# Rainbow structures in a collection of graphs with degree conditions

Luyi Li[1], Ping Li[2], Xueliang Li[1]

[1]Center for Combinatorics and LPMC

Nankai University, Tianjin 300071, China

[2]School of Mathematics and Statistics

Shaanxi Normal University, Xi'an, Shaanxi 710062, China

Emails: liluyi@mail.nankai.edu.cn, lp-math@snnu.edu.cn, lxl@nankai.edu.cn

**Abstract**

Let $\mathbf{G} = \{G_1, \ldots, G_s\}$ be a collection of not necessarily distinct $n$-vertex graphs with the same vertex set $V$. We use $\widetilde{\mathbf{G}}$ to denote an edge-colored multigraph of $\mathbf{G}$ with $V(\widetilde{\mathbf{G}}) = V$ and $E(\widetilde{\mathbf{G}})$ a multiset consisting of $E(G_1), \ldots, E(G_s)$, and the edge $e$ of $\widetilde{\mathbf{G}}$ is colored by $i$ if $e \in E(G_i)$. A graph $H$ is *rainbow* in $\mathbf{G}$ if any two edges of $H$ belong to different graphs of $\mathbf{G}$. We say that $\mathbf{G}$ is *rainbow vertex-pancyclic* if each vertex of $V$ is contained in a rainbow cycle of $\mathbf{G}$ with length $\ell$ for every integer $\ell \in [3, n]$, and that $\mathbf{G}$ is *rainbow panconnected* if for any pair of vertices $u$ and $v$ of $V$ there exists a rainbow path of $\mathbf{G}$ with length $\ell$ joining $u$ and $v$ for every integer $\ell \in [d_{\widetilde{\mathbf{G}}}(u, v), n-1]$. In this paper, we study the existences of rainbow spanning trees and rainbow Hamiltonian paths in $\mathbf{G}$ under the Ore-type conditions. Moreover, we study the rainbow vertex-pancyclicity and rainbow panconnectedness, as well as the existence of rainbow cliques in $\mathbf{G}$ under the Dirac-type conditions. We also give some examples to show the sharpness of our results.

**Keywords:** rainbow; Hamiltonian path; vertex-pancyclicity; panconnectedness

**AMS subject classification 2020:** 05C15, 05C38, 05C07.

## 1 Introduction

First, we claim that all terminology and notation on graph theory not defined in this paper are the same as those in the textbook [6]. For a positive integer $n$ we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ of integers, and for two positive integers $d < n$ we use $[d, n]$ to denote the set $\{d, d+1, \ldots, n\}$ of integers.

Hamiltonicity of graphs is a classic subject in graph theory which has been researched extensively. In 1952, Dirac [13] first gave a sufficient condition on the existence of Hamiltonian cycles in a graph using the minimum degree condition of the graph, that is, every $n$-vertex graph ($n$-graph, for short) $G$ with $\delta(G) \geq n/2$ contains a Hamiltonian cycle. Later on, Ore [19] relaxed the condition and proved that every $n$-graph $G$ with $\sigma_2(G) \geq n$ contains a Hamiltonian cycle, where $\sigma_2(G)$ is the minimum degree-sum among all pairs of nonadjacent vertices of $G$. Conditions in the above two results are usually called Dirac-type condition and Ore-type condition, respectively. In 2020, Joos and Kim [17] proved a result which can be seen as a generalization of Dirac's theorem.

**Theorem 1.1.** [17] *Suppose $\mathbf{G} = \{G_i : i \in [n]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set $V$, and $\delta(G_i) \geq \frac{n}{2}$ for $i \in [n]$. Then there exists a Hamiltonian cycle on the vertex set $V$ with edge set $\{e_1, \ldots, e_n\}$ such that $e_i \in E(G_i)$ for $i \in [n]$.*

In fact, the Hamiltonian cycle in Theorem 1.1 is a special transversal of $\mathbf{G} = \{G_i : i \in [n]\}$. In general, for a collection $\mathbf{G} = \{G_i : i \in [t]\}$ of not necessarily distinct graphs with common vertex set $V$, a simple graph $H$ is a *partial transversal* of $\mathbf{G}$ if $V(H) \subseteq V$, $|E(H)| \leq t$ and there exists an injection $\theta : E(H) \to [t]$ such that $e \in E(G_{\theta(e)})$ for every $e \in E(H)$. In particular, $H$ is a *(total) transversal* of $\mathbf{G}$ if $H$ is a partial transversal of $\mathbf{G}$ with $|E(H)| = t$.

From another perspective, we can view $\mathbf{G}$ as an edge-colored multigraph $\widetilde{\mathbf{G}}$ with $V(\widetilde{\mathbf{G}}) = V$ and $E(\widetilde{\mathbf{G}})$ a multiset consisting of $E(G_1), \ldots, E(G_t)$, and an edge $e$ of $\widetilde{\mathbf{G}}$ is colored by $i$ if $e \in E(G_i)$. Therefore, $H$ is a partial $\mathbf{G}$-transversal if and only if $H$ is a rainbow subgraph of $\widetilde{\mathbf{G}}$. In this way we also say that $H$ is a rainbow subgraph of $\mathbf{G}$. Then Theorem 1.1 can be restated as follows: If $\mathbf{G} = \{G_i : i \in [n]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set and $\delta(G_i) \geq \frac{n}{2}$ for $i \in [n]$, then there exists a rainbow Hamiltonian cycle in $\widetilde{\mathbf{G}}$. Note that the bound of the minimum degree in Theorem 1.1 agrees with that in the Dirac's theorem.

Another classic result in extremal graph theory is the Mantel's theorem, which says that an $n$-graph $G$ contains a triangle if $|E(G)| > \frac{n^2}{4}$. Aharoni, DeVos, de la Maza, Montejano and Šámal in [3] proved a rainbow version of the Mantel's theorem: A collection $\mathbf{G} = \{G_1, G_2, G_3\}$ of $n$-graphs with $|E(G_i)| > \frac{1+\tau^2}{4}n^2$ for all $1 \leq i \leq 3$ contains a rainbow triangle, where $\tau = \frac{4-\sqrt{7}}{9}$. They also proved that there is a collection $\mathbf{G} = \{G_1, G_2, G_3\}$ of $n$-graphs that does not have a rainbow triangle but satisfies that $|E(G_i)| > (\frac{1+\tau^2}{4} - \varepsilon)n^2$ for each $1 \leq i \leq 3$ and $\varepsilon > 0$ when $n$ is sufficiently large. So, $\tau^2$ cannot be replaced by a smaller constant, which means that the two bounds on the numbers of edges in Mantel's theorem and the rainbow version of Mantel's theorem are different.

Motivated by the above results, we want to explore whether other classical results can be extended in a rainbow version. In fact, there are some scholars who started to investigate on this topic and produced some beautiful results. In 2021, Cheng, Wang and Zhao in [11] considered the rainbow pancyclicity and the existence of rainbow Hamiltonian paths. Later on, Cheng, Han, Wang and Wang in [10] proved an asymptotical result of the rainbow version of Hajnal-Szemerédi theorem by using probabilistic method. Recently, Montgomery, Müyesser and Pehova in [18] gave some asymptotically-tight minimum degree conditions for a collection $\mathbf{G}$ of $n$-graphs to have a rainbow $F$-factor or a rainbow tree with maximum degree $o(\frac{n}{\log n})$. For more results about this topic, please see [1, 2, 4, 7, 8, 14, 21] for examples.

In this paper, we continue to study the rainbow versions of some extremal results. Before stating our results, let us introduce some notation and preliminaries. For a vertex $v$ of a graph $G$ and a subgraph $H$ of $G$, we use $N_G(v, H)$ to denote the set of neighbours of $v$ in $H$. Given a vertex partition $\{V_1, \ldots, V_k\}$ of $G$, we use $E_G[V_1, \ldots, V_k]$ to denote the set of edges of $G$ whose endpoints are in different $V_i$. For any two vertex-disjoint graphs $H_1$ and and $H_2$, we use $H_1 \vee H_2$ to denote a graph obtained by adding an edge between each vertex of $H_1$ and each vertex of $H_2$, and $H_1 \cup H_2$ to denote the union of $H_1$ and $H_2$. Let $d_G(u, v)$ denote the distance of $u$ and $v$ in $G$. If $G$ has exactly $k$ components and each component of $G$ is isomorphic to $H$, then we denote $G$ by $kH$. The number of components in $G$ is denoted by $c(G)$. We say that $\mathbf{G} = \{G_i : i \in [t]\}$ consists of $t$ *copies* of $G$ if $G_1 = G_2 = \ldots = G_t = G$.

Inspired by Theorem 1.1, it is natural to consider rainbow spanning structures in $\mathbf{G}$ under Ore-type condition. The following two theorems focus on the existence of rainbow spanning forests and Hamiltonian paths, respectively.

**Theorem 1.2.** *Suppose $\mathbf{G} = \{G_i : i \in [n-1]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set $V$, and $\sigma_2(G_i) \geq \frac{2n}{3} - 2$ for $i \in [n-1]$. Then, one of the following statements holds:*

*(1) there exists a rainbow spanning forest in $\mathbf{G}$ with $n - c(\widetilde{\mathbf{G}})$ edges;*

*(2) by renumbering $[n-1]$, we have that $3|n$, $\widetilde{\mathbf{G}}$ is connected and $\{G_i : i \in [2, n-1]\}$ consists of $n-2$ copies of $3K_{\frac{n}{3}}$.*

**Theorem 1.3.** *Suppose $\mathbf{G} = \{G_i : i \in [n]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set $V$, and $\sigma_2(G_i) \geq n-2$ for $i \in [n]$. Then, one of the following statements holds:*

*(1) $\mathbf{G}$ has a rainbow Hamiltonian path;*

*(2) $\mathbf{G}$ consists of $n$ copies of $K_\ell \cup K_{n-\ell}$, where $\ell \in [n-1]$;*

*(3) $n$ is even and there is a partition $(H, I)$ of $V$ with $|H| = \frac{n-2}{2}$ and $|I| = \frac{n+2}{2}$. For each $i \in [n]$, $G_i = G_i[H] \vee G_i[I]$, where $G_i[I]$ is an independent set and $G_i[H]$ is an arbitrary graph.*

Next, we consider the existence of rainbow cliques under the minimum degree conditions.

**Theorem 1.4.** *Suppose* $\mathbf{G} = \{G_i : i \in [\binom{s}{2}]\}$ *is a collection of not necessarily distinct* $n$*-graphs with the same vertex set* $V$*. If* $\delta(G_i) \geq (1 - \frac{1}{s-1})n$ *for* $i \in [\binom{s}{2} - 1]$ *and* $\delta(G_{\binom{s}{2}}) > (1 - \frac{1}{s-1})n$*, then* $\mathbf{G}$ *has a rainbow clique* $K_s$*, and the bound on the minimum degrees is sharp.*

A cycle or a path on $\ell$ vertices is called an $\ell$-cycle or an $\ell$-path, respectively. An $n$-graph $G$ is called *vertex pancyclic* if each vertex of $G$ is contained in an $\ell$-cycle for all $\ell \in [3, n]$. Similarly, a collection $\mathbf{G} = \{G_1, \ldots, G_n\}$ of $n$-graphs is called *rainbow vertex-pancyclic* if each vertex of $\mathbf{G}$ is contained in a rainbow $\ell$-cycle for all $\ell \in [3, n]$. In 1990, Hendry [16] proved that every $n$-graph with $\delta(G) \geq \frac{n+1}{2}$ is vertex pancyclic. We obtain a rainbow version of this result as follows.

**Theorem 1.5.** *Suppose* $\mathbf{G} = \{G_i : i \in [n]\}$ *is a collection of not necessarily distinct* $n$*-graphs with the same vertex set* $V$*, and* $\delta(G_i) \geq \frac{n+1}{2}$ *for all* $i \in [n]$*. Then* $\mathbf{G}$ *is rainbow vertex-pancyclic.*

We say that a collection $\mathbf{G}$ of $n$-graphs is *rainbow panconnected* if for any two vertices $u$ and $v$ of $\mathbf{G}$, there exists a rainbow $\ell$-path of $\mathbf{G}$ joining $u$ and $v$ for every integer $\ell \in [d_{\widetilde{\mathbf{G}}}(u, v) + 1, n]$. Now we define a special collection of graphs as follows.

**Definition 1.** *Let* $n$ *be an odd integer and* $H_1$ *be an empty graph with* $\frac{n-1}{2}$ *vertices, and let* $H_2$ *be a graph on* $\frac{n+1}{2}$ *vertices with* $\delta(H_2) \geq 1$ *such that one component of* $H_2$ *is a single edge* $ww'$*. Define* $\mathbf{F}_n$ *as a collection of* $n$ *copies of* $H_1 \vee H_2$*.*

It is easy to verify that $H_1 \vee H_2$ is not panconnected, since $d_{H_1 \vee H_2}(w, w') = 1$ but there is no 4-path joining $w$ and $w'$. Hence, $\mathbf{F}_n$ is not rainbow panconnected. However, it is worth noticing that $\mathbf{F}_n$ has rainbow $\ell$-paths joining $w$ and $w'$ for $\ell \in [3, n] - \{4\}$. We further discuss the rainbow panconnectedness of a collection of graphs.

**Theorem 1.6.** *Suppose* $\mathbf{G} = \{G_i : i \in [n]\}$ *is a collection of not necessarily distinct* $n$*-graphs with the same vertex set* $V$*, and* $\delta(G_i) \geq \frac{n+1}{2}$ *for each* $i \in [n]$*. Then either* $\mathbf{G}$ *is rainbow panconnected or* $\mathbf{G} = \mathbf{F}_n$*.*

The rest of this paper is organized as follows. In Section 2, we prove the existences of rainbow spanning trees (Theorem 1.2) and rainbow Hamiltonian paths (Theorem 1.3) in $\mathbf{G}$ under Ore-type conditions, and then give characterizations of the extremal graphs. At the end of this section, we show the existence of rainbow cliques (Theorem 1.4) in $\mathbf{G}$ under minimum degree conditions. In Section 3, we prove that $\mathbf{G}$ is rainbow vertex-pancyclic (Theorem 1.5) and rainbow panconnected (Theorem 1.6) under minimum degree conditions. In Section 4, we discuss some related problems for further study.

## 2 Proofs of Theorems 1.2, 1.3 and 1.4

In a collection $\mathbf{G} = \{G_i : i \in [t]\}$ of graphs on the vertex set $V$, any two vertices $u, v \in V$ may be adjacent in different graphs of $\mathbf{G}$. In order to distinguish these parallel edges, we use $C(uv) = i$ to denote that the edge $uv$ comes from $G_i$. If $F$ is a rainbow subgraph of $\mathbf{G}$, we use $C(F)$ to denote the set of colors appearing on $F$. For a tree $T$ and an edge $xy \in E(T)$, we use $T_x$ to denote the component of $T - xy$ containing $x$. Now we begin to our proofs.

**Proof of Theorem 1.2:** Since $\sigma_2(G_i) \geq \frac{2n}{3} - 2$, each $G_i$ has at most three components. If $G_i$ has exactly three components, say $D_1, D_2$ and $D_3$, then each $|V(D_i)|$ is either $\lfloor \frac{n}{3} \rfloor$ or $\lceil \frac{n}{3} \rceil$. Otherwise, suppose $|V(D_1)| < \lfloor \frac{n}{3} \rfloor$. Then $|V(D_2)| + |V(D_3)| > n - \lfloor \frac{n}{3} \rfloor \geq \frac{2n}{3}$. Since $\sigma_2(G_i) \geq \frac{2n}{3} - 2$, it follows that $|V(D_1)| + |V(D_2)| \geq \frac{2n}{3}$ and $|V(D_1)| + |V(D_3)| \geq \frac{2n}{3}$. Thus, $|V(D_1)| + |V(D_2)| + |V(D_3)| > n$, a contradiction. Now we consider the following two cases:

**Case 1.** $\widetilde{\mathbf{G}}$ is connected.

Assume that $T$ is a maximum rainbow tree in $\mathbf{G}$ and $U = V - V(T)$. If $U = \emptyset$, then the statement (1) holds. Otherwise, we will prove that the statement (2) holds below. Assume that $|V(T)| = k < n$ and $C(T) = [k-1]$. It follows from the maximality of $T$ that $E_{G_i}[V(T), U] = \emptyset$ for each $i \in [k, n-1]$. Thus, without loss of generality, suppose that $e \in E_{G_1}[V(T), U]$ and $xy$ is an edge of $T$ with $C(xy) = 1$. We first prove the following claim.

**Claim 1.** *$N_{G_i}(v) \subseteq V(T_x)$ for each vertex $v \in V(T_x)$ and $N_{G_i}(v') \subseteq V(T_y)$ for each vertex $v' \in V(T_y)$, where $i \in [2, n-1]$.*

*Proof.* By symmetry, we only need to prove that $N_{G_i}(v) \subseteq V(T_x)$ for each vertex $v \in V(T_x)$ and $i \in [2, n-1]$. We first show that the result holds for $i \in [k, n-1]$. Fix an integer $i \in [k, n-1]$. The maximality of $T$ implies that $N_{G_i}(v) \cap U = \emptyset$. We assert that $E_{G_i}[V(T_x), V(T_y)] = \emptyset$. Otherwise, there is an edge $f \in E_{G_i}[V(T_x), V(T_y)]$, and we can observe that $T - xy + e + f$ is a rainbow $(k+1)$-tree, a contradiction. Therefore, we have $N_{G_i}(v) \subseteq V(T_x)$.

We prove below that the result holds for $i \in [2, k-1]$. Fix an integer $i \in [2, k-1]$. There is an edge $f' = zz'$ of $T$ with $C(f') = i$. Without loss of generality, suppose $f' \in E(T_y)$. Let $F_z, F_{z'}$ be two components of $T_y - f'$. Since $N_{G_k}(v) \subseteq V(T_x)$ for each vertex $v \in V(T_x)$, it follows that $E(G_k) \cap E_{G_k}[V(T_x), V(T_y)] = \emptyset$. If $E(G_k) \cap E_{G_k}[V(F_z), V(F_{z'})] = \emptyset$, then $E(G_k) \cap E_{G_k}[V(T_x), V(F_z), V(F_{z'})] = \emptyset$, which implies that $G_k[V(T)]$ has at least three components. Recall that $U \neq \emptyset$. Then $G_k$ has at least four components, a contradiction. Hence, there is an edge $g$ of $G_k$ such that $T_y \cup g$ has a unique cycle containing $f'$ and $g$. If $N_{G_i}(v) \cap U \neq \emptyset$, say $w \in N_{G_i}(v) \cap U$, then $T - f' + g + vw$ is a rainbow $(k+1)$-tree, a contradiction. Note that $T' = T - f' + g$ is a rainbow tree with $V(T) = V(T')$, $T'_x = T_x$ and $T'_y = T_y - f' + g$. If $N_{G_i}(v) \cap V(T_y) \neq \emptyset$, say $w' \in N_{G_i}(v) \cap V(T_y)$, then $T' + vw'$

contains a unique cycle $C'$ such that $xy, vw' \in E(C')$. Since $e \in E_{G_1}[V(T), U]$, it follows that $T' + e + vw' - xy$ is a rainbow $(k+1)$-tree, a contradiction. $\square$

For $i \in [2, n-1]$, it follows from Claim 1 that $G_i$ has at least three components. Then $G_i$ has exactly three components $G_i[V_1]$, $G_i[V_2]$ and $G_i[V_3]$. We can see that $|V_j| \in \{\lfloor \frac{n}{3} \rfloor, \lceil \frac{n}{3} \rceil\}$ for $j \in [3]$. It is easy to verify that $3|n$ and $G_i[V_1] = G_i[V_2] = G_i[V_3] = K_{\frac{n}{3}}$ since $\sigma_2(G_j) \geq \frac{2n}{3} - 2$. Then the statement (2) holds.

**Case 2.** $\widetilde{\mathbf{G}}$ is disconnected.

Suppose $D$ is a component of $\widetilde{\mathbf{G}}$. Let $G'_i = G_i[D]$ and $G''_i = G_i - D$ for $i \in [n-1]$, and let $\mathbf{G}' = \{G'_i : i \in [n-1]\}$ and $\mathbf{G}'' = \{G''_i : i \in [n-1]\}$. Note that $\sigma_2(G'_i) \geq \frac{2n}{3} - 2 > \frac{2|V(D)|}{3} - 2$ and $\sigma_2(G''_i) \geq \frac{2n}{3} - 2 > \frac{2(|V| - |V(D)|)}{3} - 2$ for $i \in [n-1]$. By induction, $\mathbf{G}'$ contains a rainbow spanning tree $T'$ with $C(T') = [|V(D)| - 1]$ and $\mathbf{G}''$ contains a rainbow forest $F''$ with $e(F'') = n - |V(D)| - c(\widetilde{\mathbf{G}}'')$ and $C(F'') \subseteq \{|V(D)|, \ldots, n-1\}$. Thus, $T = T' \cup F''$ is a rainbow spanning forest of $\mathbf{G}$ with $e(T) = n - c(\widetilde{\mathbf{G}})$. Then the statement (1) holds.

The proof is now complete. $\square$

If we add an extra condition "$\widetilde{\mathbf{G}}$ is connected" in Theorem 1.2, then either $\mathbf{G}$ contains a rainbow spanning tree or the statement (2) holds. However, we can observe that $\sigma_2(G_i) = \frac{2n}{3} - 2$ for $i \in [2, n-1]$ when the statement (2) holds. Hence, the following result can be deduced from Theorem 1.2 immediately.

**Corollary 2.1.** *Suppose $\mathbf{G} = \{G_i : i \in [n-1]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set such that $\widetilde{\mathbf{G}}$ is connected and $\sigma_2(G_i) \geq \frac{2n}{3} - 1$ for $i \in [n-1]$. Then there is a rainbow spanning tree in $\mathbf{G}$.*

Before we proceed our discussion, let us introduce some additional notation. Let $C = v_1 v_2 \ldots v_k v_1$ be a cycle and $v_{k+1} = v_1$. For any two integers $i, j \in [k]$, we use $v_i \overrightarrow{C} v_j$ to denote the path $v_i v_{i+1} \ldots v_j$ and $v_i \overleftarrow{C} v_j$ to denote the path $v_i v_{i-1} \ldots v_j$. Similarly, we can define $v_i \overrightarrow{P} v_j$ and $v_j \overleftarrow{P} v_i$ respect to a path $P = v_1 v_2 \ldots v_k$.

**Proof of Theorem 1.3:** The proof proceeds by contradiction. Let $P = v_1 v_2 \ldots v_\ell$ be an arbitrary maximum rainbow path of $\mathbf{G}$ and $C(v_i v_{i+1}) = i$ for all $i \in [\ell - 1]$. Then $\ell \leq n - 1$. For each color $i \in [\ell, n]$ and each vertex $u \in V \setminus V(P)$, the maximality of $P$ implies that $v_1 u \notin E(G_i)$ and $v_\ell u \notin E(G_i)$. Then $N_{G_i}(v_1), N_{G_i}(v_\ell) \subseteq V(P)$ for all $i \in [\ell, n]$.

**Claim 2.** *There is no rainbow $\ell$-cycle in $\mathbf{G}$.*

*Proof.* Suppose to the contrary that there is a rainbow cycle $C = u_1 u_2 \ldots u_\ell u_1$ in $\mathbf{G}$ and $C(u_i u_{i+1}) = i$ for all $i \in [\ell]$, where $u_{\ell+1} = u_1$. If there is a color $k \in [\ell + 1, n]$ and a vertex $u \in V \setminus V(C)$ such that $N_{G_k}(u, C) \neq \emptyset$, choosing an arbitrary vertex $u_i \in N_{G_k}(u, C)$, then $uu_i\overrightarrow{C}u_{i-1}$ is a rainbow $(\ell+1)$-path, a contradiction. Hence, $N_{G_k}(u,C)=\emptyset$ for each color $k\in[\ell+1,n]$ and each vertex $u\in V\setminus V(C)$. Since $\sigma_2(G_k)\geq n-2$ for each color $k\in[\ell+1,n]$, it follows that $G_k[V(C)]\cong K_\ell$ and $G_k[V\setminus V(C)]\cong K_{n-\ell}$. Hence, the collection $\{G_i : i\in[\ell+1,n]\}$ of graphs consists of $n-\ell$ copies of $K_\ell\cup K_{n-\ell}$.

Fix an integer $k\in[\ell]$. If $N_{G_k}(u,C)\neq\emptyset$, then there is a vertex $u_i\in N_{G_k}(u,C)$. Recall that $e=u_ku_{k+1}$ is an edge of $C$ with $C(e)=k$. Since $G_{\ell+1}[V(C)]$ is a complete subgraph, there is an edge $f=u_ku_{k+1}$ in $G_{\ell+1}$. If $i\neq k$, then $C-e-u_iu_{i+1}+f+u_iu$ is a rainbow $(\ell+1)$-path with $C(u_iu_{i+1})=i$ and $C(u_iu)=k$, a contradiction. If $i=k$, then $uu_i\overleftarrow{C}u_{i+1}$ is a rainbow $(\ell+1)$-path with $C(u_iu)=k$, a contradiction. Thus, $N_{G_k}(u,C)=\emptyset$ for each color $k\in[\ell]$ and each vertex $u\in V\setminus V(C)$. By a similar discussion, we can deduce that $G_i[V(C)]\cong K_\ell$ and $G_i[V\setminus V(C)]\cong K_{n-\ell}$ for each integer $i\in[\ell]$, which implies that the collection $\{G_i : i\in[\ell]\}$ of graphs consists of $\ell$ copies of $K_\ell\cup K_{n-\ell}$.

In conclusion, the collection $\{G_i : i\in[n]\}$ of graphs consists of $n$ copies of $K_\ell\cup K_{n-\ell}$. Then the statement (2) holds, a contradiction. The claim thus follows. □

Choosing two colors $s,t\in[\ell,n]$, it follows from Claim 2 that $v_1v_\ell\notin E(G_s)$ and $v_1v_\ell\notin E(G_t)$. Then $d_{G_s}(v_1)+d_{G_s}(v_\ell)\geq n-2$ and $d_{G_t}(v_1)+d_{G_t}(v_\ell)\geq n-2$, which implies that either $d_{G_s}(v_1)+d_{G_t}(v_\ell)\geq n-2$ or $d_{G_s}(v_\ell)+d_{G_t}(v_1)\geq n-2$. Without loss of generality, set $d_{G_s}(v_1)+d_{G_t}(v_\ell)\geq n-2$ and define

$$A_0=\{i\in[\ell-2] : v_1v_{i+1}\in E(G_s)\} \text{ and } B_0=\{i\in[2,\ell-1] : v_iv_\ell\in E(G_t)\}.$$

Since $N_{G_s}(v_1), N_{G_t}(v_\ell)\subseteq V(P)-\{v_1,v_\ell\}$, we have $|A_0|+|B_0|\geq n-2$.

We assert that $A_0\cap B_0=\emptyset$. Otherwise, choosing an integer $i\in A_0\cap B_0$, we get that $v_1v_{i+1}\overrightarrow{P}v_\ell v_i\overleftarrow{P}v_1$ is a rainbow $\ell$-cycle with $C(v_1v_{i+1})=s$ and $C(v_\ell v_i)=t$, a contradiction. It is easy to verify that $A_0\cup B_0=[n-2]$, $d_{G_s}(v_1)+d_{G_t}(v_\ell)=n-2$, $\ell=n-1$ and $\{s,t\}=\{n-1,n\}$. Without loss of generality, set

$$d_{G_{n-1}}(v_1)+d_{G_n}(v_{n-1})=n-2. \tag{1}$$

Since $v_nv_1\notin E(G_{n-1})\cup E(G_n)$, we have $d_{G_{n-1}}(v_1)+d_{G_{n-1}}(v_n)\geq n-2$ and $d_{G_n}(v_1)+d_{G_n}(v_n)\geq n-2$. Then one of the following two statements holds:

(*a*) $d_{G_{n-1}}(v_1)+d_{G_n}(v_n)\geq n-2$;

(*b*) $d_{G_{n-1}}(v_n)+d_{G_n}(v_1)\geq n-2$.

Since $v_nv_{n-1}\notin E(G_{n-1})\cup E(G_n)$, similarly, one of the following two statements holds:

(*c*) $d_{G_{n-1}}(v_n)+d_{G_n}(v_{n-1})\geq n-2$;

(*d*) $d_{G_{n-1}}(v_{n-1})+d_{G_n}(v_n)\geq n-2$.

By symmetry, we only consider the case that $d_{G_{n-1}}(v_1)+d_{G_n}(v_n)\geq n-2$ and $d_{G_{n-1}}(v_n)+$

$d_{G_n}(v_{n-1}) \geq n-2$. Define the following four sets:

$$\begin{aligned} A_1 &= \{i \in [n-3] : v_1 v_{i+1} \in E(G_{n-1})\}; \\ B_1 &= \{i \in [2, n-2] : v_i v_n \in E(G_n)\}; \\ A_2 &= \{i \in [3, n-1] : v_{n-1} v_{i-1} \in E(G_n)\}; \\ B_2 &= \{i \in [2, n-2] : v_i v_n \in E(G_{n-1})\}. \end{aligned}$$

If $A_1 \cap B_1 \neq \emptyset$, say $i \in A_1 \cap B_1$, then $v_n v_i \overleftarrow{P} v_1 v_{i+1} \overrightarrow{P} v_{n-1}$ is a rainbow Hamiltonian path with $C(v_n v_i) = n$ and $C(v_1 v_{i+1}) = n-1$, a contradiction. Thus, $A_1 \cap B_1 = \emptyset$. Note that $N_{G_{n-1}}(v_1) \subseteq V(P) - \{v_1, v_{n-1}\}$ and $N_{G_n}(v_n) \subseteq V(P) - \{v_1, v_{n-1}\}$. Hence,

$$d_{G_{n-1}}(v_1) + d_{G_n}(v_n) = n-2. \tag{2}$$

By a similar discussion for $A_2$ and $B_2$, we have

$$d_{G_{n-1}}(v_n) + d_{G_n}(v_{n-1}) = n-2. \tag{3}$$

From Equations (1), (2) and (3), we have $d_{G_n}(v_{n-1}) = d_{G_n}(v_n)$ and $d_{G_{n-1}}(v_1) = d_{G_{n-1}}(v_n)$. Since $v_1 v_n \notin E(G_{n-1})$ and $v_{n-1} v_n \notin E(G_n)$, it follows that $d_{G_{n-1}}(v_1) + d_{G_{n-1}}(v_n) \geq n-2$ and $d_{G_n}(v_{n-1}) + d_{G_n}(v_n) \geq n-2$. Hence,

$$d_{G_{n-1}}(v_1) = d_{G_{n-1}}(v_n) = d_{G_n}(v_{n-1}) = d_{G_n}(v_n) = \frac{n-2}{2}.$$

Similarly, since $v_n v_{n-1} \notin E(G_{n-1})$ and $v_1 v_n \notin E(G_n)$, we can also deduce that

$$d_{G_{n-1}}(v_{n-1}) = d_{G_{n-1}}(v_n) = d_{G_n}(v_1) = d_{G_n}(v_n) = \frac{n-2}{2}.$$

Then $d_{G_i}(v_j) = \frac{n-2}{2}$ for all $i \in \{n-1, n\}$ and $j \in \{1, n-1, n\}$. In addition, $n$ is even.

**Claim 3.** *$N_{G_i}(v_j) = \{v_{2k} : \ k \in [\frac{n-2}{2}]\}$ for all $i \in \{n-1, n\}$ and $j \in \{1, n-1, n\}$.*

*Proof.* Define $A = \{i \in [n-3] : v_{i+1} v_n \in E(G_n)\}$ and $B = \{i \in [2, n-2] : v_i v_n \in E(G_{n-1})\}$. Then $A \cap B = \emptyset$; otherwise, choosing an integer $i \in A \cap B$, we can find a rainbow Hamiltonian path $v_1 \overrightarrow{P} v_i v_n v_{i+1} \overrightarrow{P} v_{n-1}$ with $C(v_i v_n) = n-1$ and $C(v_n v_{i+1}) = n$, a contradiction. Since $d_{G_{n-1}}(v_n) = d_{G_n}(v_n) = \frac{n-2}{2}$, we have $|A| = |B| = \frac{n-2}{2}$. It is easy to verify that $B$ is the set of even integers in $[2, n-2]$ and $A$ is the set of odd integers in $[n-3]$. Then $N_{G_i}(v_n) = \{v_{2k} : \ k \in [\frac{n-2}{2}]\}$ for $i \in \{n-1, n\}$.

By symmetry, we only need to prove below that $N_{G_n}(v_1) = \{v_{2k} : \ k \in [\frac{n-2}{2}]\}$. Suppose to the contrary that $v_1 v_{2k+1}$ is an edge of $G_n$ for some $k \in [\frac{n-2}{2} - 1]$. Then $v_n v_{2k} \overleftarrow{P} v_1 v_{2k+1} \overrightarrow{P} v_{n-1}$ is a rainbow Hamiltonian path with $C(v_{2k} v_n) = n-1$ and $C(v_1 v_{2k+1}) = n$, a contradiction. Thus, $N_{G_n}(v_1) \subseteq \{v_{2k} : \ k \in [\frac{n-2}{2}]\}$. Combining $d_{G_n}(v_1) = \frac{n-2}{2}$, we get that $N_{G_n}(v_1) = \{v_{2k} : \ k \in [\frac{n-2}{2}]\}$. $\square$

For convenience, for a path $R = x_1x_2\dots x_{2k+1}$, we use $V_e(R)$ to denote the vertex set $\{x_{2i} : i \in [k]\}$. In fact, since $P$ is an arbitrary maximum rainbow path, it follows from Claim 3 that for a rainbow $(n-1)$-path $L$ with endpoints $x, y$ and $V - V(L) = \{z\}$,

$$N_{G_i}(x) = N_{G_i}(y) = N_{G_i}(z) = V_e(L)$$

for each $i \in [n] - C(L)$.

**Claim 4.** *Suppose $Q = x_1x_2\dots x_{n-1}$ is a rainbow $(n-1)$-path in $\mathbf{G}$. Then $N_{G_i}(x_1) = N_{G_i}(x_{n-1}) = V_e(Q)$ for each $i \in [n]$.*

*Proof.* Assume that $V - V(Q) = \{x_n\}$ and $C(x_ix_{i+1}) = i$ for $i \in [n-2]$. From Claim 3, we know that $N_{G_{n-1}}(x_n) = N_{G_n}(x_n) = V_e(Q)$. For $i \in \{3, 5, \dots, n-3\}$, let $L_i = x_i\overleftarrow{Q}x_2x_nx_{i+1}\overrightarrow{Q}x_{n-1}$ be a rainbow $(n-1)$-path with $C(x_2x_n) = n-1$ and $C(x_nx_{i+1}) = n$, and let $L_i' = x_i\overrightarrow{Q}x_{n-2}x_nx_{i-1}\overleftarrow{Q}x_1$ be a rainbow $(n-1)$-path with $C(x_nx_{n-2}) = n-1$ and $C(x_nx_{i-1}) = n$. Since $i, 1 \notin C(L_i)$ and $i-1, n-2 \notin C(L_i')$, by Claim 3 again, we have that $N_{G_i}(x_1) = N_{G_1}(x_1) = V_e(L_i) = \{x_2, x_4, \dots, x_{n-2}\}$ and $N_{G_{i-1}}(x_1) = N_{G_{n-2}}(x_1) = V_e(L_i') = \{x_2, x_4, \dots, x_{n-2}\}$. Note that $N_{G_{n-1}}(x_1) = N_{G_n}(x_1) = \{x_2, x_4, \dots, x_{n-2}\}$. Therefore, $N_{G_i}(x_1) = V_e(Q) = \{x_2, x_4, \dots, x_{n-2}\}$ for each $i \in [n]$. By symmetry, $N_{G_i}(x_{n-1}) = V_e(Q)$ for each $i \in [n]$. The claim thus follows. $\square$

Recall that $P = v_1v_2\dots v_{n-1}$ is a rainbow $(n-1)$-path and $C(v_iv_{i+1}) = i$ for $i \in [n-2]$. Then $N_{G_i}(v_1) = N_{G_i}(v_{n-1}) = V_e(P) = \{v_2, v_4, \dots, v_{n-2}\}$ for $i \in [n]$ by Claim 4. Note that $v_nv_2\overrightarrow{P}v_{n-1}$ is a rainbow $(n-1)$-path with $C(v_2v_n) = n$ from Claim 3. Then $N_{G_i}(v_n) = \{v_2, v_4, \dots, v_{n-2}\}$ for $i \in [n]$. For $j \in \{3, 5, \dots, n-3\}$, note that $L_j = v_j\overleftarrow{P}v_2v_nv_{j+1}\overrightarrow{P}v_{n-1}$ is a rainbow $(n-1)$-path with $C(v_2v_n) = n-1$ and $C(v_nv_{j+1}) = n$. Since $v_j$ is an endpoint of $L_j$ and $V_e(L_j) = V_e(P)$, it follows from Claim 4 that $N_{G_i}(v_j) = V_e(L_j) = \{v_2, v_4, \dots, v_{n-2}\}$ for $i \in [n]$. In conclusion, we have that $N_{G_i}(x) = \{v_2, v_4, \dots, v_{n-2}\}$ for $i \in [n]$ and $x \in \{v_1, v_3, \dots, v_{n-1}, v_n\}$. Moreover, $\{v_1, v_3, \dots, v_{n-1}, v_n\}$ is an independent set in each $G_i$. Set $I = \{v_1, v_3, \dots, v_{n-1}, v_n\}$ and $H = V - I$. It is easy to verify that $\sigma_2(G_i) \geq n-2$ and there is no rainbow Hamiltonian path in $\mathbf{G}$ when $G_i[H]$ is an arbitrary graph for $i \in [n]$. Then the statement (3) holds, a contradiction. The result thus follows. $\square$

Note that if $\mathbf{G}$ contains no rainbow Hamiltonian path in Theorem 1.3, then one of the statements (2) and (3) holds, which implies that $\sigma_2(G_i) = n-2$ for $i \in [n]$. Then we can obtain a sufficient condition on the existence of a rainbow Hamiltonian path in a collection $\mathbf{G} = \{G_i : i \in [n]\}$ of $n$-graphs.

**Corollary 2.2.** *Suppose $\mathbf{G} = \{G_i : i \in [n]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set, and $\sigma_2(G_i) \geq n-1$ for $i \in [n]$. Then there exists a rainbow Hamiltonian path in $\mathbf{G}$.*

The following result can be deduced from Corollary 2.2 directly.

**Theorem 2.3.** [11] *Suppose* $\mathbf{G} = \{G_i : i \in [n]\}$ *is a collection of not necessarily distinct* $n$*-graphs with the same vertex set, and* $\delta(G_i) \geq \frac{n-1}{2}$ *for* $i \in [n]$*. Then there exists a rainbow Hamiltonian path in* $\mathbf{G}$*.*

Now we prove the last result of this section.

**Proof of Theorem 1.4:** For $2 \leq j \leq s$, let $\mathbf{H}_j = \{G_i : \ i \in [\binom{j}{2}]\}$. The proof proceeds by induction on $s$. It is obvious that $\mathbf{H}_2$ contains a rainbow $K_2$. Since the minimum degree of each graph in $\mathbf{H}_{s-1}$ is larger than $(1-\frac{1}{s-2})n$, it follows that $\mathbf{H}_{s-1}$ contains a rainbow complete graph $R$ with $|V(R)| = s-1$. Without loss of generality, assume $V(R) = \{v_1, \cdots, v_{s-1}\}$. We prove that $\mathbf{H}_s$ contains a rainbow clique $K_s$ below. First, we construct an auxiliary digraph $D$ with $V(D) = V$ and

$$A(D) = \{(v_i, u) : \ v_i u \text{ is an edge of } G_{\binom{s-1}{2}+i}, \text{ where } i \in [s-1]\}.$$

Observe that $\mathbf{H}_s$ contains a rainbow clique $K_s$ if there is a vertex in $V - V(R)$ with in-degree at least $s-1$. Now we verify that the vertex exists. Note that

$$\sum_{v \in V(D)} d_D^-(v) = |A(D)| = \sum_{v \in V(D)} d_D^+(v) = \sum_{v \in V(R)} d_D^+(v) > (s-1)(1 - \frac{1}{s-1})n.$$

Then there is at least one vertex $w$ in $V(D)$ such that $d_D^-(w) \geq s-1$. Since $d^-(v) \leq s-2$ for each $v \in V(R)$, we have $w \in V - V(R)$. Hence, $\mathbf{H}_s$ contains a rainbow clique $K_s$.

Note that if $(k-1)|n$ and $\mathbf{G} = \{G_1, \cdots, G_{\binom{k}{2}}\}$ consists of $\binom{k}{2}$ copies of $T_{n,k-1}$, then $\delta(G_i) = (1 - \frac{1}{k-1})n$ for each $i \in [\binom{k}{2}]$ and $\mathbf{G}$ does not contain a rainbow clique $K_k$, where $T_{n,k-1}$ is a balanced complete $(k-1)$-partite graph. This implies that the bound of Theorem 1.4 is sharp. The result thus follows. $\square$

# 3 Proofs of Theorems 1.5 and 1.6

Given an edge-colored multigraph $G$, a subgraph $H$ of $G$ and a vertex $u \in V(G) - V(H)$, let

$$N^i(u, H) = \{v \in V(H) : \text{ there is an edge with color } i \text{ between } u \text{ and } v\}$$

and $d^i(u, H) = |N^i(u, H)|$. The following lemma is useful for our later proofs.

**Lemma 3.1.** *Let* $G$ *be an edge-colored multigraph and* $C = v_1 \dots v_p v_1$ *be a rainbow cycle in* $G$*. For each vertex* $y \in V(G) \setminus V(C)$ *and any two colors* $g, f$ *that are not used in* $C$*, if* $d^g(y, C) \geq \frac{p}{2}$ *and* $d^f(y, C) \geq \frac{p}{2}$*, then one of the following two statement holds:*

*(1) there is a rainbow* $(p+1)$*-cycle in* $G$*;*

*(2)* $p$ *is even and either* $N^g(y, C) = N^f(y, C) = \{v_1, v_3, \dots, v_{p-1}\}$ *or* $N^g(y, C) = N^f(y, C) = \{v_2, v_4, \dots, v_p\}$*.*

*Proof.* Suppose that there is no rainbow $(p+1)$-cycle in $G$. Define

$$A = \{i \in [p] : \text{ there is an edge } e \text{ between } y \text{ and } v_{i+1} \text{ such that } C(e) = g\}$$

and

$$B = \{i \in [p] : \text{ there is an edge } e \text{ between } y \text{ and } v_i \text{ such that } C(e) = f\}.$$

Since $d^g(y, C) \geq \frac{p}{2}$ and $d^f(y, C) \geq \frac{p}{2}$, we have $|A| \geq \frac{p}{2}$ and $|B| \geq \frac{p}{2}$. If $A \cap B \neq \emptyset$, choosing an integer $i \in A \cap B$, then $v_i y v_{i+1} \overrightarrow{C} v_i$ is a rainbow $(p+1)$-cycle, a contradiction.

If $A \cap B = \emptyset$, then $A \cup B = [p]$, which implies that $p$ is even and $d^g(y, C) = d^f(y, C) = \frac{p}{2}$. We assert that $v_i$ and $v_j$ are not adjacent in $C$ for any two integers $i, j \in A$. Otherwise, choosing a maximal subset $\{i, i+1, \ldots, i+k\} \subseteq A$ such that $i-1, i+k+1 \in B$, we get that $v_{i+k} y v_{i+k+1} \overrightarrow{C} v_{i+k}$ is a rainbow $(p+1)$-cycle, a contradiction. Hence, we have that either $A = \{1, 3, \ldots, p-1\}$ or $A = \{2, 4, \ldots, p\}$. If $A = \{1, 3, \ldots, p-1\}$, then $B = \{2, 4, \ldots, p\}$, which implies that $N^g(y, C) = N^f(y, C) = \{v_2, v_4, \ldots, v_p\}$. If $A = \{2, 4, \ldots, p\}$, then $B = \{1, 3, \ldots, p-1\}$, which implies that $N^g(y, C) = N^f(y, C) = \{v_1, v_3, \ldots, v_{p-1}\}$. The claim thus follows. □

In [17], the authors proved that $\mathbf{G} = \{G_i : i \in [n]\}$ contains a rainbow Hamiltonian cycle if the minimum degree of each $G_i$ is at least $\frac{n}{2}$ (see Theorem 1.1). The following result focuses on the existence of rainbow $(n-1)$-cycles in $\mathbf{G}$.

**Lemma 3.2.** *Suppose $\mathbf{G} = \{G_i : i \in [n]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set $V$, and $\delta(G_i) \geq \frac{n}{2}$ for all $i \in [n]$. Then, one of the following statements holds:*

*(1) there is a rainbow $(n-1)$-cycle in $\mathbf{G}$;*

*(2) $n$ is even and $\mathbf{G}$ consists of $n$ copies of $K_{\frac{n}{2},\frac{n}{2}}$.*

*Proof.* Suppose that there is no a rainbow $(n-1)$-cycle in $\mathbf{G}$. First, we prove the following claim.

**Claim 5.** *There is a rainbow $(n-2)$-cycle in $\mathbf{G}$.*

*Proof.* From Corollary 2.2, assume that $P = v_1 v_2 \ldots v_{n-2}$ is a rainbow $(n-2)$-path of $\mathbf{G}$ with $v_i v_{i+1} \in E(G_i)$ for each $i \in [n-3]$ and define

$$A = \{i \in [n-4] : v_1 v_{i+1} \in E(G_n)\} \text{ and } B = \{i \in [2, n-3] : v_{n-2} v_i \in E(G_{n-1})\}.$$

If $v_1 v_{n-2} \in E(G_n)$ or $v_1 v_{n-2} \in E(G_{n-1})$, then there is a rainbow $(n-2)$-cycle in $\mathbf{G}$, the result thus follows. Next, we suppose $v_1 v_{n-2} \notin E(G_n)$ and $v_1 v_{n-2} \notin E(G_{n-1})$. Since there is no rainbow $(n-1)$-cycle in $\mathbf{G}$, we have that

$$|E_{G_{n-1}}(\{v_{n-2}\}, \{v_{n-1}, v_n\}) \cup E_{G_n}(\{v_1\}, \{v_{n-1}, v_n\})| \leq 2.$$

Then, $|A|+|B|\geq n-2$. It follows from $A\cup B\subseteq[n-3]$ that $A\cap B\neq\emptyset$. Choosing an integer $i\in A\cap B$, we can deduce that $v_1v_{i+1}\overrightarrow{P}v_{n-2}v_i\overleftarrow{P}v_1$ is a rainbow $(n-2)$-cycle of $\mathbf{G}$. $\square$

Assume that $C=v_1v_2\dots v_{n-2}v_1$ is a rainbow $(n-2)$-cycle of $\mathbf{G}$ and $v_iv_{i+1}\in E(G_i)$ for all $i\in[n-2]$, where $v_{n-1}=v_1$. Set $V\setminus V(C)=\{x,y\}$ and define

$$A=\{i\in[n-2]: xv_{i+1}\in E(G_n)\}\text{ and }B=\{i\in[n-2]: xv_i\in E(G_{n-1})\}.$$

Note that $|N_{G_n}(x,C)|\geq\frac{n}{2}-1$ and $|N_{G_{n-1}}(x,C)|\geq\frac{n}{2}-1$. From Lemma 3.1 and the assumption that there is no rainbow $(n-1)$-cycle in $\mathbf{G}$, we know that $n$ is even and either $N_{G_n}(x)=N_{G_{n-1}}(x)=\{v_1,v_3,\dots,v_{n-3},y\}$ or $N_{G_n}(x)=N_{G_{n-1}}(x)=\{v_2,v_4,\dots,v_{n-2},y\}$. Similarly, we have either $N_{G_n}(y)=N_{G_{n-1}}(y)=\{v_1,v_3,\dots,v_{n-3},x\}$ or $N_{G_n}(y)=N_{G_{n-1}}(y)=\{v_2,v_4,\dots,v_{n-2},x\}$.

**Claim 6.** $N_{G_n}(x)\cup N_{G_n}(y)=V$.

*Proof.* Otherwise, by symmetry, assume that $N_{G_n}(x)=N_{G_{n-1}}(x)=\{v_1,v_3,\dots,v_{n-3},y\}$ and $N_{G_n}(y)=N_{G_{n-1}}(y)=\{v_1,v_3,\dots,v_{n-3},x\}$. If $v_2x\in E(G_1)$, then $v_1xv_2\overrightarrow{C}v_1$ is a rainbow $(n-1)$-cycle of $\mathbf{G}$ with $C(xv_1)=n$ and $C(xv_2)=1$, a contradiction. So, $v_2x\notin E(G_1)$. Similarly, we have $v_2y\notin E(G_1)$, which implies $N_{G_1}(v_2)\subseteq V(C)$. Since $d_{G_1}(v_2)\geq\frac{n}{2}$ and $|\{v_1,v_3,\dots,v_{n-3}\}|=\frac{n-2}{2}$, we know that there is an even integer $i\in[n-2]$ such that $v_iv_2\in E(G_1)$. Hence, $v_iv_2\overrightarrow{C}v_{i-1}xv_1\overleftarrow{C}v_i$ is a rainbow $(n-1)$-cycle of $\mathbf{G}$ with $C(v_iv_2)=1$, $C(v_{i-1}x)=n-1$ and $C(xv_1)=n$, a contradiction. The claim thus follows. $\square$

Without loss of generality, assume that $N_{G_n}(x)=N_{G_{n-1}}(x)=\{v_1,v_3,\dots,v_{n-3},y\}$ and $N_{G_n}(y)=N_{G_{n-1}}(y)=\{v_2,v_4,\dots,v_{n-2},x\}$. Choosing an arbitrary vertex $v_i\in\{v_2,v_4,\dots,v_{n-2}\}$, we get that $v_{i-1}xv_{i+1}\overrightarrow{C}v_{i-1}$ is a rainbow $(n-2)$-cycle of $\mathbf{G}$ with $C(v_{i-1}x)=n-1$ and $C(xv_{i+1})=n$. Note that this cycle contains no edges of $G_{i-1}$ or $G_i$. By a similar argument, we have $N_{G_i}(v_i)=N_{G_{i-1}}(v_i)=\{v_1,v_3,\dots,v_{n-3},y\}$, and so $N_{G_i}(y)=N_{G_{i-1}}(y)=\{v_2,v_4,\dots,v_{n-2},x\}$. Consequently, we have $N_{G_i}(y)=\{v_2,v_4,\dots,v_{n-2},x\}$ for all $i\in[n]$. By symmetry, we have $N_{G_i}(x)=\{v_1,v_3,\dots,v_{n-3},y\}$ for all $i\in[n]$.

For an odd integer $a$ and an even integer $b$ of $[n-2]$, $v_{a-1}yv_{a+1}\overrightarrow{C}v_{a-1}$ and $v_{b-1}xv_{b+1}\overrightarrow{C}v_{b-1}$ are rainbow $(n-2)$-cycles with $C(v_{a-1}y)=a-1$, $C(yv_{a+1})=a$, $C(v_{b-1}x)=b-1$ and $C(xv_{b+1})=b$. We can regard $v_a$ as $y$ and $v_b$ as $x$ in the above discussion, and hence the following two statements hold:

(1) $N_{G_j}(v_i)=\{v_1,v_3,\dots,v_{n-3},y\}$ for any two integers $i\in\{2,4,\dots,n-2\}$ and $j\in[n]$;

(2) $N_{G_j}(v_i)=\{v_2,v_4,\dots,v_{n-2},x\}$ for any two integers $i\in\{1,3,\dots,n-3\}$ and $j\in[n]$.

We can easily see that $G_i$ is a complete bipartite graph with bipartition $\{v_1,v_3,\dots,v_{n-3},y\}$ and $\{v_2,v_4,\dots,v_{n-2},x\}$ for all $i\in[n]$. The proof is now complete. $\square$

Now, we start to prove the rainbow vertex-pancyclicity of a collection $\mathbf{G}$ of graphs.

**Proof of Theorem 1.5:** From Theorem 1.1, each vertex is contained in a rainbow Hamiltonian cycle of $\mathbf{G}$. Choose an arbitrary vertex $x \in V$, and set $H_i = G_i \setminus \{x\}$ for each $i \in [n-1]$ and $\mathbf{H} = \{H_1, H_2, \ldots, H_{n-1}\}$. Then $|V(H_i)| = n-1$ and $\delta(H_i) \geq \frac{n-1}{2}$ for each $i \in [n-1]$. Using Theorem 1.1 again, we know that there is a rainbow $(n-1)$-cycle in $\mathbf{H}$. This also implies that each vertex of $\mathbf{G}$ is contained in a rainbow $(n-1)$-cycle.

From Lemma 3.2, there is a rainbow $(n-2)$-cycle in $\mathbf{H}$, or $n-1$ is even and $\mathbf{H}$ consists of $n-1$ copies of $K_{\frac{n-1}{2},\frac{n-1}{2}}$. If the latter holds, then $H_i$ is an $\frac{n-1}{2}$-regular graph for each $i \in [n-1]$. Recall that $\delta(G_i) \geq \frac{n}{2}$ for each $i \in [n-1]$. Then $G_i = H_i \vee \{x\}$ for each $i \in [n-1]$. Hence, we can easily verify that each vertex of $\mathbf{G}$ is contained in a rainbow $\ell$-cycle for each integer $\ell \in [3, n-1]$. The result thus follows. If the former holds, then assume that $C = v_1v_2\ldots v_{n-2}v_1$ is a rainbow $(n-2)$-cycle of $\mathbf{H}$ with $v_iv_{i+1} \in E(G_i)$ for all $i \in [n-2]$ and $v_{n-1} = v_1$. Next, we prove that $x$ is contained in a rainbow $a$-cycle for each integer $a \in [3, \ldots, n-2]$. Define

$$A = \{i \in [n-2] : v_i \in N_{G_n}(x)\} \text{ and } B = \{i \in [n-2] : v_{i+a-2} \in N_{G_{n-1}}(x)\}.$$

Since $d_{G_n}(x) \geq \frac{n+1}{2}$ and $d_{G_{n-1}}(x) \geq \frac{n+1}{2}$, we have $|A| \geq \frac{n-1}{2}$ and $|B| \geq \frac{n-1}{2}$. Note that $A \cup B \subseteq [n-2]$ and $|A| + |B| \geq n-1$. Then $A \cap B \neq \emptyset$. Choosing an integer $i \in A \cap B$, we can see that $xv_i\overrightarrow{C}v_{i+a-2}x$ is a rainbow $a$-cycle with $C(xv_i) = n$ and $C(v_{i+a-2}x) = n-1$. Consequently, by the arbitrariness of $x$, each vertex of $\mathbf{G}$ is contained in a rainbow $\ell$-cycle for each integer $\ell \in [3, n-1]$. The result thus follows. $\square$

In order to prove the panconnectedness of $\mathbf{G}$, we first give the characterization of the existence of a rainbow $(n-1)$-cycle or $n$-cycle in $\mathbf{G}$.

**Lemma 3.3.** *Suppose $\mathbf{G} = \{G_i : i \in [n]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set $V$, and $\delta(G_i) \geq \frac{n-1}{2}$ for all $i \in [n]$. Then, one of the following statements holds:*

*(1) $\mathbf{G}$ has a rainbow cycle of length at least $n-1$;*

*(2) $n$ is odd and $\mathbf{G}$ consists of $n$ copies of $K_1 \vee (2K_{\frac{n-1}{2}})$.*

*Proof.* If $n$ is even, then by Theorem 1.1, $\mathbf{G}$ has a rainbow Hamiltonian cycle. Thus, we only need to prove the result when $n$ is an odd integer. Suppose that the statement (1) does not hold. We prove below that the statement (2) holds.

**Claim 7.** *Either $\mathbf{G}$ has a rainbow $(n-2)$-cycle, or $\mathbf{G}$ consists of $n$ copies of $K_1 \vee (2K_{\frac{n-1}{2}})$.*

*Proof.* From Theorem 2.3, there is a rainbow Hamiltonian path $x_1x_2\ldots x_n$ in $\mathbf{G}$, and set $x_ix_{i+1} \in E(G_i)$ for $i \in [n-1]$. Let $P = x_1x_2\ldots x_{n-1}$, $U = \{x_1, x_2, \ldots, x_{\frac{n+1}{2}}\}$ and $W =$

$\{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \dots, x_n\}$. Define

$$A_1 = \{i \in [n-3] : x_1x_{i+1} \in E(G_n)\} \text{ and } B_1 = \{i \in [2, n-2] : x_{n-1}x_i \in E(G_{n-1})\}.$$

The assumption that $\mathbf{G}$ has no rainbow cycle of length at least $n-1$ implies that $x_1x_n \notin G_n$, $x_1x_{n-1} \notin G_n$ and $x_1x_{n-1} \notin G_{n-1}$. Hence, $N_{G_n}(x_1) \subseteq V(P) - \{x_{n-1}\}$ and $N_{G_{n-1}}(x_{n-1}) \subseteq V(P) \cup \{x_n\} - \{x_1\}$. Moreover, $|A_1| \geq \frac{n-1}{2}$ and $|B_1| \geq \frac{n-3}{2}$.

If $A_1 \cap B_1 \neq \emptyset$, then we can find a rainbow $(n-1)$-cycle in $\mathbf{G}$, a contradiction. If $A_1 \cap B_1 = \emptyset$, then $A_1 \cup B_1 = [n-2]$, which means that $|A_1| = \frac{n-1}{2}$ and $|B_1| = \frac{n-3}{2}$. If there is an integer $i$ such that $i \in B_1$ and $i+1 \in A_1$, then $x_1x_{i+2}\overrightarrow{P}x_{n-1}x_i\overleftarrow{P}x_1$ is a rainbow $(n-2)$-cycle of $\mathbf{G}$ with $C(x_1x_{i+2}) = n$ and $C(x_{n-1}x_i) = n-1$, the claim thus follows. Otherwise, $A_1 = \{1, 2, \dots, \frac{n-1}{2}\}$ and $B_1 = \{\frac{n+1}{2}, \frac{n+3}{2}, \dots, n-2\}$. From the definitions of $A_1$ and $B_1$, we have

$$N_{G_n}(x_1) = U - \{x_1\} \text{ and } N_{G_{n-1}}(x_{n-1}) = W - \{x_{n-1}\}. \tag{4}$$

By symmetry, considering the rainbow $(n-1)$-path $x_2x_3 \dots x_n$, we can conclude that $N_{G_n}(x_n) = \{x_{\frac{n-1}{2}}, \dots, x_{n-1}\}$ and $N_{G_1}(x_2) = \{x_1, x_3, x_4, \dots, x_{\frac{n-1}{2}}\}$. Then $x_{n-1}x_n \in E(G_n)$.

Now let us continue to consider the rainbow path $P = x_1x_2 \dots x_{n-1}$. Define

$$A_2 = \{i \in [n-3] : x_1x_{i+1} \in E(G_{n-1})\} \text{ and } B_2 = \{i \in [2, n-2] : x_{n-1}x_i \in E(G_n)\}.$$

If $x_1x_n \in E(G_{n-1})$, then $x_1\overrightarrow{P}x_{n-1}x_nx_1$ is a rainbow Hamiltonian cycle of $\mathbf{G}$ with $C(x_1x_n) = n-1$ and $C(x_{n-1}x_n) = n$, a contradiction. Then we have $x_1x_n \notin E(G_{n-1})$. Recall that $x_nx_{n-1} \in E(G_n)$. By a similar discussion, we can get that

$$N_{G_{n-1}}(x_1) = U - \{x_1\} \text{ and } N_{G_n}(x_{n-1}) = W - \{x_{n-1}\}. \tag{5}$$

Combining (4) and (5), we have $N_{G_n}(x_1) = N_{G_{n-1}}(x_1) = U - \{x_1\}$ and $N_{G_{n-1}}(x_{n-1}) = N_{G_n}(x_{n-1}) = W - \{x_{n-1}\}$.

For each integer $i \in \{2, 3, \dots, \frac{n-1}{2}\}$, consider the rainbow $(n-1)$-path $x_i\overleftarrow{P}x_1x_{i+1}\overrightarrow{P}x_{n-1}$ with $C(x_1x_{i+1}) = n-1$ (resp. $C(x_1x_{i+1}) = n$). One can deduce that $d_{G_n}(x_i) = \frac{n-1}{2}$ and $N_{G_n}(x_i) = U - \{x_i\}$ (resp. $d_{G_{n-1}}(x_i) = \frac{n-1}{2}$ and $N_{G_{n-1}}(x_i) = U - \{x_i\}$). Similarly, one can also deduce that $d_{G_n}(x_j) = \frac{n-1}{2}$ and $N_{G_n}(x_j) = W - \{x_j\}$ (resp. $d_{G_{n-1}}(x_j) = \frac{n-1}{2}$ and $N_{G_{n-1}}(x_j) = W - \{x_j\}$) for $j \in \{\frac{n+3}{2}, \dots, n-1\}$. In conclusion, we know that $G_n[U] = G_{n-1}[U] = K_{\frac{n+1}{2}}$ and $G_n[W] = G_{n-1}[W] = K_{\frac{n+1}{2}}$. Hence, $G_n = G_{n-1} = K_1 \vee (2K_{\frac{n-1}{2}})$. By symmetry of $P$ and $v_2v_3 \cdots v_n$, we can obtain that $G_1 = G_n = K_1 \vee (2K_{\frac{n-1}{2}})$.

For $i \in \{2, 3, \dots, \frac{n-1}{2}\}$ and $j \in \{\frac{n+1}{2}, \dots, n-2\}$, $P' = x_i\overleftarrow{P}x_1x_{i+1}\overrightarrow{P}x_jx_{n-1}\overleftarrow{P}x_{j+1}$ is a rainbow $(n-1)$-path of $\mathbf{G}$ with $C(x_1x_{i+1}) = n$ and $C(x_jx_{n-1}) = n-1$. Then $C(P') = [n] - \{i, j\}$. By a similar analysis, one can show that $G_i[U] = G_j[U] = K_{\frac{n+1}{2}}$ and $G_i[W] = G_j[W] = K_{\frac{n+1}{2}}$, which implies that $G_i = G_j = K_1 \vee (2K_{\frac{n-1}{2}})$. The claim thus follows. $\square$

By Claim 7 and the hypothesis that the statement (1) does not hold, the maximum rainbow cycle in $\mathbf{G}$ is an $(n-2)$-cycle. In order to complete the proof, suppose $C = x_1x_2\dots x_{n-2}x_1$ is a rainbow $(n-2)$-cycle of $\mathbf{G}$ with $x_ix_{i+1} \in E(G_i)$ for all $i \in [n-2]$, where $x_{n-1} = x_1$. Set $V \setminus V(C) = \{y, z\}$ and define

$$A_1 = \{i \in [n-2] : yx_{i+1} \in E(G_n)\} \text{ and } B_1 = \{i \in [n-2] : yx_i \in E(G_{n-1})\}.$$

The assumption that there is no rainbow $(n-1)$-cycle in $\mathbf{G}$ implies that $A_1 \cap B_1 = \emptyset$. Then $A_1 \cup B_1 \subseteq [n-2]$ and $|A_1| + |B_1| \le n-2$. Recall that $n$ is odd and $\delta(G_i) \ge \frac{n-1}{2}$ for all $i \in [n]$. Then $\frac{n-3}{2} \le |A_1| \le \frac{n-1}{2}$ and $\frac{n-3}{2} \le |B_1| \le \frac{n-1}{2}$.

If $|A_1| = \frac{n-1}{2}$, then $|B_1| = \frac{n-3}{2}$ and $A_1 \cup B_1 = [n-2]$. Hence, there is a subset $\{i, i+1, \dots, i+k\} \subseteq A_1$ such that $k \ge 1$ and $i-1, i+k+1 \in B_1$, which means that $x_{i+k}yx_{i+k+1}\overrightarrow{C}x_{i+k}$ is a rainbow $(n-1)$-cycle of $\mathbf{G}$ with $C(x_{i+k}y) = n$ and $C(yx_{i+k+1}) = n-1$, a contradiction. By symmetry, we have $|A_1| = |B_1| = \frac{n-3}{2}$, which implies that $yz \in E(G_n) \cap E(G_{n-1})$. It follows from $A_1 \cap B_1 = \emptyset$ that $|A_1 \cup B_1| = n-3$. Let $V(C) - (N_{G_{n-1}}(y) \cup N_{G_n}(y)) = \{w\}$.

We assert that there are no two successive integers in $A_1$. Otherwise, without loss of generality, let $A_1' = \{1, 2, \dots, a\}$ be a subset of $A_1$ such that $|A_1'| \ge 2$ and $n-2, a+1 \notin A_1$. Since $x_ayx_{a+1}\overrightarrow{C}x_a$ is not a rainbow $(n-1)$-cycle, it follows that $a+1 \notin B_1$. Hence, $w = x_{a+1}$. By the uniqueness of $w$, we get that there is only one subset of order at least two in $A_1$. Hence, the elements of $A_1 - A_1'$ and those of $B_1$ appear alternately in $[a+2, n-2]$, which implies $|A_1 - A_1'| + 1 \ge |B_1|$. Recall $|A_1'| \ge 2$, we have $|A_1| \ge |B_1| + 1$, a contradiction.

Therefore, the elements of $A_1$ and those of $B_1$ appear alternately in $C - w$. Without loss of generality, let $w = x_{n-2}$. Since $n$ is odd, $A_1, B_1 \subseteq [n-3]$ and $|A_1| = |B_1| = \frac{n-3}{2}$, it follows that either $A_1 = \{1, 3, \dots, n-4\}$ or $A_1 = \{2, 4, \dots, n-3\}$. By symmetry, suppose $A_1 = \{2, 4, \dots, n-3\}$. Since $d_{G_i}(y) \ge \frac{n-1}{2}$ for each $i \in [n]$, it follows that $N_{G_n}(y) = N_{G_{n-1}}(y) = \{x_1, x_3, \dots, x_{n-4}, z\}$.

Let $C' = yx_3\overrightarrow{C}x_1y$ be a rainbow $(n-2)$-cycle of $\mathbf{G}$ with $C(yx_3) = n-1$ and $C(yx_1) = n$. Note that the colors 1 and 2 do not appear in $C'$. Similarly, we can also get that $x_2z \in E(G_1) \cap E(G_2)$. Recall that $N_{G_n}(y) = N_{G_{n-1}}(y) = \{x_1, x_3, \dots, x_{n-4}, z\}$. Then $x_1yzx_2\overrightarrow{C}x_1$ is a rainbow Hamiltonian cycle of $\mathbf{G}$ with $C(x_1y) = n-1$, $C(yz) = n$ and $C(zx_2) = 1$, a contradiction. $\square$

**Proof of Theorem 1.6:** Suppose $\mathbf{G} \ne \mathbf{F}_n$. Choose two vertices $x$ and $y$ in $V$ arbitrarily. We prove that there is a rainbow $k$-path in $\mathbf{G}$ between them for each $k \in [3, n]$. Assume that $H_i = G_i - \{x, y\}$ for all $i \in [n]$ and $\mathbf{H} = \{H_1, \dots, H_{n-2}\}$. Note that $\delta(H_i) \ge \frac{|V(H_i)|-1}{2}$ for each $i \in [n]$. Suppose that $C = v_1v_2\dots v_\ell v_1$ is a maximum rainbow cycle of $\mathbf{H}$. Clearly, $\ell \le n-2$. Without loss of generality, assume that $v_iv_{i+1} \in E(G_i)$ for all $i \in [\ell]$ and $v_{\ell+1} = v_1$.

From Lemma 3.3, we know that one of the following statements holds:

(1) $\ell \in \{n-3, n-2\}$;

(2) $n-2$ is odd and $\mathbf{H}$ consists of $n-2$ copies of $K_1 \vee (2K_{\frac{n-3}{2}})$.
Hence, we consider the following two cases:

**Case 1.** The statement (1) holds, i.e., $\ell = n-2$ or $\ell = n-3$.

Let $C = v_1 v_2 \dots v_\ell v_1$ be a rainbow $\ell$-cycle of $\mathbf{H}$ with $C(v_i v_{i+1}) = i$, where $v_{\ell+1} = v_1$. For each $k \in [3, n]$, we define

$$A_1 = \{i \in [\ell] : xv_{i+k-3} \in E(G_n)\} \text{ and } B_1 = \{i \in [\ell] : yv_i \in E(G_{n-1})\}.$$

If $\ell = n-2$, then $|A_1| \geq \frac{n-1}{2}$, $|B_1| \geq \frac{n-1}{2}$ and $|A_1 \cup B_1| \leq n-2$, and hence $A_1 \cap B_1 \neq \emptyset$. Choosing an integer $i \in A_1 \cap B_1$, we get that $yv_i \overrightarrow{C} v_{i+k-3}x$ is a rainbow $k$-path with $C(xv_{i+k-3}) = n$ and $C(yv_i) = n-1$, the result then follows. Thus, suppose that $\mathbf{H}$ has a rainbow $(n-3)$-cycle $C$ but has no rainbow $(n-2)$-cycles. For convenience, let $V \setminus V(C) = \{x, y, z\}$.

Since $|N_{G_n}(z, C)| \geq \frac{n-3}{2}$ and $|N_{G_{n-1}}(z, C)| \geq \frac{n-3}{2}$, by Lemma 3.1 we have that $n$ is odd, and either $N_{G_n}(z, C) = N_{G_{n-1}}(z, C) = \{v_1, v_3, \dots, v_{n-4}\}$ or $N_{G_n}(z, C) = N_{G_{n-1}}(z, C) = \{v_2, v_4, \dots, v_{n-3}\}$. It is obvious that $N_{G_n}(z) = N_{G_{n-1}}(z) = I_0 \cup \{x, y\}$, where $I_0$ is either $\{v_1, v_3, \dots, v_{n-4}\}$ or $\{v_2, v_4, \dots, v_{n-3}\}$. By symmetry, we have

$$N_{G_n}(z) = N_{G_{n-1}}(z) = N_{G_{n-2}}(z) = I_0 \cup \{x, y\}.$$

By the same discussion, we have that

$$N_{G_n}(x) = N_{G_{n-1}}(x) = N_{G_{n-2}}(x) = I_1 \cup \{z, y\}$$

and

$$N_{G_n}(y) = N_{G_{n-1}}(y) = N_{G_{n-2}}(y) = I_2 \cup \{x, z\},$$

where $I_j$ is either $\{v_1, v_3, \dots, v_{n-4}\}$ or $\{v_2, v_4, \dots, v_{n-3}\}$ for $j \in [2]$.

**Case 1.1.** $I_1 \neq I_2$.

Without loss of generality, suppose $I_0 = I_1 = \{v_1, v_3, \dots, v_{n-4}\}$ and $I_2 = \{v_2, v_4, \dots, v_{n-3}\}$. If $k$ is even, then $xv_1 \overrightarrow{C} v_{k-2}y$ is a rainbow $k$-path joining $x$ and $y$, where $C(xv_1) = n-1$ and $C(v_{k-2}y) = n$. If $k$ is odd, then $xzv_1 \overrightarrow{C} v_{k-3}y$ is a rainbow $k$-path joining $x$ and $y$, where $C(xz) = n-2$, $C(zv_1) = n-1$ and $C(v_{k-3}y) = n$.

**Case 1.2.** $I_1 = I_2 = I_0$.

Suppose $I_1 = \{v_1, v_3, \dots, v_{n-4}\}$. If $k$ is even, then $xzv_1 \overrightarrow{C} v_{k-3}y$ is a rainbow $k$-path joining $x$ and $y$, where $C(xz) = n-2$, $C(zv_1) = n-1$ and $C(v_{k-3}y) = n$. If $k$ is odd and $k \neq n$, then $xv_1 \overrightarrow{C} v_{k-2}y$ is a rainbow $k$-path joining $x$ and $y$, where $C(xv_1) = n-1$ and $C(v_{k-2}y) = n$. Now, we consider $n = k$. If $|N_{G_1}(v_2) \cap \{x, y, z\}| \geq 2$, then either $x \in N_{G_1}(v_2)$ or $y \in N_{G_1}(v_2)$, say $x \in N_{G_1}(v_2)$. We can find a rainbow Hamiltonian path $xv_2\overrightarrow{C}v_1zy$ of $\mathbf{G}$ with $C(xv_2) = 1$, $C(v_2z) = n-1$ and $C(yz) = n$. If $|N_{G_1}(v_2) \cap \{x, y, z\}| \leq 1$, then since $d_{G_1}(v_2) \geq \frac{n+1}{2}$, it follows that there is a vertex $v_{2j} \in N_{G_1}(v_2)$, where $j \in [\frac{n-3}{2}]$. Then $xv_1\overleftarrow{C}v_{2j}v_2\overrightarrow{C}v_{2j-1}zy$ is a rainbow Hamiltonian path of $\mathbf{G}$ with $C(xv_1) = n-2$, $C(v_{2j}v_2) = 1$, $C(v_{2j-1}z) = n-1$ and $C(zy) = n$. Thus, the result follows.

**Case 1.3.** $I_1 = I_2$ and $I_1 \neq I_0$.

Suppose $I_1 = I_2 = \{v_1, v_3, \ldots, v_{n-4}\}$ and $I_0 = \{v_2, v_4, \ldots, v_{n-3}\}$. If $k$ is odd and $k \neq n$, then $xv_1\overrightarrow{C}v_{k-2}y$ is a rainbow $k$-path joining $x$ and $y$, where $C(xv_1) = n-1$ and $C(v_{k-2}y) = n$. If $k = n$, then $yzv_2\overrightarrow{C}v_1x$ is a rainbow Hamiltonian path of $\mathbf{G}$ joining $x$ and $y$, where $C(yz) = n-2$, $C(zv_2) = n-1$ and $C(v_1x) = n$. Recall that $n$ is odd. We only need to consider that $k$ is even and $k \neq n$ below.

If $N_{G_1}(v_2) \cap \{x, y\} \neq \emptyset$, say $x \in N_{G_1}(v_2) \cap \{x, y\}$, then $xv_2\overrightarrow{C}v_{k-1}y$ is a rainbow $k$-path with $C(xv_2) = 1$ and $C(v_{k-1}y) = n$. If $N_{G_1}(v_2) \cap \{x, y\} = \emptyset$, then there is a vertex $v_{2j}$ of $C$ such that $v_{2j} \in N_{G_1}(v_2)$, where $j \in [\frac{n-3}{2}]$. If $k \geq 6$, then let $P_1 = v_2\overrightarrow{C}v_{2j}$ and $P_2 = v_{2j}\overrightarrow{C}v_2$. Thus, it is not difficult to show that there is a vertex $v_{2s+1}$ of $V(P_1)$ and a vertex $v_{2t+1}$ of $V(P_2)$ such that $xv_{2s+1}\overrightarrow{C}v_{2j}v_2\overleftarrow{C}v_{2t+1}y$ is a rainbow $k$-path with $C(xv_{2s+1}) = n-1$, $C(v_2v_{2j}) = 1$ and $C(yv_{2t+1}) = n$. Therefore, we only need to consider $k = 4$.

Let $U_1 = \{x, y, v_2, v_4, \ldots, v_{n-3}\}$ and $U_2 = \{z, v_1, v_3, \ldots, v_{n-4}\}$. Suppose to the contrary that there is no rainbow 4-path in $\mathbf{G}$ joining $x$ and $y$. Recall that $U_2 \subseteq N_{G_i}(x) \cap N_{G_i}(y)$ for $i \in \{n-2, n-1, n\}$. Choosing an integer $i \in [n]$ arbitrarily, it is obvious that $U_2$ is an independent set in $G_i$; for otherwise there is a rainbow 4-path joining $x$ and $y$, a contradiction. Since $d_{G_i}(a) \geq \frac{n+1}{2}$ for each $a \in U_2$, it follows that every vertex of $U_2$ is adjacent to every vertex of $U_1$ in $G_i$. Since $d_{G_i}(b) \geq \frac{n+1}{2}$ for each $b \in U_1$, it follows that $\delta(G_i[U_1]) \geq 1$. Since there is no rainbow 4-path in $\mathbf{G}$ joining $x$ and $y$, $xy$ is a component of $G_i[U_1]$. Otherwise, there is an edge $yy'$ (resp. $xx'$) of $G_i[U_1]$ such that $y' \neq x$ (resp. $x' \neq y$). Choose two integers $\{p, q\} = \{n-2, n-1, n\} - \{i\}$. Then $yy'zx$ (resp. $xx'zy$) is a rainbow 4-path of $\mathbf{G}$ with $C(yy') = i$, $C(y'z) = p$ and $C(zx) = q$ (resp. $C(xx') = i$, $C(x'z) = p$ and $C(zy) = q$), a contradiction. Therefore, $G_i = G_i[U_1] \vee G_i[U_2]$ for each $i \in [n]$. According to the above discussion, we have $\mathbf{G} = \mathbf{F}_n$, which contradicts the hypothesis that $\mathbf{G} \neq \mathbf{F}_n$.

**Case 2.** The statement (2) holds.

Set $\mathbf{H}' = \{H_3, H_4, \ldots, H_n\}$. By symmetry of $\mathbf{H}$ and $\mathbf{H}'$, we know that $\mathbf{H}$ consists of $n-2$ copies of $K_1 \vee (2K_{\frac{n-3}{2}})$. It follows that $\{H_1, H_2, \ldots, H_n\}$ consists of $n$ copies of $K_1 \vee (2K_{\frac{n-3}{2}})$. Hence, there are two subsets $U, W$ of $V - \{x, y\}$ such that both $H_i[U]$ and $H_i[W]$ are complete graphs of order $\frac{n-1}{2}$ for $i \in [n]$, and $U \cap W = \{z\}$. For each vertex $u \in U \cup W - \{z\}$ and each $i \in [n]$, we have $|N_{G_i}(u) \cap (U \cup W)| = \frac{n-3}{2}$. Since $d_{G_i}(u) \geq \frac{n+1}{2}$, it follows that $ux, uy \in E(G_i)$ for $i \in [n]$. Thus, either $G_i = H_i \vee \{x, y\}$ or $G_i = H_i \vee xy$. It is easy to verify that there is a rainbow $s$-path in $\mathbf{G}$ joining $x$ and $y$ for each integer $s \in [3, n]$. The proof is now complete. $\square$

A collection $\mathbf{G}$ of graphs on the same vertex set $V$ is called *rainbow Hamiltonian connected* if for any two vertices of $V$, $\mathbf{G}$ has a rainbow Hamiltonian path joining them. From Theorem 1.6, we can deduce the following result.

**Corollary 3.1.** *Suppose $\mathbf{G} = \{G_i : i \in [n]\}$ is a collection of not necessarily distinct $n$-graphs with the same vertex set, and $\delta(G_i) \geq \frac{n+1}{2}$ for all $i \in [n]$. Then $\mathbf{G}$ is rainbow Hamiltonian connected.*

Corollary 3.1 is best possible, since the collection $\mathbf{G}$ of graphs consisting of $n$ copies of $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is not rainbow Hamiltonian connected.

# 4 Concluding remarks

Motivated by the rainbow version of Dirac's theorem (Theorem 1.1), it is natural to generalize the Ore's theorem into rainbow version, that is, to find a rainbow Hamiltonian cycle in a collection $\mathbf{G}$ of graphs under the Ore-type condition. Corollary 2.2 says that a rainbow Hamiltonian path can be found in $\mathbf{G}$ under Ore-type condition, which can be seen as a weak generalization of the Ore's theorem. More generally, there are many other sufficient conditions to guarantee the Hamiltonicity of a graph, such as Pósa's condition [20], Bondy's condition [5], Chvátal's condition [12] and Fan's condition [15]. It is also interesting to obtain the rainbow versions of the results under these conditions.

Theorem 1.3 provides a sufficient condition for the existence of a rainbow Hamiltonian path in $\mathbf{G} = \{G_i : i \in [n]\}$. However, the edges of a rainbow Hamiltonian path only come from $n-1$ graphs in the collection $\mathbf{G}$ of graphs. It is worth studying whether $n-1$ graphs are sufficient to guarantee the existence of a rainbow Hamiltonian path in a collection $\mathbf{G} = \{G_i : i \in [n-1]\}$ of graphs.

We have considered rainbow panconnectedness of the collection $\mathbf{G}$ of graphs in Theorem 1.6. Another problem is the rainbow connectedness of a collection $\mathbf{G}$ of graphs. Let $\mathbf{G} = \{G_i : i \in [t]\}$ be a collection of $n$-graphs with the same vertex set such that each $G_i$ has a property $\mathcal{P}$. What is the smallest positive integer $t_{\mathcal{P}}$ such that $\mathbf{G}$ is rainbow connected when $t = t_{\mathcal{P}}$? Bradshaw and Mohar in [9] studied the threshold functions for a collection of random graphs to be rainbow connected.

# 5 Acknowledgment

The authors are very grateful to the anonymous referees for their insightful and valuable comments. The work was supported by the National Natural Science Foundation of China (Nos. 12131013, 11871034 and 12201375).

# References

[1] R. Aharoni, E. Berger, Rainbow matchings in $r$-partite $r$-graphs, Electron. J. Combin. **16** (2009).

[2] R. Aharoni, E. Berger, M. Chudnovsky, D. Howard, P. Seymour, Large rainbow matchings in general graphs, Europ. J. Combin. **79** (2019), 222–227.

[3] R. Aharoni, M. DeVos, S. González Hermosillo de la Maza, A. Montejano, R. Šámal, A rainbow version of Mantel's theorem, Adv. Comb. **2** (2020), 12 pp.

[4] R. Aharoni, D. Howard, A rainbow $r$-partite version of the Erdös-Ko-Rado theorem, Combin. Probab. Comput. **26** (2017), 321–337.

[5] J. A. Bondy, Large cycles in graphs, Discrete Math. **1** (1971), 121–132.

[6] J. A. Bondy, U.S.R. Murty, Graph Theory, GTM No.244, Springer, Berlin, 2008.

[7] P. Bradshaw, Transversals and bipancyclicity in bipartite graph families, Electron. J. Combin. **28** (2021).

[8] P. Bradshaw, K. Halasz, L. Stacho, From one to many rainbow Hamiltonian cycles, arXiv:2104.07020.

[9] P. Bradshaw, B. Mohar, A rainbow connectivity threshold for random graph families, arXiv:2107.05670.

[10] Y. Cheng, J. Han, B. Wang, G. Wang, Rainbow spanning structures in graph and hypergraph system, arXiv:2105.10219.

[11] Y. Cheng, G. Wang, Y. Zhao, Rainbow pancyclicity in graph systems, Electron. J. Combin. **28** (2021).

[12] V. Chvátal. On Hamilton's ideals, J. Comb. Theory, Ser. B **12** (1972), 163–168.

[13] G. A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. **2** (1952), 69–81.

[14] A. A. Drisko, Transversals in row-Latin rectangles, J. Comb. Theory, Ser. A **84** (1998), 181–195.

[15] G. Fan, New sufficient conditions for cycles in graphs, J. Comb. Theory, Ser. B **37** (1984), 221–227.

[16] G. R. T. Hendry, Extending cycles in graphs, Discrete Math. **85** (1990), 59–72.

[17] F. Joos, J. Kim, On a rainbow version of Dirac's theorem, Bull. London Math. Soc. **52** (2020), 498–504.

[18] R. Montgomery, A. Müyesser, Y. Pehova, Transversal factors and spanning trees, Adv. Comb. **3** (2020), 25 pp.

[19] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly **67** (1960), 55.

[20] L. Pósa, A theorem concerning Hamilton lines, Magyar Tud. Akad. Mat. Kutato Int. Közl. **7** (1962), 225–226.

[21] I. M. Wanless, Transversals in Latin squares, Quasigroups and Related Systems **15** (2007), 169–190.