The saturation number of $K_{3,3}$

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Abstract

A graph G is called F-saturated if G does not contain F as a subgraph (not necessarily induced) but the addition of any missing edge to G creates a copy of F. The saturation number of F, denoted by $sat(n, F)$, is the minimum number of edges in an n-vertex F-saturated graph. Determining the saturation number of complete bipartite graphs is one of the most important problems in the study of saturation numbers. The value of $sat(n, K_{2,2})$ was shown to be $\lfloor \frac{3n-5}{2} \rfloor$ by Ollmann, and a shorter proof was later given by Tuza. For $K_{2,3}$, there has been a series of study aiming to determine $sat(n, K_{2,3})$ over the years. This was finally achieved by Chen who confirmed a conjecture of Bohman, Fonoberova, and Pikhurko that $sat(n, K_{2,3}) = 2n - 3$ for all $n \geq 5$. Pikhurko and Schmitt conjectured that $sat(n, K_{3,3}) = (3 + o(1))n$. In this paper, for $n \geq 9$, we give an upper bound of $3n - 9$ for $sat(n, K_{3,3})$, and prove that $3n - 9$ is also a lower bound when the minimum degree of a $K_{3,3}$-saturated graph is 2 or 5, where it is trivial when the minimum degree is greater than 5.

Keywords: saturation number; complete bipartite graph; minimum degree

1 Introduction

All graphs in this paper are finite and simple. Throughout the paper we use the terminology and notation of [11]. Given a graph G, we use $|G|$, $e(G)$, $\delta(G)$, and $\Delta(G)$ to denote the number of vertices, the number of edges, the minimum degree and the maximum degree of G, respectively. Let $\overline{G}$ denote the complement graph of G. For any $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and neighborhood of v in G, respectively, and let $N_G[v] = N_G(v) \cup \{v\}$. We shall omit

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the subscript G when the context is clear. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, let $A \sim B$ denote that each vertex in A is adjacent to each vertex in B and $G[A, B]$ be the subgraph with vertex set $A \cup B$ and edge set $E(G[A, B]) = \{xy \in E(G) : x \in A, y \in B\}$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. Let n be a positive integer. For a positive integer k, we let $[k] = \{1, 2, \ldots, k\}$. We denote a path, a cycle, a star, and a complete graph with n vertices by P_n, C_n, S_n, and K_n, respectively. For $r \geq 2$ and positive integers $s_1, \ldots, s_r$, let $K_{s_1,\ldots,s_r}$ denote the complete r-partite graph with part sizes $s_1, \ldots, s_r$. Let G and H be two disjoint graphs. Denote by $G \cup H$ the union of G and H. The *join* $G \vee H$ is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

Given a family of graphs $\mathcal{F}$, a graph G is $\mathcal{F}$-*saturated* if no member of $\mathcal{F}$ is a subgraph of G, but for any $e \in E(\overline{G})$, some member of $\mathcal{F}$ is a subgraph of $G + e$. The *saturation number* of $\mathcal{F}$, denoted by $sat(n, \mathcal{F})$, is the minimum number of edges in an n-vertex $\mathcal{F}$-saturated graph. Define $sat_\delta(n, \mathcal{F})$ to be the minimum number of edges in a graph with n vertices and minimum degree δ that is $\mathcal{F}$-saturated. If $\mathcal{F} = \{F\}$, then we also write $sat(n, \{F\})$ and $sat_\delta(n, \{F\})$ as $sat(n, F)$ and $sat_\delta(n, F)$, respectively.

Saturation numbers were first studied in 1964 by Erdős, Hajnal, and Moon [4], who proved that $sat(n, K_{k+1}) = (k-1)n - \binom{k}{2}$. Furthermore, they proved that equality holds only for the graph $K_{k-1} \vee \overline{K_{n-k+1}}$. In 1986, Kászonyi and Tuza in [6] determined $sat(n, F)$ for $F \in \{S_k, kK_2, P_k\}$, and they proved that $sat(n, \mathcal{F}) = O(n)$ for any family $\mathcal{F}$ of graphs. Since then, there has been extensive research on saturation numbers for various graph families $\mathcal{F}$.

We now mention some results for complete multipartite graphs. When all but at most one parts have size 1, Pikhurko [8] and Chen, Faudree, and Gould [2] independently determined the saturation number of complete multipartite graphs with sufficiently large order. When there are at least two parts of size at least 2, the exact values were only known for $K_{2,2}$ and $K_{2,3}$. The exact value for $K_{2,2}$ was first determined by Ollmann [7]. Later on, a shorter proof was given by Tuza [10]. For $K_{2,3}$, there have been several papers aiming to determine $sat(n, K_{2,3})$ over the years. This was finally achieved by Chen [3] who confirmed a conjecture of Bohman, Fonoberova, and Pikhurko [1] that $sat(n, K_{2,3}) = 2n - 3$ for all $n \geq 5$. For the case where the graph has r parts and all parts have size 2, Gould and Schmitt [5] conjectured that $sat(n, K_{2,\ldots,2}) = \lceil ((4r-5)n - 4r^2 + 6r - 1)/2 \rceil$, and they proved the conjecture when the minimum degree of the $K_{2,\ldots,2}$-saturated graphs is $2r-3$. For general complete multipartite graphs $K_{s_1,\ldots,s_r}$ with $s_r \geq \cdots \geq s_1 \geq 1$, Bohman, Fonoberova, and Pikhurko [1] determined the asymptotic bound on $sat(n, K_{s_1,\ldots,s_r})$ as $n \to \infty$.

Theorem 1.1 ([1]) *Let $r \geq 2$ and $s_r \geq \cdots \geq s_1 \geq 1$. Define $p = s_1 + \cdots + s_{r-1} - 1$. Then, for all large n,*

$$(p + \frac{s_r - 1}{2})n - O(n^{3/4}) \leq sat(n, K_{s_1,\ldots,s_r}) \leq \binom{p}{2} + p(n-p) + \lceil \frac{(s_r - 1)(n-p)}{2} - \frac{s_r^2}{8} \rceil.$$

In particular, $sat(n, K_{s_1,\ldots,s_r}) = (s_1 + \ldots + s_{r-1} + 0.5s_r - 1.5)n + O(n^{3/4})$.

We continue to study the saturation number for complete multipartite graphs. In light of the known results, studying $sat(n, K_{3,3})$ is the natural next step. In 2008, Pikhurko and Schmitt [9] conjectured that $sat(n, K_{3,3}) = (3 + o(1))n$.

In this paper, we give an upper bound on $sat(n, K_{3,3})$. Moreover, we consider its lower bound. In particular, we determine the exact value of $sat(n, K_{3,3})$ for $6 \le n \le 8$ and provide a lower bound on $sat(n, K_{3,3})$ when the minimum degree of a $K_{3,3}$-saturated graph is 2 or 5. The main results are the following theorems.

Theorem 1.2 *Let n be a positive integer and $n \ge 6$. Then* $sat(n, K_{3,3}) \le \begin{cases} 2n, & 6 \le n \le 8, \\ 3n-9, & n \ge 9. \end{cases}$

Theorem 1.3 **(i)** *For $6 \le n \le 8$, $sat(n, K_{3,3}) = 2n$.*

(ii) *For $n \ge 9$, $sat_2(n, K_{3,3}) = 3n - 9$ and $sat_5(n, K_{3,3}) \ge 3n - 9$.*

Let G be a $K_{3,3}$-saturated graph with n vertices and $n \ge 9$. If $\delta(G) \ge 6$, then $e(G) \ge 3n \ge 3n-9$. Hence, for $n \ge 9$, in order to determine the exact value of $sat(n, K_{3,3})$, we only need to consider $K_{3,3}$-saturated graphs with the minimum degree at most 5.

An outline of this paper is as follows. To prove Theorem 1.2, we construct an n-vertex $K_{3,3}$-saturated graph with $2n$ edges when $6 \le n \le 8$ and $3n - 9$ edges when $n \ge 9$ in Section 2. In Section 3, we first prove that $sat(n, K_{3,3}) \ge 2n$ when $6 \le n \le 8$ in Section 3.1, then we prove $sat_\delta(n, K_{3,3}) \ge 3n - 9$ when $\delta \in \{2, 5\}$ in Section 3.2.

2 Proof of Theorem 1.2

In this section, for $n \ge 6$, we construct an n-vertex $K_{3,3}$-saturated graph G_n with $2n$ edges when $6 \le n \le 8$, and $3n - 9$ edges when $n \ge 9$. Let G_{11} be a graph as depicted in Figure 1. Then $G_n = G_{11}[\{v_1, \ldots, v_n\}]$ for $6 \le n \le 11$.

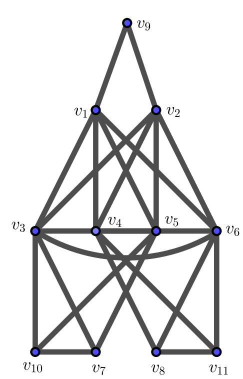

Figure 1: The graph G_{11}.

Proposition 2.1 *For $6 \le n \le 11$, the graph G_n is $K_{3,3}$-saturated and*

$$e(G_n) = \begin{cases} 2n, & 6 \le n \le 8, \\ 3n-9, & 9 \le n \le 11. \end{cases}$$

Proof. It is easy to verify that $e(G_n) = 2n$ when $6 \le n \le 8$, and $e(G_n) = 3n - 9$ when $9 \le n \le 11$. Next we show that G_n contains no copy of $K_{3,3}$ for $6 \le n \le 11$. Suppose R is a copy of $K_{3,3}$ of G_{11}. Then $v_9 \notin V(R)$ because $d_{G_{11}}(v_9) = 2$. For $u \in \{v_7, v_8, v_{10}, v_{11}\}$, since $d_{G_{11}}(u) = 3$ and there exists $v \in N_{G_{11}}(u)$ such that $d_{G_{11}}(v) = 3$ and $|N_{G_{11}}(u) \cap N_{G_{11}}(v)| = 2$, we have $u \notin V(R)$. Thus $R \subseteq G_6$. Since $v_1v_2 \notin E(G_6)$, v_1 and v_2 lie in the same part of R. Then $R[\{v_3, v_4, v_5, v_6\}]$ contains a copy of $K_{1,3}$, a contradiction. So G_{11} contains no copy of $K_{3,3}$. Note that G_n $(6 \le n \le 10)$ is a subgraph of G_{11}. Hence G_n contains no copy of $K_{3,3}$ for any $6 \le n \le 11$.

Let xy be an edge in the complement of G_n. It remains to show that the graph G'_n obtained by adding xy to G_n has a copy of $K_{3,3}$. We consider the following cases.

(a) If $\{x, y\} \cap \{v_1, v_2\} \ne \emptyset$ or $x, y \in \{v_7, v_8, v_{10}, v_{11}\}$, then the subgraph of G'_n induced by $\{x, y\} \cup \{v_3, v_5\} \cup \{v_4, v_6\}$ contains a copy of $K_{3,3}$.

(b) If $\{x, y\} \cap \{v_3, v_5\} \ne \emptyset$ or $x = v_9$, $y \in \{v_8, v_{11}\}$, then the subgraph of G'_n induced by $\{x, y\} \cup \{v_1, v_2\} \cup \{v_4, v_6\}$ contains a copy of $K_{3,3}$.

(c) If $\{x, y\} \cap \{v_4, v_6\} \ne \emptyset$ or $x = v_9$, $y \in \{v_7, v_{10}\}$, then the subgraph of G'_n induced by $\{x, y\} \cup \{v_1, v_2\} \cup \{v_3, v_5\}$ contains a copy of $K_{3,3}$.

For $6 \le n \le 11$, in all cases, G'_n contains a copy of $K_{3,3}$, hence G_n is $K_{3,3}$-saturated. ∎

Definition 2.2 For $n \ge 12$, let $H = \overline{K}_2 \vee (C_4 \cup C_{n-9} \cup K_1)$, where $V(\overline{K}_2) = \{v_1, v_2\}$, $C_4 = v_3v_4v_5v_6v_3$, $C_{n-9} = v_7v_8 \ldots v_{n-3}v_7$, $V(K_1) = \{v_{n-2}\}$. Let G_n be the graph obtained from H by adding new vertices $\{v_{n-1}, v_n\}$ and new edges $\{v_{n-1}v_3, v_{n-1}v_5, v_nv_4, v_nv_6\}$.

Proposition 2.3 *For $n \ge 12$, the graph G_n defined in Definition 2.2 is $K_{3,3}$-saturated and has $3n - 9$ edges.*

Proof. Clearly, $e(G) = 2(n-4) + (n-5) + 4 = 3n - 9$. Firstly, We show that G_n has no subgraph isomorphic to $K_{3,3}$. Suppose R is a copy of $K_{3,3}$ of G_n. From the structure of G_n, we see that $d(v_{n-1}) = d(v_n) = 2$ and hence $v_{n-1}, v_n \notin V(R)$. Thus $R \subseteq H$. Since each vertex of $C_4 \cup C_{n-9} \cup K_1$ has at most two neighbors in $C_4 \cup C_{n-9} \cup K_1$, $v_1, v_2 \in V(R)$ and they lie in different parts of R. This contradicts $v_1v_2 \notin E(G_n)$. So G_n contains no copy of $K_{3,3}$.

Let xy be an edge in the complement of G_n. It remains to show that the graph G'' obtained by adding xy to G_n has a copy of $K_{3,3}$. We consider the following cases.

(a) If $x, y \in \{v_1, v_2, v_{n-1}, v_n\}$, then the subgraph of G'' induced by $\{x, y\} \cup \{v_3, v_5\} \cup \{v_4, v_6\}$ contains a copy of $K_{3,3}$.

(b) If $x = v_{n-1}$, $y \in \{v_4, v_6, v_7, \ldots, v_{n-2}\}$ or $x = v_4$, $y = v_6$ or $x \in \{v_4, v_6\}$, $y \in \{v_7, \ldots, v_{n-2}\}$, then the subgraph of G'' induced by $\{x, v_1, v_2\} \cup \{y, v_3, v_5\}$ contains a copy of $K_{3,3}$.

(c) If $x = v_n$, $y \in \{v_3, v_5, v_7, \ldots, v_{n-2}\}$ or $x = v_3$, $y = v_5$ or $x \in \{v_3, v_5\}$, $y \in \{v_7, \ldots, v_{n-2}\}$, then the subgraph of G'' induced by $\{x, v_1, v_2\} \cup \{y, v_4, v_6\}$ contains a copy of $K_{3,3}$.

(d) If $x, y \in \{v_7, \ldots, v_{n-2}\}$ and $x \neq v_{n-2}$, let $N(x) \cap \{v_7, \ldots, v_{n-3}\} = \{x', x''\}$, then the subgraph of G'' induced by $\{x, v_1, v_2\} \cup \{y, x', x''\}$ contains a copy of $K_{3,3}$.

In all cases, G'' contains a copy of $K_{3,3}$. Hence G_n is $K_{3,3}$-saturated. ■

By Proposition 2.1 and Proposition 2.3, we complete the proof of Theorem 1.2.

3 Proof of Theorem 1.3

In the rest of the paper, we consider the lower bound on $sat(n, K_{3,3})$. Let $G = (V, E)$ be a $K_{3,3}$-saturated graph. We firstly choose a vertex a such that $d(a) = \delta(G)$ and $e(G[N(a)])$ is as small as possible. We partition V into four parts V_1, V_2, V_3 and V_4, where $V_1 = N[a]$, $V_2 = \{x \in V \backslash V_1 : |N(x) \cap N(a)| \geq 2\}$, $V_3 = \{y \in V \backslash (V_1 \cup V_2) : |N(y) \cap N(a)| = 1\}$ and $V_4 = V \backslash (V_1 \cup V_2 \cup V_3)$. Let $N_G(a) = \{a_1, a_2, \ldots, a_{d(a)}\}$. For $i_1, i_2, \ldots, i_s \in [d(a)]$, let $V_{i_1 i_2 \ldots i_s} = \{x \in V_2 : N(x) \cap V_1 = \{a_{i_1}, a_{i_2}, \ldots, a_{i_s}\}\}$.

In the following, we will first describe some useful properties of the $K_{3,3}$-saturated graph G.

Proposition 3.1 *The following statements hold.*

(i) *For any $x, y \in V$, if $xy \notin E$, then there are $\{x_1, x_2\} \subseteq N(x)$ and $\{y_1, y_2\} \subseteq N(y)$ such that $\{x_1, x_2\} \sim \{y_1, y_2\}$. (We usually say there is a copy of $K_{2,2}$ between $N(x)$ and $N(y)$.)*

(ii) *For any $x \in V \setminus V_1$, we have $|N(x) \cap N(a_i) \cap N(a_j)| \leq 2$ for any $i, j \in [d(a)]$ with $i \neq j$, and there exist $i, j \in [d(a)]$ with $i \neq j$ such that $|N(x) \cap N(a_i) \cap N(a_j)| = 2$.*

(iii) *For any $x \in V_3$, we have $|N(x) \cap V_2| \geq 1$. For any $x \in V_4$, we have $|N(x) \cap V_2| \geq 2$.*

(iv) *When $G[V_1 \backslash \{a\}]$ contains no copy of $K_{1,2}$, we have $|N(x) \cap V_2| \geq 2$ for any $x \in V \setminus V_1$, and $|V_2| \geq 3$. When $G[V_1 \setminus \{a\}]$ contains no copy of $K_{2,2}$, we have $|N(x) \cap V_2| \geq 1$ for any $x \in V_2$, and $|V_2| \geq 2$.*

Proof. Suppose $xy \notin E$. Then there is a copy of $K_{3,3}$ in $G + xy$, and (i) follows. For any $x \in V \setminus V_1$, if there is a vertex $x \in V \setminus V_1$ such that $|N(x) \cap N(a_i) \cap N(a_j)| \geq 3$ for some $i, j \in [d(a)]$ with $i \neq j$, then we would obtain a copy of $K_{3,3}$ of G, a contradiction. So $|N(x)\cap N(a_i)\cap N(a_j)|\le 2$ for any $x\in V\setminus V_1$ and $i,j\in[d(a)]$ with $i\ne j$. Since $ax\notin E$ for any $x\in V\backslash V_1$, there exist $i,j\in[d(a)]$ such that $|N(x)\cap N(a_i)\cap N(a_j)|=2$ by (i). This proves (ii). Let $x\in V\setminus V_1$ and $i,j\in[d(a)]$ with $i\ne j$ such that $|N(x)\cap N(a_i)\cap N(a_j)|=2$, we say $\{u,v\}=N(x)\cap N(a_i)\cap N(a_j)$. Then $u,v\in(V_1\cup V_2)\setminus\{a\}$. If $x\in V_3$, then we have $|N(x)\cap V_2|\ge 1$ by the definition of V_3. If $x\in V_4$, then we have $|N(x)\cap V_2|\ge 2$ by the definition of V_4. This proves (iii). Suppose $G[V_1\backslash\{a\}]$ contains no copy of $K_{1,2}$. Then $u,v\in V_2$. Hence we have $|N(x)\cap V_2|\ge 2$ for any $x\in V\setminus V_1$, and $|V_2|\ge 3$. Suppose $G[V_1\backslash\{a\}]$ contains no copy of $K_{2,2}$. Then $\{u,v\}\cap V_2\ne\emptyset$. Hence we have $|N(x)\cap V_2|\ge 1$ for each $x\in V_2$, and $|V_2|\ge 2$. This proves (iv). ■

Proposition 3.1(i) implies $\delta(G)\ge 2$ for each $K_{3,3}$-saturated graph G. Thus we consider $\delta(G)\ge 2$.

3.1 Proof of Theorem 1.3(i)

By Theorem 1.2, to prove $sat(n,K_{3,3})=2n$ for $6\le n\le 8$, it suffices to prove $sat(n,K_{3,3})\ge 2n$. We consider the minimum degree of G. If $\delta(G)\ge 4$, then we have $e(G)\ge 2n$. So we assume that $2\le\delta(G)\le 3$. For $i\in\{2,3,4\}$ and $x\in V_i$, we define $f(x)=|N(x)\cap(V_1\cup\cdots\cup V_{i-1})|+0.5|N(x)\cap V_i|-2$. Let $s_i=\sum_{x\in V_i}f(x)$, where $i\in\{2,3,4\}$.

We first observe that one can relate the number of edges to s_2, s_3 and s_4 in the following way:

$$\begin{aligned}
e(G)&=e(G[V_1])+e(G[V_2])+e(G[V_1,V_2])+e(G[V_3])+e(G[V_1,V_3])+e(G[V_2,V_3])+e(G[V_4])\\
&\quad+e(G[V_4,V_2\cup V_3])\\
&=e(G[V_1])+2(|V_2|+|V_3|+|V_4|)+s_2+s_3+s_4\\
&=e(G[V_1])+2(n-|V_1|)+s_2+s_3+s_4. \qquad (1)
\end{aligned}$$

Lemma 3.2 *For $6\le n\le 8$,*

(i) *if $\delta(G)=2$, then $s_2+s_3+s_4\ge|V_2|+|V_3|$.*

(ii) *if $\delta(G)=3$, then $s_2+s_3+s_4\ge|V_2|+|V_3|+|V_4|$ when $e(G[V_1\backslash\{a\}])\le 1$ and $s_2+s_3+s_4\ge 0.5(|V_2|+|V_3|+|V_4|)$ when $e(G[V_1\backslash\{a\}])\ge 2$.*

Proof. Suppose that $\delta(G)=2$. Then $G[V_1\backslash\{a\}]$ contains no $K_{1,2}$. Thus $f(x)\ge 1$ for each $x\in V_2\cup V_3$ and $f(x)\ge 0$ for each $x\in V_4$ by Proposition 3.1 (iii). So $s_2+s_3+s_4\ge|V_2|+|V_3|$. Suppose that $\delta(G)=3$. If $e(G[V_1\backslash\{a\}])\le 1$, then $|V_4|\le 1$ because $n\le 8$ and $|V_2|\ge 3$ by Proposition 3.1(iv). Thus $f(x)\ge 1$ for each $x\in V\setminus V_1$ by Proposition 3.1 (iii). So $s_2+s_3+s_4\ge|V_2|+|V_3|+|V_4|$. If $e(G[V_1\backslash\{a\}])\ge 2$, then we have $|N(x)\cap V_2|\ge 1$ for each $x\in V_2\cup V_3$ and $|N(x)\cap V_2|\ge 2$ for each $x\in V_4$ by Proposition 3.1 (iii). Thus for $x\in V_2$, $f(x)\ge 0.5$; for $y\in V_3$, $f(y)\ge 0.5$ or $f(y)=0$ and there exists a vertex $z\in V_4$ such that $f(z)=1$; for $z\in V_4$, $f(z)\ge 0.5$. Proposition 3.1(iv) implies $|V_2|\ge 2$ and so $|V_3\cup V_4|\le 2$, we have $s_2+s_3+s_4\ge 0.5(|V_2|+|V_3|+|V_4|)$. ■

Suppose that $\delta(G) = 2$. If $a_1a_2 \in E$, then $e(G) \geq 2n+|V_2|+|V_3|-3$ by Lemma 3.2(i) and (1). By Proposition 3.1(iii), we have $|V_2| \geq 3$. So $e(G) \geq 2n$. If $a_1a_2 \notin E(G)$, then $e(G) \geq 2n+|V_2|+|V_3|-4$ by Lemma 3.2(i) and (1). Proposition 3.1(i) implies that there is a copy of $K_{2,2}$ between $N(a_1)$ and $N(a_2)$, we have $|V_2 \cup V_3| \geq 4$. So $e(G) \geq 2n$.

Suppose that $\delta(G) = 3$. If $n = 6$, then $|V_2| = 2$, $|V_3| = |V_4| = 0$ and $e(V_1) = 6$ by Proposition 3.1(i). Otherwise, $a_ia_j \notin E$ where $i, j \in [3]$ with $i \neq j$, Proposition 3.1(i) implies that there is a copy of $K_{2,2}$ between $N(a_i)$ and $N(a_j)$, which contradicts the fact that $|V_2 \cup V_3| = 2$. Let $V_2 = \{x_1, x_2\}$. Proposition 3.1(iv) implies $x_1x_2 \in E$. If $x_1a_i \notin E$ for some $i \in [3]$, then $x_2 \in V_{123}$ by Proposition 3.1 (i). Thus $e(G) \geq 12 = 2n$.

If $n = 7$ and $e(G[V_1]) \leq 4$, then $G[V_1 \setminus \{a\}]$ contains no copy of $K_{1,2}$. Proposition 3.1(iv) implies $|V_2| = 3$ and $e(G[V_2]) = 3$. Since $a_ia_j \notin E$ for some $i, j \in [3]$, Proposition 3.1(i) implies there is a copy of $K_{2,2}$ between $N(a_i)$ and $N(a_j)$. Since $|V_2 \cup V_3| = |V_2| = 3$, $e(G[V_1]) \geq 4$. We see $|V_{123}| \leq 1$, else G contains a copy of $K_{3,3}$. There exists a vertex x such that $|N(x) \cap V_1| = 2$ and $xa_k \notin E$ for some $k \in [3]$. Proposition 3.1(i) implies that there is a copy of $K_{2,2}$ between $N(x)$ and $N(a_k)$, say $\{x_1, x_2\} \sim \{a_{k1}, a_{k2}\}$. When $\{a_{k1}, a_{k2}\} \subseteq V_2$, then $\{x_1, x_2\} \subseteq V_1$ and $\{a_{k1}, a_{k2}\} \subseteq V_{123}$, a contradiction. When $\{a_{k1}, a_{k2}\} \cap V_1 \neq \emptyset$, since $e(G[V_1]) \leq 4$, $|\{a_{k1}, a_{k2}\} \cap \{a_1, a_2, a_3\}| \leq 1$. If $a_{k1} \in \{a_1, a_2, a_3\}$, then $a_{k2} \in V_2$. By $|V_2| = 3$, $\{x_{k1}, x_{k2}\} \cap V_1 \neq \emptyset$, which contradicts $e(G[V_1]) \leq 4$. If $a \in \{a_{k1}, a_{k2}\}$, say $a_{k1} = a$, then $\{x_1, x_2\} \subseteq V_1$, $a_{k2} \in V_2$ and $a_{k2} \in V_{123}$. Then $e(G) = e(G[V_1]) + e(G[V_2]) + e(G[V_1, V_2]) \geq 4 + 3 + 7 = 14 = 2n$.

If $n = 7$ and $e(G[V_1]) = 6$, by Lemma 3.2(ii), then $e(G) \geq 2n - 0.5$, that is $e(G) \geq 2n$. Suppose $n = 7$ and $e(G[V_1]) = 5$. Let $E(G[V_1 \setminus \{a\}]) = \{a_1a_2, a_1a_3\}$. If $|V_2| = 2$, then let $V_2 = \{x_1, x_2\}$. Applying Proposition 3.1(i) to $ax_1 \notin E$ ($ax_2 \notin E$), we have the $K_{2,2}$ between $N(a)$ and $N(x_1)$ ($N(x_2)$) is $\{a_2, a_3\} \sim \{a_1, x_2\}(\{a_1, x_1\})$. Then $\{x_1, x_2\} \subseteq V_{123}$, and so $\{a_1, a_2, a_3\} \sim \{a, x_1, x_2\}$ is a copy of $K_{3,3}$ of G, a contradiction. If $|V_2| \geq 3$, then $|V_2| = 3$ by $n = 7$. Let $V_2 = \{x_1, x_2, x_3\}$. Note that $f(x_i) \geq 0.5$ for each $i \in [3]$. If there exists a vertex $x_i \in V_{123}$ or there are two vertices $x_i, x_j \in V_2$ such that $f(x_i) \geq 1$ and $f(x_j) \geq 1$, then $e(G) \geq 2n - 0.5$ by (1), and so $e(G) \geq 2n$. Thus we may assume $V_{123} = \emptyset$ and there is at most one vertex $x_i \in V_2$ such that $f(x_i) \geq 1$. Since there is a copy of $K_{2,2}$ between $N(x)$ and $N(a)$ for each $x \in V_2$, there is some vertex $x_i \in V_2$ with $f(x_i) = 1$, say x_1. Then $x_1 \in V_{23}$ and $\{x_2, x_3\} \subseteq V_{1i}$ for some $i \in \{2, 3\}$, say $i = 2$. Then $N(a_3) = \{a, a_1, x_1\}$, but $e(G[N(a_3)]) \leq 1$, which contradicts the minimality of $e(G[N(a)])$. So $e(G) \geq 2n$.

If $n = 8$, then $e(G) \geq 2n$ when $e(G[V_1 \setminus \{a\}]) = 1$ or 3 by Lemma 3.2(ii). Suppose $n = 8$ and $e(G[V_1 \setminus \{a\}]) = 0$, then $e(G) = 2n + s_2 + s_3 + s_4 - 5$. So we need to show $s_2 + s_3 + s_4 \geq 4.5$. If $|V_{123}| \geq 1$, then $f(x) \geq 2$ for each $x \in V_{123}$. So $s_2 + s_3 + s_4 \geq |V_2| + |V_3| + |V_4| + 1 \geq 5$ by the proof of Lemma 3.2(ii). Now we consider $|V_{123}| = 0$. Since $a_1a_2 \notin E$, Proposition 3.1(i) implies that there is a copy of $K_{2,2}$ between $N(a_1)$ and $N(a_2)$, say $\{x_1, x_2\} \sim \{x_3, x_4\}$. Then $\{x_1, x_2, x_3, x_4\} \subseteq V_2 \cup V_3$. Since $n = 8$, $|V_2 \cup V_3| = 4$. If $x_1 \in V_3$, then we can not find a copy of $K_{2,2}$ between $N(a_2)$ and $N(a_3)$ because $|(N(a_2) \cup N(a_3)) \cap (V_2 \cup V_3)| \leq 3$, a contradiction. By symmetry, we have $\{x_1, x_2, x_3, x_4\} \subseteq V_2$. If there exists $i \in [4]$ such that $|N(x_i) \cap V_2| \geq 3$, then $e(G) = e(G[V_1]) + e(G[V_2]) + e(G[V_1, V_2]) \geq 3 + 5 + 8 = 16 = 2n$. If $|N(x_i) \cap V_2| = 2$ for each $i \in [4]$, then $E(G[V_2]) = \{x_ix_j | i \in \{1,2\}, j \in \{3,4\}\}$. Since $x_1x_2 \notin E$, Proposition 3.1(i) implies that there is a copy of $K_{2,2}$ between $N(x_1)$ and $N(x_2)$. Note that $N(x_1) \cup N(x_2) \subseteq \{a_1, a_2, a_3, x_3, x_4\}$, $e(G[\{a_1, a_2, a_3\}]) = 0$ and $x_3x_4 \notin E$. So the $K_{2,2}$ between $N(x_1)$ and $N(x_2)$ must be $\{a_1, a_2\} \sim \{x_3, x_4\}$. Then $d(a_3) = 1$, this contradicts $\delta(G) \geq 2$. Suppose $n = 8$ and $e(G[V_1 \backslash \{a\}]) = 2$. Then $e(G) = 2n + s_2 + s_3 + s_4 - 3$. So we need to show $s_2 + s_3 + s_4 \geq 2.5$. If $f(x) \geq 1$ for some $x \in V_2$, then $s_2 + s_3 + s_4 \geq 2.5$ by the proof of Lemma 3.2(ii). If $f(x) = 0.5$ for some $x \in V_2$, then $f(x') \geq 1$ where $\{x'\} = N(x) \cap V_2$. So $s_2 + s_3 + s_4 \geq 2.5$.

This completes the proof of the lower bound on $sat(n, K_{3,3})$ for $6 \leq n \leq 8$. ■

3.2 Proof of Theorem 1.3(ii)

Note that for $n \geq 9$, the minimum degree of the $K_{3,3}$-saturation graph we constructed in Section 2 with $3n-9$ edges is 2. Thus $sat_2(n, K_{3,3}) \leq 3n-9$ for $n \geq 9$. Hence, to prove $sat_2(n, K_{3,3}) = 3n-9$, it suffices to prove $sat_2(n, K_{3,3}) \geq 3n - 9$ for $n \geq 9$. In this section, we give the lower bound of $3n - 9$ for $sat_\delta(n, K_{3,3})$ for $\delta \in \{2, 5\}$ and $n \geq 9$. We first consider the case where the minimum degree of G is 2.

3.2.1 $\delta(G) = 2$

We prove $sat_2(n, K_{3,3}) \geq 3n - 9$ for $n \geq 9$ in this part. According to the partition of V, we define $h(x) = |N(x) \cap (V_1 \cup \ldots \cup V_{i-1})| + 0.5|N(x) \cap V_i| - 3$ for each $x \in V_i$ and $q_i = \sum_{x \in V_i} h(x)$ where $i \in \{2, 3, 4\}$. For each $x \in V$, we say that the h-value of x is k if $h(x) = k$.

$$\begin{aligned}
e(G) &= e(G[V_1]) + e(G[V_2]) + e(G[V_1, V_2]) + e(G[V_3]) + e(G[V_1, V_3]) + e(G[V_2, V_3]) + e(G[V_4]) \\
&\quad + e(G[V_4, V_2 \cup V_3]) \\
&= e(G[V_1]) + 3(|V_2| + |V_3| + |V_4|) + q_2 + q_3 + q_4 \\
&= e(G[V_1]) + 3(n - |V_1|) + q_2 + q_3 + q_4. \qquad (2)
\end{aligned}$$

By (2), we have $e(G) \geq 3n - 7 + q_2 + q_3 + q_4$. Therefore, it suffices to prove

$$q_2 + q_3 + q_4 \geq -2.5. \qquad (3)$$

By Proposition 3.1(iv), we have $|N(x) \cap V_2| = 2$ for each $x \in V \setminus V_1$. So $h(z) \geq 0$ for each $z \in V_2 \cup V_3$ and $h(z) \geq -1$ for each $z \in V_4$. Thus, $q_2 \geq 0$ and $q_3 \geq 0$. Therefore, to prove (3), it suffices to show $q_4 \geq -2.5$.

Let $V_4^- = \{z \in V_4 : h(z) < 0\} = \{z_1, z_2, \ldots, z_{|V_4^-|}\}$ and $n_4^-(x) = |N(x) \cap V_4^-|$ for each $x \in V$. By Proposition 3.1(iii), each vertex $z \in V_4^-$ has exactly two neighbors in V_2, so we let $N(z_i) \cap V_2 = \{x_{i1}, x_{i2}\}$. Note that if $h(z_i) = -1$, then $N(z_i) = \{x_{i1}, x_{i2}\}$ and so z_i has no neighbor in V_4^-, and if $h(z_i) = -0.5$, then $d(z_i) = 3$ and z_i has one neighbor in V_4, saying $N_4(z_i) = \{c_i\}$.

For each $z_i, z_j \in V_4^-$ with $z_iz_j \notin E$, there is a $K_{2,2}$ between $N(z_i)$ and $N(z_j)$ by Proposition 3.1(i), we define four different types of $K_{2,2}$ as follows.

Type 1 : $\{x_{i1}, x_{i2}\} \sim \{x_{j1}, x_{j2}\}$;

Type 2 : $\{x_{i1}, x_{i2}\} \sim \{x_{jt}, c_j\}$, where $t \in \{1, 2\}$;

Type 3 : $\{x_{is}, c_i\} \sim \{x_{j1}, x_{j2}\}$, where $s \in \{1, 2\}$;

Type 4 : $\{x_{is}, c_i\} \sim \{x_{jt}, c_j\}$, where $s, t \in \{1, 2\}$.

If there are three vertices in V_4 with an h-value of -1, then there are six distinct vertices $x_1, x_2, \ldots, x_6 \in V_2$ such that $\{x_1, x_2\} \sim \{x_3, x_4\}$, $\{x_3, x_4\} \sim \{x_5, x_6\}$ and $\{x_1, x_2\} \sim \{x_5, x_6\}$. Thus G contains a copy of $K_{3,3}$ as $\{a_1, a_2, x_1\} \sim \{x_3, x_4, x_5\}$, a contradiction. So there are at most two vertices in V_4 with an h-value of -1. Thus $q_4 \geq -2.5$ when $|V_4^-| \leq 3$. In the following, we assume that $|V_4^-| \geq 4$.

Claim 1 There is at most one vertex in V_4^- with an h-value of -1.

Proof. Suppose that, by contradiction, there are exactly two vertices with an h-value -1, say z_1 and z_2. Then $z_1 z_2 \notin E$ and the $K_{2,2}$ between $N(z_1)$ and $N(z_2)$ is Type 1. Since $|V_4^-| \geq 4$, there exists a vertex, say z_3, such that $d(z_3) = 3$ and $z_1 z_3 \notin E$, $z_2 z_3 \notin E$. Applying Proposition 3.1(i) to $z_1 z_3 \notin E$ and $z_2 z_3 \notin E$, we obtain that there exists $b \in N(z_3)$ such that $b \sim \{x_{11}, x_{12}, x_{21}, x_{22}\}$. Then G contains a copy of $K_{3,3}$ as $\{a_1, a_2, b\} \sim \{x_{11}, x_{12}, x_{21}\}$, a contradiction. Hence there is at most one vertex in V_4^- with an h-value of -1. ∎

By Claim 1, if $|V_4^-| \leq 4$, then $q_4 \geq -2.5$. So we assume that $|V_4^-| \geq 5$ in the following.

Claim 2 If there exists a vertex in V_4^- with an h-value of -1, then $q_2 + q_3 + q_4 \geq -2.5$.

Proof. Without loss of generality, we assume that $h(z_1) = -1$. For each $z_i \in V_4^- \setminus \{z_1\}$, since $z_1 z_i \notin E$, $\{x_{11}, x_{12}\} \not\subseteq N(z_i)$. We first prove that there is at most one vertex $z_i \in V_4^-$ such that $\{x_{11}, x_{12}\} \sim \{x_{i1}, x_{i2}\}$. Suppose not. Then there exist two vertices, say z_2 and z_3, such that $\{x_{11}, x_{12}\} \sim \{x_{t1}, x_{t2}\}$ for each $t \in \{2, 3\}$. Since $|N(x) \cap V_2| = 2$ for each $x \in V \setminus V_1$, $\{x_{21}, x_{22}\} = \{x_{31}, x_{32}\}$. Note that $z_2 z_3 \in E$ for otherwise the non-edge $z_2 z_3$ contradicts Proposition 3.1(i). Since $|V_4^-| \geq 5$, there exists a vertex, say z_4, such that $z_4 z_p \notin E$ for each $p \in [3]$. By applying Proposition 3.1(i) to $z_1 z_4$, we have $\{x_{4i}, c_4\} \sim \{x_{11}, x_{12}\}$ for some $i \in \{1, 2\}$ and thus $x_{4i} \in \{x_{21}, x_{22}\}$. Since $c_2 = z_3$, there is no $K_{2,2}$ between $N(z_2)$ and $N(z_4)$, contradicting Proposition 3.1(i). This proves the statement. Thus for $i \in \{3, 4, \ldots, |V_4^-|\}$, without loss of generality, we assume $\{x_{11}, x_{12}\} \sim \{x_{ij}, c_i\}$, where $j \in [2]$. Applying Proposition 3.1(i) to $c_i z_1 \notin E$, we know that c_i has at least two neighbors other than x_{11}, x_{12} and z_i and thus $h(c_i) \geq 0.5$. Now we show that $c_i \neq c_j$ for $i, j \in \{3, 4, \ldots, |V_4^-|\}$ with $i \neq j$. Since $c_i \notin V_4^-$, we have $z_i z_j \notin E$. By Proposition 3.1 (i), there is a $K_{2,2}$ between $N(z_i)$ and $N(z_j)$. By considering the $K_{2,2}$ between $N(z_k)$ and $N(z_1)$

for $k \in \{i, j\}$, we see $N(c_k) \cap V_2 = \{x_{11}, x_{12}\}$. It follows that the $K_{2,2}$ between $N(z_i)$ and $N(z_j)$ must be Type 4. So $c_i \neq c_j$. Now we have

$$q_4 \geq h(z_1) + h(z_2) + \sum_{i=3}^{|V_4^-|} (h(z_i) + h(c_i)) \geq -1.5.$$

This completes the proof. ■

By Claim 2, we assume $h(z) = -0.5$ for each vertex $z \in V_4^-$. If $|V_4^-| \leq 5$, then $q_4 \geq -2.5$. So we assume $|V_4^-| \geq 6$ in the following.

Claim 3 If $h(z) = -0.5$ for each vertex $z \in V_4^-$ and there exist two non-adjacent vertices in V_4^- satisfying the $K_{2,2}$ between their neighborhood is Type 1, then $q_2 + q_3 + q_4 \geq -2.5$.

Proof. Suppose $z_1 z_2 \notin E$ and the $K_{2,2}$ between $N(z_1)$ and $N(z_2)$ is Type 1. Let $U = \{z \in V_4^- \setminus \{z_1, z_2\}$ with $zz_1, zz_2 \notin E\}$. Since $|V_4^-| \geq 6$, we have $|U| \geq 2$. Let $z_i \in U$. By applying Proposition 3.1 (i) to $z_i z_1 \notin E$, there is a copy of $K_{2,2}$ between $N(z_1)$ and $N(z_i)$. Note that $|N(v) \cap V_2| = 2$ for each $v \in V \setminus V_1$. If the $K_{2,2}$ is Type 1 or Type 3, then $\{x_{i1}, x_{i2}\} = \{x_{21}, x_{22}\}$. If the $K_{2,2}$ is Type 2, then $N(c_3) \cap V_2 = \{x_{11}, x_{12}\}$ and $x_{is} \in \{x_{21}, x_{22}\}$ for some $s \in [2]$. In each case, we cannot find a $K_{2,2}$ between $N(z_2)$ and $N(z_i)$. So the $K_{2,2}$ between $N(z_1)$ and $N(z_i)$ is Type 4. Similarly, the $K_{2,2}$ between $N(z_2)$ and $N(z_i)$ is Type 4. So we have $x_{is} \in \{x_{11}, x_{12}\}$ and $x_{it} \in \{x_{21}, x_{22}\}$, where $\{s, t\} = [2]$, and $c_1, c_2, c_i \notin V_4^-$. Hence for each $z_i, z_j \in U$, the $K_{2,2}$ between $N(z_i)$ and $N(z_j)$ is Type 4. So $c_i \neq c_j$. This means that for each $z \in U$, its unique neighbor $c \in V_4$ has at least 3 neighbors in $V_4 \setminus V_4^-$, so $h(z) + h(c) \geq 0$. And for any $z_i, z_j \in U$, $c_i \neq c_j$, so $q_4 \geq -2$. ■

By Claim 3, we suppose there are no two vertices $z_i, z_j \in V_4^-$ with $z_i z_j \notin E$ such that the $K_{2,2}$ between $N(z_i)$ and $N(z_j)$ is Type 1. Suppose that $c \in V_4^-$ for each $z \in V_4^-$. Let $z_i, z_j \in V_4^-$ with $z_i z_j \notin E$. By Proposition 3.1(i), Claim 3, and $c_i, c_j \in V_4^-$, we may assume the $K_{2,2}$ between $N(z_i)$ and $N(z_j)$ is Type 2. Then there is no copy of $K_{2,2}$ between $N(z_i)$ and $N(c_j)$, a contradiction. So we choose $z \in V_4^-$ with $c \notin V_4^-$ as z_1. Let $A_0 = \emptyset$. Let $A_\ell = \{z | z \in V_4^- \setminus (A_0 \cup \ldots \cup A_{\ell-1})$ and the $K_{2,2}$ between $N(z_1)$ and $N(z)$ is Type ℓ $\}$ and $B_\ell = \{c_i : z_i \in A_\ell\}$ for $\ell \in [4]$. By Claim 3, we have $A_1 = B_1 = \emptyset$. Thus $|A_2| + |A_3| + |A_4| = |V_4^-| - 1$. Let $B = \{c_1\} \cup B_2 \cup B_3 \cup B_4$ and $B' = \{c_1\} \cup B_2 \cup B_4$. Note that B_j and B_k may intersect when $j \neq k$ and $j, k \in [4]$.

For any $z \in A_2$, we have $c \notin V_4^-$ for otherwise there is no copy of $K_{2,2}$ between $N(z_1)$ and $N(c)$. Thus for each $z_i, z_j \in A_2$, we have $z_i z_j \notin E$. Since $z_i, z_j \in A_2$, we have $N(c_i) \cap V_2 = N(c_j) \cap V_2 = \{x_{11}, x_{12}\}$ and there exist $s, t \in [2]$ such that $x_{is} \notin \{x_{11}, x_{12}\}$ and $x_{jt} \notin \{x_{11}, x_{12}\}$. If the $K_{2,2}$ between $N(z_i)$ and $N(z_j)$ is Type 2 or Type 3, then $N(c_j) \cap V_2 = \{x_{i1}, x_{i2}\}$ or $N(c_i) \cap V_2 = \{x_{j1}, x_{j2}\}$, a contradiction. So the $K_{2,2}$ between $N(z_i)$ and $N(z_j)$ is Type 4. This implies that C_2 is a clique. For each two vertices $z_i, z_j \in A_3$, we have $N(z_i) \cap V_2 = N(z_j) \cap V_2$ since $|N(c_1) \cap V_2| = 2$. If

$z_iz_j \notin E$, then the $K_{2,2}$ between $N(z_i)$ and $N(z_j)$ is Type 4. If $z_iz_j \in E$, then $c_ic_j \in E$. This implies that B_3 is a clique. Thus if $|A_3| \geq 3$, then for each $z \in A_3$, we have $c \notin V_4^-$.

Let $|B_2| = p$, $|B_3 \setminus B_2| = q$ and $|B_4 \setminus (B_3 \cup B_2)| = r$. Note that $|B| \leq p+q+r+1$ and the equation $|B| = p+q+r+1$ implies that $c_1 \notin C_2 \cup C_3$. Note that

$$q_4 \geq \sum_{v\in C\setminus V_4^-} h(v) + \sum_{v\in V_4^-} h(v) = \sum_{v\in C\setminus V_4^-} h(v) - 0.5|V_4^-|. \tag{4}$$

To prove $q_4 \geq -2.5$, it suffices to prove $\sum_{v\in C\setminus V_4^-} h(v) \geq 0.5|V_4^-| - 2.5$ by (4). Recall that B_2 and B_3 are two cliques of G, $(B_2 \cup B_4) \cap V_4^- = \emptyset$ and $B_3 \cap B_4^- = \emptyset$ if $|A_3| \geq 3$.

Case 1: $|B_3| = |A_3| \geq 3$.

In this case, we have $(B_2 \cup B_3 \cup B_4) \cap V_4^- = \emptyset$. Thus $\sum_{v\in B\setminus V_4^-} h(v) = \sum_{v\in B} h(v)$.

If $B_2 \cap B_3 \neq \emptyset$, then

$$\begin{aligned}
\sum_{v\in B} h(v) \geq & 2|B| + e(G[B]) + 0.5e(G[B, V_4^-]) - 3|B| \\
\geq & 2|B| + \binom{p}{2} + \binom{q}{2} + q + r + 0.5|V_4^-| - 3|B| \\
= & \binom{p}{2} + \binom{q}{2} + q + r + 0.5|V_4^-| - |B| \\
\geq & \max\{0, p-1\} + \max\{0, q-1\} + q + r + 0.5|V_4^-| - (p+q+r+1) \\
\geq & 0.5|V_4^-| - 2.
\end{aligned}$$

If $B_2 \cap B_3 = \emptyset$, then $q \geq 3$ and

$$\begin{aligned}
\sum_{v\in B} h(v) \geq & 2|B| + e(G[B]) + 0.5e(G[B, V_4^-]) - 3|B| \\
\geq & 2|B| + \binom{p}{2} + \binom{q}{2} + r + 0.5|V_4^-| - 3|B| \\
= & \binom{p}{2} + \binom{q}{2} + r + 0.5|V_4^-| - (p+q+r+1) \\
\geq & p - 1 + q + r + 0.5|V_4^-| - (p+q+r+1) \\
= & 0.5|V_4^-| - 2.
\end{aligned}$$

Case 2: $|A_3| \leq 2$ and $|A_2| = p \geq 3$.

$$\begin{aligned}\sum_{v \in B \setminus V_4^-} h(v) \geq \sum_{v \in B'} h(v) &\geq 2|B'| + e(G[B']) + 0.5e(G[B', V_4^-]) - 3|B'| \\ &\geq 2|B'| + \binom{p}{2} + |B_4 \setminus B_2| + 0.5(|V_4^-| - 2) - 3|B'| \\ &\geq \binom{p}{2} + |B_4 \setminus B_2| + 0.5|V_4^-| - 1 - (p + |B_4 \setminus B_2| + 1) \\ &\geq \binom{p}{2} - p + 0.5|V_4^-| - 2 \\ &\geq 0.5|V_4^-| - 2.\end{aligned}$$

Case 3: $|A_2| \leq 2$ and $|A_3| \leq 2$.

Note that $(\{c_1\} \cup B_4) \cap (\{z_1\} \cup A_4) = \emptyset$. We have

$$\begin{aligned}\sum_{v \in \{c_1\} \cup B_4} h(v) &\geq 2(|B_4| + 1) + e(G[\{c_1\} \cup B_4]) + 0.5e(G[\{c_1\} \cup B_4, \{z_1\} \cup A_4]) - 3(|B_4| + 1) \\ &\geq 2(|B_4| + 1) + |B_4| + 0.5(|A_4| + 1) - 3(|B_4| + 1) = 0.5(|A_4| - 1). \qquad (5)\end{aligned}$$

Then

$$\begin{aligned}q_4 &\geq \sum_{v \in \{c_1\} \cup B_4} h(v) + \sum_{v \in V_4^-} h(v) \\ &\geq 0.5(|A_4| - 1) - 0.5(|A_2| + |A_3| + |A_4| + 1) = -0.5(|A_2| + |A_3|) - 1.\end{aligned}$$

Observe that $q_4 \geq -2.5$ when $|A_2|+|A_3| \leq 3$. Thus we just need to consider the case $|A_2| = |A_3| = 2$.

Note that $B' \cap V_4^- = \emptyset$. Suppose $B_2 \cap (\{c_1\} \cup B_4) \neq \emptyset$. Then $G[B']$ is a connected graph, and so $e(G[B']) \geq |B'| - 1$. We see

$$\begin{aligned}\sum_{v \in B \setminus V_4^-} h(v) \geq \sum_{v \in B'} h(v) &\geq 2|B'| + e(G[B']) + 0.5e(G[B', V_4^- \setminus A_3]) - 3|B'| \\ &\geq e(G[B']) - |B'| + 0.5(|V_4^-| - 2) \\ &\geq |B'| - 1 - |B'| + 0.5|V_4^-| - 1 \\ &\geq 0.5|V_4^-| - 2.\end{aligned}$$

Suppose $B_2 \cap (\{c_1\} \cup B_4) = \emptyset$. Let $B_2 = \{c_2, c_3\}$. If $h(c_2) > 0$ or $h(c_3) > 0$, by (5), then

$$\begin{aligned}q_4 &\geq \sum_{v \in \{c_1\} \cup B_4} h(v) + \sum_{v \in B_2} h(v) + \sum_{v \in V_4^-} h(v) \\ &\geq 0.5(|A_4| - 1) + 0.5 - 0.5(1 + 4 + |A_4|) = -2.5.\end{aligned}$$

If $h(c_2) = h(c_3) = 0$, then $N(c_2) = \{x_{11}, x_{12}, c_3, z_2\}$ and $N(c_3) = \{x_{11}, x_{12}, c_2, z_3\}$. Since $z_1 c_2 \notin E$, the $K_{2,2}$ between $N(z_1)$ and $N(c_2)$ must be Type 4, which contradicts $c_3 c_1 \notin E$.

In a conclusion, $q_4 \geq -2.5$ and so $e(G) \geq 3n - 9$. This completes the proof of the lower bound on $sat_2(n, K_{3,3})$ for ≥ 9. ∎

3.2.2 $\delta(G) = 5$

We prove $sat_5(n, K_{3,3}) \geq 3n-9$ for $n \geq 9$ in this part. Since $\delta(G) = 5$, we have $e(G) \geq 2.5n$. Then $e(G) \geq 3n-9$ when $n \leq 19$. Thus we assume $n \geq 20$ in the following.

We define a new function g as follows.

- For $x \in V_2$, let $g(x) = |N(x) \cap V_1| + 0.5|N(x) \cap (V_2 \cup V_3)| + 0.25|N(x) \cap V_4| - 3$.
- For $x \in V_3$, let $g(x) = |N(x) \cap V_1| + 0.5|N(x) \cap (V_2 \cup V_3 \cup V_4)| - 3$.
- For $x \in V_4$, let $g(x) = 0.75|N(x) \cap V_2| + 0.5|N(x) \cap (V_3 \cup V_4)| - 3$.

Observe that

$$\begin{aligned} e(G) &= e(G[V_1]) + e(G[V_2]) + e(G[V_1, V_2]) + e(G[V_3]) + e(G[V_1, V_3]) + e(G[V_2, V_3]) + e(G[V_4]) \\ &\quad + e(G[V_4, V_2 \cup V_3]) \\ &= e(G[V_1]) + 3(|V_2| + |V_3| + |V_4|) + \sum_{x \in V \setminus V_1} g(x) \\ &= e(G[V_1]) + 3(n - |V_1|) + \sum_{x \in V \setminus V_1} g(x). \end{aligned} \tag{6}$$

Note that $\delta(G) = 5$. Then $g(x) \geq -0.25$ for each $x \in V_2$ because $|N(x) \cap V_1| \geq 2$; $g(x) \geq 0$ for each $x \in V_3$ because $|N(x) \cap V_1| = 1$; $g(x) \geq 0$ for each $x \in V_4$ because $|N(x) \cap V_2| \geq 2$. If there exists a vertex $x_0 \in V_2$ such that $g(x_0) < 0$, then $g(x_0) = -0.25$, $d(x_0) = 5$, $N(x_0) \cap (V_2 \cup V_3) = \emptyset$, $|N(x_0) \cap V_1| = 2$ and $|N(x_0) \cap V_4| = 3$. We may assume that $N(x_0) = \{a_i, a_j, z_1, z_2, z_3\}$, where $i, j \in [5]$, $i \neq j$ and $\{z_1, z_2, z_3\} \subseteq V_4$. Since $ax_0 \notin E(G)$, Proposition 3.1(ii) implies that there is a copy of $K_{2,2}$ in $G[V_1 \setminus \{a\}]$. Let $s = 1$ if $a_i a_j \in E$ and $s = 0$ if $a_i a_j \notin E$. Thus $e(G[V_1 \setminus \{a\}]) \geq 4 + s$. But $e(G[N(x_0)]) \leq 3 + s$ because $N(z_i) \cap V_1 = \emptyset$ for each $i \in [3]$, which contradicts the minimality of $e(G[N(a)])$. Hence, $g(x) \geq 0$ for each $x \in V \setminus V_1$ and so $\sum_{x \in V \setminus V_1} g(x) \geq 0$. When $e(G[V_1]) \geq 9$, by (6), we have $e(G) \geq 3n - 9$. Thus we next consider $e(G[V_1]) \leq 8$. Note that $|N(x) \cap V_2| \geq 1$ for each $x \in V_2$ when $e(G[V_1]) \leq 8$. The following discussion is split into three cases below.

Case 1: $e(G[V_1]) = 8$.

If $\sum_{x \in V \setminus V_1} g(x) > 0$, then $e(G) = 3n - 10 + \sum_{x \in V \setminus V_1} g(x) > 3n - 10$ by (6) and so $e(G) \geq 3n - 9$ because $e(G)$ is an integer. Next we prove $\sum_{x \in V \setminus V_1} g(x) > 0$. If there exists a vertex $x \in V_2$ with $|N(x) \cap V_1| \geq 3$, then $g(x) > 0$ and so $\sum_{x \in V \setminus V_1} g(x) > 0$. So we may assume that $|N(x) \cap V_1| = 2$ for each $x \in V_2$. Since $e(G[V_1 \setminus \{a\}]) = 3$, there is a vertex a_i such that $N(a_i) \cap N(a) = \emptyset$ or $N(a_i) \cap N(a) = \{a_j\}$ with $N(a_j) \cap N(a) = \{a_i\}$, where $i, j \in [5]$ and $i \neq j$. We denote such a vertex by a_1. There is a vertex a_k such that $a_1 a_k \notin E$ for $k \in [5]$ and $k \neq 1$. Since $a_1 a_k \notin E$, by Proposition 3.1(i), $N(a_1) \cap (V_2 \cup V_3) \neq \emptyset$. Let $x \in N(a_1) \cap (V_2 \cup V_3)$ and $x_1 \in N(x) \cap V_2$. If $x \in V_3$, then $|N(x_1) \cap (V_2 \cup V_3)| \geq 2$. If $x \in V_2$, by the choice of a_1, then we have $|N(x_1) \cap V_2| \geq 2$, else there is no $K_{2,2}$ between $N(x_1)$ and $N(a)$. So $g(x_1) \geq 0.25$, which implies $\sum_{x \in V \setminus V_1} g(x) > 0$. Hence $e(G) \geq 3n - 9$.

Case 2: $e(G[V_1]) = 7$ and there is a copy of $K_{1,2}$ in $G[V_1 \setminus \{a\}]$.

We may assume that $E(G[V_1 \setminus \{a\}]) = \{a_1a_2, a_1a_3\}$. If $\sum_{x\in V\setminus V_1} g(x) > 1$, by (6), then

$$e(G) = e(G[V_1]) + 3(n - |V_1|) + \sum_{x\in V\setminus V_1} g(x) > 7 + 3(n - 6) + 1 = 3n - 10.$$

Since $e(G)$ is an integer, $e(G) \geq 3n - 9$. Thus we just need to prove $\sum_{x\in V\setminus V_1} g(x) > 1$. Let $V_2^1 = \{x \in V_2 : |N(x) \cap V_2| = 1\}$ and $V_2^2 = \{x \in V_2 : |N(x) \cap V_2| \geq 2\}$. Let $x \in V_2^1$ and $xx_1 \in E(G[V_2])$. Applying Proposition 3.1(i) to $ax \notin E(G)$, we have $x \in N(a_1)$ and $x_1 \in N(a_2) \cap N(a_3)$. If $x_1 \in V_2^1$, then $x_1 \in N(a_1)$ and $x \in N(a_2) \cap N(a_3)$ by $x_1a \notin E(G)$. Thus $\{a_1, a_2, a_3\} \subseteq (N(x) \cap V_1) \cap (N(x_1) \cap V_1)$. There is a copy of $K_{3,3}$ in G, that is $\{a, x, x_1\} \sim \{a_1, a_2, a_3\}$, a contradiction. This implies that $e(G[V_2^1]) = 0$, $V_2^2 \neq \emptyset$ and $|V_2| \geq 3$. Since $a_4a_5 \notin E$, there is a copy of $K_{2,2}$ between $N(a_4)$ and $N(a_5)$, say $\{x_{41}, x_{42}\} \sim \{x_{51}, x_{52}\}$. Notice that $N(a_4) \cap V_1 = N(a_5) \cap V_1 = \{a\}$. Thus $\{x_{41}, x_{42}, x_{51}, x_{52}\} \subseteq V_2 \cup V_3$. For each $y \in \{x_{41}, x_{42}, x_{51}, x_{52}\} \cap V_3$, by Proposition 3.1(i), then $|N(y) \cap V_2| \geq 2$. By the definition of g-function, for each $x \in V_2$, we have

$$\begin{aligned}g(x) =& |N(x) \cap V_1| + 0.25|N(x) \cap (V_2 \cup V_3 \cup V_4)| + 0.25|N(x) \cap (V_2 \cup V_3)| - 3\\ =& |N(x) \cap V_1| + 0.25|N(x) \cap (V_2 \cup V_3 \cup V_4)| + 0.25|N(x) \cap V_2| - 3 + 0.25|N(x) \cap V_3|.\end{aligned}$$

If $x \in V_2^1$, then

$$g(x) \geq 2 + 0.25 \times 3 + 0.25 \times 1 - 3 + 0.25|N(x) \cap V_3| = 0.25|N(x) \cap V_3|.$$

If $x \in V_2^2$, then

$$g(x) \geq 2 + 0.25 \times 3 + 0.25 \times 2 - 3 + 0.25|N(x) \cap V_3| = 0.25 + 0.25|N(x) \cap V_3|.$$

If $|N(x) \cap V_1| \geq 3$, then

$$g(x) \geq 3 + 0.25 \times 2 + 0.25 \times 1 - 3 + 0.25|N(x) \cap V_3| = 0.75 + 0.25|N(x) \cap V_3|.$$

Suppose $|\{x_{41}, x_{42}, x_{51}, x_{52}\} \cap V_3| \geq 2$. Then $e(G[V_2, V_3]) \geq 2|\{x_{41}, x_{42}, x_{51}, x_{52}\} \cap V_3| \geq 4$. Note that $V_2^2 \neq \emptyset$. Thus

$$\sum_{x\in V\setminus V_1} g(x) \geq \sum_{x\in V_2} g(x) \geq 0.25 + \sum_{x\in V_2} 0.25|N(x) \cap V_3| = 0.25 + 0.25e(G[V_2, V_3]) \geq 1.25.$$

Suppose $|\{x_{41}, x_{42}, x_{51}, x_{52}\} \cap V_3| = 1$, say $x_{41} \in V_3$. Then $\{x_{42}, x_{51}, x_{52}\} \subseteq V_2$ and $x_{42} \in V_2^2$. We see $\{x_{51}, x_{52}\} \subseteq V_2^2$ or $x_{42} \in N(a_2) \cap N(a_3)$. Note that $|N(x_{42}) \cap V_1| \geq 3$ when $x_{42} \in N(a_2) \cap N(a_3)$. Thus

$$\sum_{x\in V\setminus V_1} g(x) \geq \sum_{x\in V_2} g(x) \geq 0.75 + \sum_{x\in V_2} 0.25|N(x) \cap V_3| = 0.75 + 0.25e(G[V_2, V_3]) \geq 1.25.$$

It remains to consider the case $\{x_{41}, x_{42}, x_{51}, x_{52}\} \subseteq V_2$, that is $\{x_{41}, x_{42}, x_{51}, x_{52}\} \subseteq V_2^2$. If $V_3 \neq \emptyset$, then $e(G[V_2, V_3]) \geq 1$ and

$$\sum_{x\in V\setminus V_1} g(x) \geq \sum_{x\in V_2} g(x) \geq 0.25|V_2^2| + \sum_{x\in V_2} 0.25|N(x)\cap V_3| \geq 1 + 0.25e(G[V_2, V_3]) \geq 1.25.$$

If $|N(x) \cap V_1| \geq 3$ for some $x \in V_2$, then

$$\sum_{x\in V\setminus V_1} g(x) \geq \sum_{x\in V_2} g(x) \geq 0.75 + 0.25(|V_2^2| - 1) + \sum_{x\in V_2} 0.25|N(x)\cap V_3| \geq 1.5.$$

Next we assume that $|N(x) \cap V_1| = 2$ for each $x \in V_2$ and $|V_3| = 0$. Note that for each $x \in V_2^1$, let $xx_1 \in E(G[V_2])$, we have $x_1 \in N(a_2) \cap N(a_3)$. Thus $x_1 \notin \{x_{41}, x_{42}, x_{51}, x_{52}\}$. If $|V_2| \geq 5$, then $V_2^2 \setminus \{x_{41}, x_{42}, x_{51}, x_{52}\} \neq \emptyset$. Thus $|V_2^2| \geq 5$ and $\sum_{x\in V\setminus V_1} g(x) \geq 1.25$. If $|V_2| \leq 4$, that is $V_2 = \{x_{41}, x_{42}, x_{51}, x_{52}\}$, then we have $|V_4| \geq n - |V_2| - |V_3| - 6 = n - 10$ because $|V_3| = 0$. Note that $n \geq 20$. Thus

$$\begin{aligned} e(G) =& e(G[V_1]) + e(G[V_2]) + e(G[V_1 \cup V_4, V_2]) + e(G[V_4]) \\ \geq& 7 + 4 + 8 + 2|V_4| + \frac{3|V_4|}{2} > 3n - 9 \end{aligned}$$

Case 3: $e(G[V_1]) = 7$ and there is no copy of $K_{1,2}$ in $G[V_1 \setminus \{a\}]$ or $5 \leq e(G[V_1]) \leq 6$.

In this case, we define a new function g' as follows.

- For $x \in V_2$, let $g'(x) = |N(x) \cap V_1| + 0.5|N(x) \cap V_2| - 3$.
- For $x \in V_3 \cup V_4$, let $g'(x) = |N(x) \cap (V_1 \cup V_2)| + 0.5|N(x) \cap (V_3 \cup V_4)| - 3$.

We see

$$\begin{aligned} e(G) &= e(G[V_1]) + e(G[V_2]) + e(G[V_1, V_2]) + e(G[V_3]) + e(G[V_1, V_3]) + e(G[V_2, V_3]) + e(G[V_4]) \\ &\quad + e(G[V_4, V_2 \cup V_3]) \\ &= e(G[V_1]) + 3(|V_2| + |V_3| + |V_4|) + \sum_{x\in V\setminus V_1} g'(x) \\ &= e(G[V_1]) + 3(n - |V_1|) + \sum_{x\in V\setminus V_1} g'(x). \end{aligned} \tag{7}$$

For each $x \in V_2$, by Proposition 3.1(iv), $|N(x) \cap V_2| \geq 2$. Thus $g(x) \geq 0.25$ because $d(x) \geq 5$. It follows that $\sum_{x\in V\setminus V_1} g(x) \geq 0.25|V_2|$. It suffices to consider the following two subcases.

Subcase 3.1: $|V_2| \geq 13$ or $|V_3 \cup V_4| \geq 7$

Suppose $|V_2| \geq 13$. Then

$$e(G) = e(G[V_1]) + 3(n - |V_1|) + \sum_{x\in V\setminus V_1} g(x) \geq 5 + 3n - 18 + 0.25|V_2| \geq 3n - 9.75$$

and so $e(G) \geq 3n - 9$ because $e(G)$ is an integer.

Suppose $|V_3 \cup V_4| \geq 7$. By Proposition 3.1(iv), $|N(x) \cap V_2| \geq 2$ for each $x \in V \setminus V_1$. Thus $g'(x) \geq 0$ for each $x \in V_2$, $g'(x) \geq 1$ for each $x \in V_3$, and $g'(x) \geq 0.5$ for each $x \in V_4$. It follows that

$$e(G) = e(G[V_1]) + 3(n - |V_1|) + \sum_{x \in V \setminus V_1} g'(x) \geq 5 + 3n - 18 + 0.5|V_3 \cup V_4| \geq 3n - 9.5.$$

Since $e(G)$ is an integer, $e(G) \geq 3n - 9$.

Subcase 3.2: $|V_2| \leq 12$ or $|V_3 \cup V_4| \leq 6$

Since $n \geq 20$, $|V_3 \cup V_4| \geq 2$. We first prove the following claim.

Claim 4 If there is no copy of $K_{1,2}$ in $G[V_1 \setminus \{a\}]$ and $|V_3 \cup V_4| \geq 2$, then $\sum_{x \in V_3 \cup V_4} g'(x) \geq 2$. In particular, if $|V_3| \geq 1$ or $|N(z) \cap V_2| \geq 3$ for some $z \in V_4$, then $\sum_{x \in V_3 \cup V_4} g'(x) \geq 3$.

Proof. By the definition of g'-function and $\delta(G) = 5$, we have for each $x \in V_3$, $g'(x) \geq 1$ and for each $x \in V_4$, $g'(x) \geq 0.5$. When $|V_3 \cup V_4| \geq 4$, $\sum_{x \in V_3 \cup V_4} g'(x) \geq 2$. When $2 \leq |V_3 \cup V_4| \leq 3$, for each $x \in V_3 \cup V_4$, we have $|N(x) \cap (V_1 \cup V_2)| \geq 5 - (|V_3 \cup V_4| - 1)$. Thus

$$g'(x) \geq (6 - |V_3 \cup V_4|) + 0.5(|V_3 \cup V_4| - 1) - 3 = 2.5 - 0.5(|V_3 \cup V_4|$$

and

$$\sum_{x \in V_3 \cup V_4} g'(x) \geq (2.5 - 0.5(|V_3 \cup V_4|))|V_3 \cup V_4| \geq 3.$$

Next we assume that $|V_3| \geq 1$ or $|N(z) \cap V_2| \geq 3$ for some $z \in V_4$. To prove $\sum_{x \in V_3 \cup V_4} g'(x) \geq 3$, it suffices to consider the case $|V_3 \cup V_4| \geq 4$ by the above discussion. If $|V_3 \cup V_4| \geq 5$ or $|V_3| \geq 2$, then $\sum_{x \in V_3 \cup V_4} g'(x) \geq 3$. Suppose $|V_3 \cup V_4| = 4$ and $|V_3| \leq 1$. Let $V_3 \cup V_4 = \{y_1, y_2, y_3, y_4\}$ and $\{y_1, y_2, y_3\} \subseteq V_4$. Let $y_4 \in V_3$ or $|N(y_4) \cap V_2| \geq 3$ when $y_4 \in V_4$. If $g'(y_i) \geq 1$ for some $i \in [3]$, then $\sum_{x \in V_3 \cup V_4} g'(x) \geq 3$. So we assume $g'(y_i) = 0.5$ for each $i \in [3]$, then we have $|N(y_i) \cap (V_3 \cup V_4)| = 3$. Thus $G[\{y_1, y_2, y_3, y_4\}]$ is a clique. It follows that $g'(y_4) \geq 1.5$ and $\sum_{x \in V_3 \cup V_4} g'(x) \geq 3$. ■

Since $|V_3 \cup V_4| \geq 2$, $\sum_{v \in V_3 \cup V_4} g'(v) \geq 2$ by Claim 4. When $e(G[V_1]) \geq 7$, by inequality (7), $e(G) \geq 3n - 9$. Now we consider the case $e(G[V_1]) = 6$. If we can show $\sum_{v \in V_2} g'(v) > 0$ or $\sum_{v \in V_3 \cup V_4} g'(v) > 2$, by Claim 4 and (7), then $e(G) > 3n - 10$ and so $e(G) \geq 3n - 9$. If there exists a vertex $u \in V_2$ such that $|N(u) \cap V_1| \geq 3$, then $g'(u) \geq 1$ and so $\sum_{v \in V_2} g'(v) > 0$. If $V_3 \neq \emptyset$, then $\sum_{v \in V_3 \cup V_4} g'(v) \geq 3$ by Claim 4. Thus we may assume that $|N(v) \cap V_1| = 2$ for each $v \in V_2$ and $V_3 = \emptyset$. We choose a vertex $x \in V_2$. Without loss generality, suppose $x \in N(a_1) \cap N(a_2)$. Since $xa_i \notin E$ for each $i \in \{3, 4, 5\}$, there is a copy of $K_{2,2}$ between $N(a_i)$ and $N(x)$, say $\{a_{i1}, a_{i2}\} \sim \{x_{i1}, x_{i2}\}$. Note that there is no copy of $K_{1,2}$ in $G[V_1 \setminus \{a\}]$. Thus $\{a_{i1}, a_{i2}\} \cap V_2 \neq \emptyset$ for each $i \in \{3, 4, 5\}$. Recall that $|N(v) \cap V_1| = 2$ for each $v \in V_2$ and $V_3 = \emptyset$. We have $\{x_{i1}, x_{i2}\} \cap (V_2 \cup V_4) \neq \emptyset$ for each $i \in \{3, 4, 5\}$. By Proposition 3.1(ii), we have

$|N(w) \cap V_2| \geq 3$ for each $w \in (\bigcup_{i \in \{3,4,5\}} \{x_{i1}, x_{i2}\}) \cap (V_2 \cup V_4)$. Thus $\sum_{v \in V_3 \cup V_4} g'(v) \geq 3$ by Claim 4 or $\sum_{v \in V_2} g'(v) \geq 0.5$.

Next we consider $e(G[V_1]) = 5$. If $\sum_{v \in V \setminus V_1} g'(v) > 3$, by (7), then $e(G) > 3n - 10$ and so $e(G) \geq 3n - 9$. Thus we prove $\sum_{v \in V \setminus V_1} g'(v) > 3$ in the following. Recall $\sum_{v \in V_3 \cup V_4} g'(v) \geq 2$. If there is a vertex $x \in V_2$ with $|N(x) \cap V_1| \geq 4$, then $g'(x) \geq 2$ and

$$\sum_{v \in V \setminus V_1} g'(v) \geq g'(x) + \sum_{v \in V_3 \cup V_4} g'(v) \geq 4.$$

If there are two different vertices $x, y \in V_2$ with $|N(x) \cap V_1| = |N(y) \cap V_1| = 3$, then $g'(x) \geq 1$, $g'(y) \geq 1$ and

$$\sum_{v \in V \setminus V_1} g'(v) \geq g'(x) + g'(y) + \sum_{v \in V_3 \cup V_4} g'(v) \geq 4.$$

Suppose $x \in V_2$ with $|N(x) \cap V_1| = 3$, and $|N(v) \cap V_1| = 2$ for each $v \in V_2 \setminus \{x\}$. Let $N(x) \cap V_1 = \{a_1, a_2, a_3\}$. Since $xa_4 \notin E$, there is a copy of $K_{2,2}$ between $N(x)$ and $N(a_4)$, say $\{x_{11}, x_{12}\} \sim \{a_{41}, a_{42}\}$. If $V_3 \neq \emptyset$, by Claim 4, then $\sum_{v \in V_3 \cup V_4} g'(v) \geq 3$. Thus

$$\sum_{v \in V \setminus V_1} g'(v) \geq g'(x) + \sum_{v \in V_3 \cup V_4} g'(v) \geq 4.$$

So we may assume that $V_3 = \emptyset$. Then $\{a_{41}, a_{42}\} \subseteq V_2$ and $\{x_{11}, x_{12}\} \cap (V_2 \cup V_4) \neq \emptyset$. Let $w \in \{x_{11}, x_{12}\} \cap (V_2 \cup V_4)$. By Proposition 3.1(ii), $|N(w) \cap V_2| \geq 3$. Thus $g'(w) \geq 0.5$. If $w \in V_2$, then

$$\sum_{v \in V \setminus V_1} g'(v) \geq g'(x) + g'(w) + \sum_{v \in V_3 \cup V_4} g'(v) \geq 3.5.$$

If $w \in V_4$, by Claim 4, then $\sum_{v \in V \setminus V_1} g'(v) \geq 4$.

Suppose $|N(v) \cap V_1| = 2$ for each $v \in V_2$. Since $|V_3 \cup V_4| \leq 6$ and $n \geq 20$, $|V_2| \geq 8$. Recall the definition of g-function, for each $v \in V_2$, we have $g(v) \geq 0.25$ and if $g(v) > 0.25$, then $g(v) \geq 0.5$. We see there exists a vertex $x \in V_2$ such that $g(x) = 0.25$, otherwise, $g(v) \geq 0.5$ for each $v \in V_2$ and so $\sum_{v \in V_2} g(v) \geq 0.5|V_2| \geq 4$. By (6), $e(G) \geq 5 + 3(n-6) + 4 = 3n - 9$. We choose such a vertex $x \in V_2$ such that $g(x) = 0.25$. Then $d(x) = 5$ and let $N(x) = \{a_1, a_2, x_{11}, x_{12}, z\}$, where $\{a_1, a_2\} \subseteq V_1$, $\{x_{11}, x_{12}\} \subseteq V_2$ and $z \in V_4$. Note that $xa_j \notin E$ for each $j \in \{3, 4, 5\}$. By Proposition 3.1(i), there is a copy of $K_{2,2}$ between $N(x)$ and $N(a_j)$, say $\{x_{j1}, x_{j2}\} \sim \{a_{j1}, a_{j2}\}$. We see $\{a_{j1}, a_{j2}\} \subseteq V_2 \cup V_3$ for each $j \in \{3, 4, 5\}$. Since $|N(v) \cap V_1| = 2$ for each $v \in V_2$, $\{x_{j1}, x_{j2}\} \nsubseteq V_1$ for each $j \in \{3, 4, 5\}$. Otherwise, $\{x_{j1}, x_{j2}, a_j\} \subseteq N(a_{j1}) \cap V_1$, a contradiction.

Suppose $V_3 \neq \emptyset$. Then we have $\sum_{v \in V_3 \cup V_4} g'(v) \geq 3$ by Claim 4. We have $|V_3| \leq 3$, otherwise $\sum_{v \in V_3} g'(v) \geq 4$ and we are done. Note that $\{a_{j1}, a_{j2}\} \subseteq V_2 \cup V_3$ for any $j \in \{3, 4, 5\}$. When $|\bigcup_{j \in \{3,4,5\}} \{a_{j1}, a_{j2}\}| \leq 5$, we may assume $a_{31} = a_{41}$. When $|\bigcup_{j \in \{3,4,5\}} \{a_{j1}, a_{j2}\}| = 6$, we have $|\bigcup_{j \in \{3,4,5\}} \{a_{j1}, a_{j2}\} \cap V_2| \geq 3$ because $|V_3| \leq 3$, so we may assume $\{a_{31}, a_{41}\} \subseteq \bigcup_{j \in \{3,4,5\}} \{a_{j1}, a_{j2}\} \cap V_2$. In two cases, we have $\{a_{31}, a_{41}\} \subseteq V_2$. Let $k \in \{3, 4\}$. If $\{x_{k1}, x_{k2}\} \cap V_2 \neq \emptyset$, let $w \in$

$\{x_{k1}, x_{k2}\} \cap V_2$, then $\{x, a_{k1}\} \subseteq N(w) \cap V_2$, Proposition 3.1(ii) implies that $|N(w) \cap V_2| \geq 3$ and so $g'(w) \geq 0.5$. Thus

$$\sum_{v \in V \setminus V_1} g'(v) \geq g'(w) + \sum_{v \in V_3 \cup V_4} g'(v) \geq 3.5.$$

So we assume $\{x_{k1}, x_{k2}\} \cap V_2 = \emptyset$. Note that $\{x_{k1}, x_{k2}\} \nsubseteq V_1$. Since $d(x) = 5$, $\{x_{k1}, x_{k2}\} = \{a_1, z\}$ or $\{a_2, z\}$, and so $N(a_{k1}) \cap N(a_{k2}) \cap V_1 = \{a_k, a_{\ell_k}\}$ for some $\ell_k \in [2]$. Then $\{x, a_{31}, a_{32}, a_{41}, a_{42}\} \subseteq N(z) \cap V_2$. Note that $|N(v) \cap V_1| = 2$ for any $v \in V_2$. Since $x \in V_{12}$, $\{a_{31}, a_{32}\} \subseteq V_{1\ell_3}$ and $\{a_{41}, a_{42}\} \subseteq V_{1\ell_4}$, $|\{x, a_{31}, a_{32}, a_{41}, a_{42}\}| = 5$, which follows that $g'(z) \geq 2$. Note that $V_3 \neq \emptyset$ and $g'(y) \geq 1$ for each $y \in V_3$ and $g'(v) \geq 0.5$ for each $v \in V_4$. Recall $z \in V_4$ and $|V_3 \cup V_4| \geq 2$. So $\sum_{v \in V_3 \cup V_4} g'(v) > 3$ when $|V_3 \cup V_4| \geq 3$. If $|V_3 \cup V_4| = 2$, then $g'(y) > 1$ for $y \in V_3$ because $d(y) \geq 5$. Therefore $\sum_{v \in V_3 \cup V_4} g'(v) > 3$.

It remains to consider $V_3 = \emptyset$. Then $\{a_{j1}, a_{j2}\} \subseteq V_2$ for any $j \in \{3, 4, 5\}$. When $\{x_{j1}, x_{j2}\} \cap V_2 = \emptyset$ for any $j \in \{3, 4, 5\}$, then $\{x_{j1}, x_{j2}\} = \{a_1, z\}$ or $\{a_2, z\}$. Note that $|N(v) \cap V_1| = 2$ for each $v \in V_2$. Since $\{a_{j1}, a_{j2}\} \subseteq V_{j\ell_j}$ for $\ell_j \in [2]$, $|(\bigcup_{j \in \{3,4,5\}} \{a_{j1}, a_{j2}\}) \cup \{x\}| = 7$. Thus $|N(z) \cap V_2| \geq 7$, which implies that $\sum_{v \in V_2} g'(v) \geq 4$. When there exists $j \in \{3, 4, 5\}$ such that $\{x_{j1}, x_{j2}\} \cap V_2 \neq \emptyset$, then $g'(w) \geq 0.5$ for $w \in \{x_{j1}, x_{j2}\} \cap V_2$ because $|N(w) \cap V_2| \geq 3$ by Proposition 3.1(ii). In this case, we have $z \notin \{x_{j1}, x_{j2}\} \cap V_4$. Otherwise, Proposition 3.1(ii) implies $|N(z) \cap V_2| \geq 3$. By Claim 4,

$$\sum_{v \in V \setminus V_1} g'(v) \geq g'(w) + \sum_{v \in V_3 \cup V_4} g'(v) \geq 3.5.$$

Thus we are done. If $|N(x_{11}) \cap V_2| + |N(x_{12}) \cap V_2| \geq 7$, then we have

$$g'(x_{11}) + g'(x_{12}) = e(G[\{x_{11}, x_{12}\}, V_1]) + 0.5(e(G[\{x_{11}\}, V_2]) + e(G[\{x_{12}\}, V_2])) - 6 \geq 4 + 3.5 - 6 = 1.5.$$

Thus $\sum_{v \in V \setminus V_1} g'(v) \geq 3.5$, and we are done. So it suffices to prove $|N(x_{11}) \cap V_2| + |N(x_{12}) \cap V_2| \geq 7$ in the following. Since $z \notin \{x_{j1}, x_{j2}\}$, we have $\{x_{j1}, x_{j2}\} \cap V_2 \neq \emptyset$ for any $j \in \{3, 4, 5\}$. Recall $N(x) = \{a_1, a_2, x_{11}, x_{12}, z\}$ and $x \in N(x_{11}) \cap N(x_{12}) \cap V_{12}$. Then $\{a_{31}, a_{32}, a_{41}, a_{42}, a_{51}, a_{52}\} \subseteq N(x_{11}) \cup N(x_{12})$. If $|\{a_{31}, a_{32}, a_{41}, a_{42}, a_{51}, a_{52}\}| \geq 5$, then

$$|N(x_{11}) \cap V_2| + |N(x_{12}) \cap V_2| = |(N(x_{11}) \cup N(x_{12})) \cap V_2| + |(N(x_{11}) \cap N(x_{12})) \cap V_2| \geq 7.$$

Suppose that $|\{a_{31}, a_{32}, a_{41}, a_{42}, a_{51}, a_{52}\}| \leq 4$. Note that $|N(x) \cap V_1| = 2$ for each $x \in V_2$. We obtain $|\{a_{31}, a_{32}, a_{41}, a_{42}, a_{51}, a_{52}\}| \geq 3$. When $\{x_{31}, x_{32}\} \cap V_1 \neq \emptyset$, say $a_\ell \in \{x_{31}, x_{32}\}$ for some $\ell \in [2]$, then $\{a_{31}, a_{32}\} \subseteq V_{3\ell}$ and $\{a_{31}, a_{32}\} \cap \{a_{k1}, a_{k2}\} = \emptyset$ for each $k \in \{4, 5\}$. Since $|\{a_{31}, a_{32}, a_{41}, a_{42}, a_{51}, a_{52}\}| \leq 4$, we have $\{x_{k1}, x_{k2}\} = \{x_{11}, x_{12}\}$ for each $k \in \{4, 5\}$, that is $|N(x_{11}) \cap N(x_{12}) \cap \bigcup_{j \in \{3,4,5\}} \{a_{j1}, a_{j2}\}| \geq 2$. Thus

$$\begin{aligned} |N(x_{11}) \cap V_2| + |N(x_{12}) \cap V_2| =& |(N(x_{11}) \cup N(x_{12})) \cap V_2| + |(N(x_{11}) \cap N(x_{12})) \cap V_2| \\ \geq& |\bigcup_{j \in \{3,4,5\}} \{a_{j1}, a_{j2}\} \cup \{x\}| + 3 \geq 7. \end{aligned}$$

When $\{x_{j1}, x_{j2}\} \cap V_1 = \emptyset$ for each $j \in \{3,4,5\}$, then $\{x_{j1}, x_{j2}\} = \{x_{11}, x_{12}\}$ for each $j \in \{3,4,5\}$ and $\bigcup_{j\in\{3,4,5\}}\{a_{j1}, a_{j2}\} \subseteq N(x_{11}) \cap N(x_{12})$. By $|\bigcup_{j\in\{3,4,5\}}\{a_{i1}, a_{i2}\}| \geq 3$ and $x \notin \bigcup_{j\in\{3,4,5\}}\{a_{i1}, a_{i2}\}$, we have $|N(x_{11}) \cap V_2| + |N(x_{12}) \cap V_2| \geq 8$.

As a result, we have $e(G) \geq 3n-9$ for $n \geq 9$ in each case and so $sat_5(n, K_{3,3}) \geq 3n-9$. ∎

This completes the proof of Theorem 1.3.

4 Conclusion

Based on above results, we make the following conjecture, which proposes an exact value for $sat(n, K_{3,3})$.

Conjecture 4.1 *For $n \geq 9$, $sat(n, K_{3,3}) = 3n-9$.*

By Theorem 1.2, $sat(n, K_{3,3}) \leq 3n-9$ for $n \geq 9$. To confirm Conjecture 4.1, it suffices to prove $sat(n, K_{3,3}) \geq 3n-9$ for $n \geq 9$. Let G be a $K_{3,3}$-saturated graph with n vertices and $n \geq 9$. Proposition 3.1(i) implies $\delta(G) \geq 2$. If $\delta(G) \geq 6$, then $e(G) \geq 3n \geq 3n-9$. Thus we only need to consider $2 \leq \delta(G) \leq 5$. We have proved $sat_\delta(n, K_{3,3}) \geq 3n-9$ when $\delta \in \{2,5\}$. Actually, for $\delta \in \{3,4\}$, we can also apply the method in this paper, but it is more complex and there are quite a few cases to consider.

Acknowledgments. Huang was partially supported by the National Natural Science Foundation of China (No. 12171256). Lei was partially supported by the National Natural Science Foundation of China (No. 12371351). Shi and Zhang were partially supported by the National Natural Science Foundation of China (No. 12161141006), the Natural Science Foundation of Tianjin (No. 20JCJQJC00090).

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