q^3,q^5,q^2)-q^6f_{1,5,1}(q^9,q^{11},q^2).
\end{aligned}
\tag{3.28}
$$

From (1.6) and (2.11), it can be seen that

$$f_{1,5,1}(q^9,q^{11},q^2)=-q^{-6}f_{1,5,1}(q^3,q^5,q^2).$$

Thus, combining the above identity and (3.28), we find

$$\sum_{n=0}^{\infty}(-1)^nq^{3n^2+3n}\sum_{j=-n}^{n}q^{-2j^2-j}=f_{1,5,1}(q^3,q^5,q^2). \tag{3.29}$$

Therefore, substituting (3.27) and (3.29) into (3.26), we prove the theorem. □

Proof of Theorem 1.7. Setting $(u,a,b,c,d,q)\to(q^4,-q^{-3},0,0,q^3,q^2)$ in (2.8), we derive

(3.30)
$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+2}q^{n^2}}{(q;q^2)_{n+1}}\\&=\frac{1}{J_{1,4}}\sum_{n=0}^{\infty}(1+q^{2n+1})(1+q^{2n+3})(1-q^{4n+4})(-1)^nq^{4n^2+4n}\\&\quad\times\sum_{j=0}^{n}(1+q^{2j+1})q^{-2j^2-3j}\\&=\frac{1}{J_{1,4}}\sum_{n=0}^{\infty}(1+q^{2n+1})(1+q^{2n+3})(1-q^{4n+4})(-1)^nq^{4n^2+4n}\sum_{j=-n}^{n}q^{-2j^2-j+1}\\&\quad+\frac{1}{J_{1,4}}\sum_{n=0}^{\infty}(1+q^{2n+1})(1+q^{2n+3})(1-q^{4n+4})(-1)^nq^{2n^2+n}.\end{aligned}$$

Then considering the first sum on the right-hand side of (3.30), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(1+q^{2n+1})(1+q^{2n+3})(1-q^{4n+4})(-1)^nq^{4n^2+4n}\sum_{j=-n}^{n}q^{-2j^2-j+1}\\&=\sum_{n=0}^{\infty}(-1)^nq^{4n^2+4n}\sum_{j=-n}^{n}q^{-2j^2-j+1}+(1+q^2)\sum_{n=0}^{\infty}(-1)^nq^{4n^2+6n+1}\\&\quad\times\sum_{j=-n}^{n}q^{-2j^2-j+1}-(1+q^2)\sum_{n=0}^{\infty}(-1)^nq^{4n^2+10n+5}\sum_{j=-n}^{n}q^{-2j^2-j+1}\\&\quad-\sum_{n=0}^{\infty}(-1)^nq^{4n^2+12n+8}\sum_{j=-n}^{n}q^{-2j^2-j+1}\\&=2\sum_{n=0}^{\infty}(-1)^nq^{4n^2+4n}\sum_{j=-n}^{n}q^{-2j^2-j+1}+(1+q^2)\sum_{n=0}^{\infty}(-1)^nq^{4n^2+6n+1}\\&\quad\times\sum_{j=-n}^{n}q^{-2j^2-j+1}-(1+q^2)\sum_{n=0}^{\infty}(-1)^nq^{4n^2+10n+5}\sum_{j=-n}^{n}q^{-2j^2-j+1}\\&\quad-q-\sum_{n=1}^{\infty}(1+q^{2n})(-1)^nq^{2n^2+3n+1}.\end{aligned}\tag{3.31}$$

For the first term on the right-hand side of (3.31),

$$2\sum_{n=0}^{\infty}(-1)^nq^{4n^2+4n}\sum_{j=-n}^{n}q^{-2j^2-j+1}\tag{3.32}$$

$$
\begin{aligned}
&= \left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{4n^2+4n-2j^2-j+1} \\
&= \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r+s}{2}} q^{\frac{r^2+s^2+6rs+3r+5s+2}{2}} \\
&= qf_{1,3,1}(q^5,q^7,q^4) - q^9 f_{1,3,1}(q^{13},q^{15},q^4) \\
&= 2qf_{1,3,1}(q^5,q^7,q^4),
\end{aligned}
$$

where the last identity follows from (1.6) and (2.11). Next, we treat the second and third terms on the right-hand side of (3.31). Notice that

(3.33)

$$
\begin{aligned}
&(1+q^2)\sum_{n=0}^{\infty}(-1)^n q^{4n^2+6n+1}\sum_{j=-n}^{n} q^{-2j^2-j+1} \\
&\quad - (1+q^2)\sum_{n=0}^{\infty}(-1)^n q^{4n^2+10n+5}\sum_{j=-n}^{n} q^{-2j^2-j+1} \\
&= (1+q^2)\left(\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n-1}(-1)^n q^{4n^2+2n-2j^2-j} - \sum_{n=1}^{\infty}\sum_{j=-n+1}^{n-1}(-1)^n q^{4n^2-2n-2j^2-j}\right) \\
&= (1+q^2)\left(\sum_{n=1}^{\infty}\sum_{j=-n}^{n}(-1)^n q^{4n^2+2n-2j^2-j} - \sum_{n=1}^{\infty}(1+q^{2n})(-1)^n q^{2n^2+n}\right. \\
&\quad \left. - \sum_{n=1}^{\infty}\sum_{j=-n+1}^{n-1}(-1)^n q^{4n^2-2n-2j^2-j}\right) \\
&= (1+q^2)\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^n q^{4n^2+2n-2j^2-j} \\
&\quad - (1+q^2)\left(\sum_{n=0}^{\infty}(-1)^n q^{2n^2+n} + \sum_{n=1}^{\infty}(-1)^n q^{2n^2+3n}\right) \\
&= (1+q^2)\left(\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r+s}{2}} q^{\frac{r^2+s^2+6rs+r+3s}{2}}\right. \\
&\quad \left. - \sum_{n=0}^{\infty}(1-q^{6n+5})(-1)^n q^{2n^2+n}\right) \\
&= (1+q^2)\left(f_{1,3,1}(q^3,q^5,q^4) - q^6 f_{1,3,1}(q^{11},q^{13},q^4)\right. \\
&\quad \left. - \sum_{n=0}^{\infty}(1-q^{6n+5})(-1)^n q^{2n^2+n}\right)
\end{aligned}
$$

$$= \frac{1+q^2}{2}\left(f_{1,3,1}(-q,-q^2,-q) + J_{1,4} - 2\sum_{n=0}^{\infty}(1-q^{6n+5})(-1)^n q^{2n^2+n}\right),$$

where the last step follows from [CGH19, Eq. (2.36)]:

$$f_{1,3,1}(-q,-q^2,-q) = 2f_{1,3,1}(q^3,q^5,q^4) - 2q^6 f_{1,3,1}(q^{11},q^{13},q^4) - J_{1,4}.$$

So, substituting (3.32) and (3.33) into (3.31) yields

$$\begin{aligned}(3.34)\quad &\sum_{n=0}^{\infty}(1+q^{2n+1})(1+q^{2n+3})(1-q^{4n+4})(-1)^n q^{4n^2+4n}\sum_{j=-n}^{n} q^{-2j^2-j+1}\\ &= 2qf_{1,3,1}(q^5,q^7,q^4) + \frac{1+q^2}{2}f_{1,3,1}(-q,-q^2,-q) + \frac{(1+q^2)J_{1,4}}{2}\\ &\quad - (1+q^2)\sum_{n=0}^{\infty}(1-q^{6n+5})(-1)^n q^{2n^2+n} - q\\ &\quad + \sum_{n=0}^{\infty}(1+q^{2n+2})(-1)^n q^{2n^2+7n+6}.\end{aligned}$$

Finally, combining (3.30) and (3.34), we derive

(3.35)

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+2}q^{n^2}}{(q;q^2)_{n+1}}\\ &= \frac{1}{J_{1,4}}\left(2qf_{1,3,1}(q^5,q^7,q^4) + \frac{1+q^2}{2}f_{1,3,1}(-q,-q^2,-q) + \frac{(1+q^2)J_{1,4}}{2}\right)\\ &\quad + \frac{1}{J_{1,4}}\left(\sum_{n=0}^{\infty}(-1)^n\left(q^{2n^2+3n+3} - q^{2n^2+n+2} + q^{2n^2+3n+1} + q^{2n^2+7n+6}\right) - q\right).\end{aligned}$$

Since

$$\sum_{n=0}^{\infty}(-1)^n\left(q^{2n^2+3n+3} - q^{2n^2+n+2}\right) = \sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+3n+3} = -q^2 J_{1,4},$$

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n\left(q^{2n^2+3n+1} + q^{2n^2+7n+6}\right)\\ &= \sum_{n=0}^{\infty}(-1)^n q^{2n^2+3n+1} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2+3n+1}\\ &= q,\end{aligned}$$

substituting the above identities into (3.35), we complete the proof. □

Proof of Corollary 1.8. First, applying (1.7), (1.8), and (1.21) yields (1.24), and using (1.9), (1.10), and (1.22), we prove (1.25).

Next, we have [CGH19, Eq. (2.39)]

$$\begin{aligned} f_{1,3,1}(-q^2,-q^3,-q) &= f_{1,3,1}(q^5,q^7,q^4) - q^8 f_{1,3,1}(q^{13},q^{15},q^4) \\ &= 2f_{1,3,1}(q^5,q^7,q^4), \end{aligned}$$

where the second step follows from (2.11). Hence, based on (1.12) and the above identity, we derive

$$V_1(q) = \frac{q}{J_{1,4}} f_{1,3,1}(q^5,q^7,q^4). \tag{3.36}$$

Therefore, using (1.11), (1.23), and (3.36), we derive (1.26). □

Proof of Corollary 1.9. Using (1.19), (2.14), and (2.15), we obtain (1.27). Then substituting (2.14) and (2.15) into (1.20), we derive (1.28). Next, using (1.13), (1.14), and (1.24), we prove (1.29). Applying (1.15), (1.16), and (1.25) yields (1.30). Finally, substituting (1.17) and (1.18) into (1.26), we deduce (1.31). □

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