GENERALIZATIONS OF MOCK THETA FUNCTIONS AND RADIAL LIMITS

SU-PING CUI, NANCY S. S. GU, AND CHEN-YANG SU

Abstract. In the last letter to Hardy, Ramanujan introduced seventeen functions defined by q-series convergent for $|q| < 1$ with a complex variable q, and called these functions "mock theta functions". Subsequently, mock theta functions were widely studied in the literature. In the survey of B. Gordon and R. J. McIntosh, they showed that the odd (resp. even) order mock theta functions are related to the function $g_3(x, q)$ (resp. $g_2(x, q)$). These two functions are usually called "universal mock theta functions". In 2014, D. R. Hickerson and E. T. Mortenson expressed all the classical mock theta functions and the two universal mock theta functions in terms of Appell–Lerch sums. In this paper, based on some q-series identities, we find four functions, and express them in terms of Appell–Lerch sums. For example,

$$1 + (xq^{-1} - x^{-1}q) \sum_{n=0}^{\infty} \frac{(-1; q)_{2n} q^n}{(xq^{-1}, x^{-1}q; q^2)_{n+1}} = 2m(x, q^2, q).$$

Then we establish some identities related to these functions and the universal mock theta function $g_2(x, q)$. These relations imply that all the classical mock theta functions can be expressed in terms of these four functions. Furthermore, by means of q-series identities and some properties of Appell–Lerch sums, we derive four radial limit results related to these functions.

1. Introduction

Throughout the paper, we use the standard q-series notation [13]. For positive integers n and m,

$$(a; q)_0 := 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k),$$

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

Define

$$j(x; q) := (x; q)_\infty (x^{-1}q; q)_\infty (q; q)_\infty,$$

$$J_{a,m} := j(q^a; q^m), \quad \overline{J}_{a,m} := j(-q^a; q^m), \quad J_m := J_{m,3m} = \prod_{i=1}^{\infty} (1 - q^{mi}).$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is definded as

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, x\right) := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} x^n.$$

2020 *Mathematics Subject Classification.* 11B65; 11F27

Date: December 15, 2022.

Key words and phrases. mock theta functions, universal mock theta functions, Appell–Lerch sums, radial limits.

Seventeen mock theta functions were presented by Ramanujan [32] in his last letter to Hardy. These functions were assigned orders 3, 5, and 7. In 1936, Watson [34, 35] proved some identities for the third and fifth order mock theta functions. In view of Bailey's lemma and Bailey pairs, Andrews [1] established Hecke-type double sums for the fifth and seventh order mock theta functions. For example, he showed that

$$f_0(q)=\frac{1}{J_1}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j\left(1-q^{4n+2}\right)q^{n(5n+1)/2-j^2},$$

$$f_1(q)=\frac{1}{J_1}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j\left(1-q^{2n+1}\right)q^{n(5n+3)/2-j^2},$$

where the fifth order mock theta functions $f_0(q)$ and $f_1(q)$ are defined as

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}\quad\text{and}\quad f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n}.$$

Then based on these Hecke-type double sums, Hickerson [16] proved the following two identities which are known as Mock Theta Conjectures:

$$f_0(q)=\frac{J_{5,10}J_{2,5}}{J_1}-2q^2g(q^2,q^{10}),$$

$$f_1(q)=\frac{J_{5,10}J_{4,5}}{J_1}-2q^3g(q^4,q^{10}),$$

where the function $g(x,q)$ is given by

$$g(x,q)=x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(x^{-1}q;q)_n}\right)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(x,x^{-1}q;q)_{n+1}}.$$

Since then, mock theta functions have received a great deal of attention. For the development of the classical mock theta functions, one can see [3, 14] and the references therein. Gordon and McIntosh [14] stated that the odd order mock theta functions can be represented by $g_3(x,q)$ which is $g(x,q)$ and the even order mock theta functions can be expressed in terms of $g_2(x,q)$ where

$$g_2(x,q)=\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{(n^2+n)/2}}{(x,x^{-1}q;q)_{n+1}}.$$

They also obtained the following relation between $g_3(x,q)$ and $g_2(x,q)$ [14, Equation (6.1)]:

$$g_3(x^4,q^4)=\frac{qg_2(x^6q,q^6)}{x^2}+\frac{x^2g_2(x^6q^{-1},q^6)}{q}-\frac{x^2(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty j(x^2q;q^2)j(x^{12}q^6;q^{12})}{q(q^4;q^4)_\infty(q^6;q^6)_\infty^2 j(x^4;q^2)j(x^6q^{-1};q^2)}.$$

We usually call $g_2(x,q)$ and $g_3(x,q)$ the universal mock theta functions due to the fact that they are at the core of representing classical mock theta functions. In addition to these two functions, McIntosh [28] also considered

$$N(x,q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(xq,x^{-1}q;q)_n},\qquad K_1(x,q)=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{(n+1)^2}}{(xq,x^{-1}q;q^2)_{n+1}},$$

$$K(x,q)=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{n^2}}{(xq^2,x^{-1}q^2;q^2)_n},\quad K_2(x,q)=\sum_{n=0}^{\infty}\frac{(-1;q)_nq^{(n^2+n)/2}}{(xq,x^{-1}q;q)_n}.$$

Notice that some of these functions have combinatorial interpretations. For example, $N(x,q)$ is the generating function of partitions of n with rank m [9], $K(x,q)$ is the generating function of partitions of n with distinct odd parts with M_2-rank m [5,25], and $K_2(x,q)$ is the generating function of overpartitions of n with rank m [23]. A relation between $K(x,q)$ and $K_1(x,q)$ was given by Ramanujan [2, Entry 12.3.2]. Utilizing Appell–Lerch sums [20, 21], McIntosh [28] provided various linear relations for these functions. For example, he showed that

$$\frac{1+x}{1-x}K_2(x,q) = 1 + 2xg_2(x,q).$$

For more on universal mock theta functions, one can see [6, 8, 12, 19, 29].

In [17], Hickerson and Mortenson gave the following definition of Appell–Lerch sums.

Definition 1.1. *Let $x,\ z \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ with neither z nor xz an integral power of q. Then*

$$m(x,q,z) = \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}.$$

Changing r to $r+1$ in the above series gives another useful form of $m(x,q,z)$:

$$m(x,q,z) = \frac{-z}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r+1}{2}} z^r}{1-q^{r}xz}. \tag{1.1}$$

In view of the definition of Appell-Lerch sums, Hickerson and Mortenson [17] presented that

$$\begin{aligned} g(x,q) &= -x^{-1}m(x^{-3}q^2,q^3,x^2) - x^{-2}m(x^{-3}q,q^3,x^2),\\ g_2(x,q) &= -x^{-1}m(x^{-2}q,q^2,x). \end{aligned}$$

Moreover, they built some relations between Hecke-type double sums and Appell–Lerch sums, and established explicit representations of all the classical mock theta functions by means of Appell–Lerch sums. Subsequently, some new families of mock theta functions which are expressed in terms of Appell–Lerch sums were established in [15, 24, 26, 27].

In [30], according to the equations given by Ramanujan [33], Mortenson stated the following series in terms of Appell–Lerch sums, and built some mixed mock modular bilateral q-hypergeometric series.

Proposition 1.2. *[30, Proposition 2.6] We have*

$$\begin{aligned} (1+x^{-1})\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(xq,x^{-1}q;q^2)_{n+1}} &= -m(x,q^2,q),\\ \sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{n^2}}{(-x;q^2)_{n+1}(-x^{-1}q^2;q^2)_n} &= m(x,q,-1) + \frac{J_{1,2}^2}{2j(-x;q)}\\ &= 2m(x,q,-1) - m(x,q,\sqrt{-x^{-1}q})\\ &= m(-x^2q,q^4,-q^{-1}) - xq^{-1}m(-x^2q^{-1},q^4,-q),\\ \sum_{n=0}^{\infty}{}^{*}\frac{(q;q^2)_n(-1)^n}{(-x;q)_{n+1}(-x^{-1}q;q)_n} &= m(x,q,-1), &(1.2)\\ (1+x^{-1})\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{(n+1)^2}}{(-xq,-x^{-1}q;q^2)_{n+1}} &= m(x,q,-1) - \frac{J_{1,2}^2}{2j(-x;q)},\\ \sum_{n=0}^{\infty}\frac{(q^2;q^4)_n(-1)^nq^{2n^2}}{(-x;q^4)_{n+1}(-x^{-1}q^4;q^4)_n} &= m(x,q^2,q) + \frac{\overline{J}_{1,4}^2j(-xq^2;q^4)}{j(-x;q^4)j(xq;q^2)}. &(1.3) \end{aligned}$$

Notice that the symbol $\sum^*$ in (1.2) denotes convergence problems. The series on the left-hand side of (1.2) is a divergent series. Then based on the following identity [2, Equation (12.3.6)]

$$\lim_{\alpha\to 1^-}\sum_{n=0}^{\infty}\frac{(\alpha q;q)_n(q;q^2)_n(-\alpha)^n}{(q,-\alpha xq,-\alpha q/x;q)_n}=\frac{1}{\overline{J}_{0,1}}\sum_{n=-\infty}^{\infty}\frac{(1+x)(1+x^{-1})q^{n(n+1)/2}}{(1+xq^n)(1+x^{-1}q^n)}, \tag{1.4}$$

this divergent series can be replaced by the right-hand side of (1.4).

Recall that given any mock theta function $f(q)$, for every root of unity ζ, when q tends to ζ radially, there exists a theta function $\theta_\zeta(q)$ such that $f(q)-\theta_\zeta(q)$ is bounded. Moreover, there is no single theta function which works for all ζ. In light of the bilateral q-series obtained in [30], Mortenson [31] established some radial limits for the mock theta functions considered by Zudilin [36] and also derived some new radial limit results. For example, Mortenson derived the following theorem.

Theorem 1.3. *[31, Theorem 5.1] If ζ is a primitive even order $2k$ root of unity, k odd, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}\left(2\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{(n+1)^2}}{(-q;q^2)_{n+1}^2}+\frac{J_{1,2}^2}{\overline{J}_{0,1}}\right)=-2\sum_{n=0}^{(k-1)/2}\frac{(-\zeta;\zeta^2)_n^2\zeta^{2n+1}}{(\zeta;\zeta^2)_{n+1}}.$$

For more on radial limits, one can see [4, 7, 10, 11, 18].

Inspired by Mortenson's work [30, 31], we find the following four functions:

$$A(x,q):=1+(xq^{-1}-x^{-1}q)\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^n}{(xq^{-1},x^{-1}q;q^2)_{n+1}}, \tag{1.5}$$

$$B(x,q):=1+(x-x^{-1}q)\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(x,x^{-1}q;q)_{n+1}},$$

$$C(x,q):=\frac{1}{x+1}+\frac{x-1}{x+1}\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{(n^2-n)/2}}{(x,x^{-1};q)_{n+1}},$$

$$D(x,q):=-\frac{(1+x)q^2}{1-x^2q^2}-\frac{(1+xq^2)(1+x^{-1})}{1+xq}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n(-q^2;q^2)_{n+1}q^{n+1}}{(xq;q^2)_{n+2}(x^{-1}q;q^2)_{n+1}}. \tag{1.6}$$

Then we establish the Appell–Lerch sum representations of these functions.

Theorem 1.4. *We have*

$$A(x,q)=2m(x,q^2,q).$$

Theorem 1.5. *We have*

$$B(x,q)=2m(-x,q,-1).$$

Theorem 1.6. *We have*

$$C(x,q)=m(x^2q,q^2,q)+\frac{x\overline{J}_{1,4}^2}{j(x^2;q^2)}.$$

Theorem 1.7. *We have*

$$D(x,q)=m(x,q^2,q).$$

Moreover, we derive the following relations.

Theorem 1.8. *We have*

$$A(x,q)-B(-x,q^2)=-\frac{\overline{J}_{1,2}^2 j(-xq;q^2)}{j(-x;q^2)j(xq;q^2)}, \tag{1.7}$$

$$A(x^2q,q)-2C(x,q)=-\frac{2x\overline{J}_{1,4}^2}{j(x^2;q^2)}, \tag{1.8}$$

$$A(x,q)-2D(x,q)=0, \tag{1.9}$$

$$A(x^{-2}q,q)+2xg_2(x,q)=\frac{2x\overline{J}_{1,4}^2}{j(x^2;q^2)}, \tag{1.10}$$

$$B(-x^{-2}q,q^2)+2xg_2(x,q)=\frac{J_{2,4}^2 j(-x;q)}{j(x;q)j(-x^2q;q^2)}, \tag{1.11}$$

$$C(x^{-1},q)+xg_2(x,q)=0, \tag{1.12}$$

$$D(x^{-2}q,q)+xg_2(x,q)=\frac{x\overline{J}_{1,4}^2}{j(x^2;q^2)}. \tag{1.13}$$

Equations (1.10)-(1.13) imply that all the classical mock theta functions can be expressed by $A(x,q)$, $B(x,q)$, $C(x,q)$, or $D(x,q)$, respectively.

Finally, in view of an identity given by Liu [22, Theorem 6] and the properties of Appell–Lerch sums, we obtain the following radial limits.

Theorem 1.9. *If ζ is a primitive odd order $2k+1$ root of unity, then when q tends to ζ radially within the unit disk, we have*

$$\lim_{q\to\zeta}\left(A(q^3,q^2)-1-\frac{\overline{J}_{2,4}^2}{J_{1,4}}\right)=\zeta(1-\zeta^2)\sum_{n=0}^{k}\frac{(\zeta^4;\zeta^8)_n(-1)^n\zeta^{4n}}{(-\zeta;\zeta^2)_{2n+2}}.$$

Theorem 1.10. *If ζ is a primitive even order $4k$ root of unity, then when q tends to ζ radially within the unit disk, we have*

$$\lim_{q\to\zeta}\left(B(-1,q^2)-1-\frac{\overline{J}_{1,2}^3}{2J_{1,4}^2}\right)=\frac{1-\zeta^2}{\zeta}\sum_{n=0}^{k}\frac{(-1;\zeta)_{2n}\zeta^n}{(\zeta,\zeta^{-1};\zeta^2)_{n+1}}.$$

Theorem 1.11. *If ζ is a primitive odd order $2k+1$ root of unity, then when q tends to ζ radially within the unit disk, we have*

$$\lim_{q\to\zeta}\left(C(q,q^2)-q\frac{\overline{J}_{1,4}^2\overline{J}_{2,8}}{J_{2,4}\overline{J}_{4,8}}\right)=\sum_{n=0}^{k}\frac{(\zeta^4;\zeta^8)_n(-1)^n\zeta^{4n^2}}{(-\zeta^4;\zeta^8)_{n+1}(-\zeta^4;\zeta^8)_n}.$$

Theorem 1.12. *If ζ is a primitive odd order $2k+1$ root of unity, then when q tends to ζ radially within the unit disk, we have*

$$\lim_{q\to\zeta}\left(D(1,q)+\frac{2q^2}{1-q^2}+4q\frac{J_4^8}{J_2^7}\right)=\frac{2\zeta(1+\zeta^2)^2}{1-\zeta^2}\sum_{n=0}^{k}\frac{(\zeta,-\zeta^4;\zeta^2)_n(-1)^n\zeta^n}{(-\zeta,-\zeta^3;\zeta^2)_{n+1}}.$$

This paper is organized as follows. In Section 2, we state some preliminaries. In Section 3, the proofs of Theorems 1.4-1.8 are established. In Section 4, we prove Theorems 1.9-1.12.

2. Preliminaries

The following identities are frequently used in this paper.

$$j(x;q)=j(x^{-1}q;q),$$
$$j(xq;q)=-x^{-1}j(x;q),$$
$$j(x;q)\overline{J}_{1,4}=j(x;q^2)j(xq;q^2).$$

In order to prove the main theorems, the following results are needed.

Lemma 2.1. *[33, p.1] If z is not an integral power of q, then*

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{1-zq^n}=\frac{J_1^3}{j(z;q)}. \tag{2.1}$$

Proposition 2.2. *[17, Proposition 3.1] For generic $x,z\in\mathbb{C}^*$,*

$$m(x,q,z)=m(x,q,zq), \tag{2.2}$$
$$m(x,q,z)=x^{-1}m(x^{-1},q,z^{-1}), \tag{2.3}$$
$$m(xq,q,z)=1-xm(x,q,z). \tag{2.4}$$

Following [17], the term “generic” means that the parameters do not cause poles in the Appell–Lerch sums or in the quotients of theta functions.

Corollary 2.3. *[17, Corollary 3.2] We have*

$$m(-1,q^2,q)=0. \tag{2.5}$$

Lemma 2.4. *[17, Theorem 3.3] For generic $x,z_0,z_1\in\mathbb{C}^*$,*

$$m(x,q,z_1)-m(x,q,z_0)=\frac{z_0J_1^3 j(z_1/z_0;q)j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}. \tag{2.6}$$

Lemma 2.5. *[13, Appendix (III.17)] The Watson–Whipple transformation formula is given by*

$$\begin{aligned}&{}_8\phi_7\left(\begin{matrix}a, & a^{1/2}q, & -a^{1/2}q, & b, & c, & d, & e, & f\\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f\end{matrix};q,\frac{a^2q^2}{bcdef}\right)\\ &=\frac{(aq,aq/de,aq/df,aq/ef;q)_\infty}{(aq/d,aq/e,aq/f,aq/def;q)_\infty}{}_4\phi_3\left(\begin{matrix}aq/bc, & d, & e, & f\\ & aq/b, & aq/c, & def/a\end{matrix};q,q\right).\end{aligned}$$

Letting $f\to\infty$ in the above identity yields

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{1-aq^{2n}}{1-a}\frac{(a,b,c,d,e;q)_n(-1)^nq^{\binom{n}{2}}}{(q,aq/b,aq/c,aq/d,aq/e;q)_n}\left(\frac{a^2q^2}{bcde}\right)^n\\ &\qquad=\frac{(aq,aq/de;q)_\infty}{(aq/d,aq/e;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq/bc,d,e;q)_n}{(q,aq/b,aq/c;q)_n}\left(\frac{aq}{de}\right)^n.\end{aligned} \tag{2.7}$$

Lemma 2.6. *[22, Theorem 6] We have*

$$d\sum_{n=0}^{\infty}\frac{(q/bc,acdf;q)_n(bd)^n}{(ad,df;q)_{n+1}}-c\sum_{n=0}^{\infty}\frac{(q/bd,acdf;q)_n(bc)^n}{(ac,cf;q)_{n+1}}=d\frac{(q,qd/c,c/d,abcd,acdf,bcdf;q)_\infty}{(ac,ad,cf,df,bc,bd;q)_\infty}. \tag{2.8}$$

3. Proofs of Theorems 1.4-1.8

In this section, we establish the Appell–Lerch sums of $A(x,q)$, $B(x,q)$, $C(x,q)$, and $D(x,q)$.

Proof of Theorem 1.4. Setting $a=1$, $b=x^{-1}q$, $c=xq^{-1}$, $d=-1$, $e=-q$, and $q\to q^2$ in (2.7), and then dividing both sides by $(1-xq^{-1})(1-x^{-1}q)$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^n}{(xq^{-1},x^{-1}q;q^2)_{n+1}}\\
&=\frac{1}{J_{1,2}}\left(\frac{1}{(1-xq^{-1})(1-x^{-1}q)}+2\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+2n}}{(1-xq^{2n-1})(1-x^{-1}q^{2n+1})}\right)\\
&=\frac{1}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{(1-xq^{2n-1})(1-x^{-1}q^{2n+1})}\\
&=\frac{1}{(xq^{-1}-x^{-1}q)J_{1,2}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^nxq^{n^2+2n-1}}{1-xq^{2n-1}}-\frac{(-1)^nx^{-1}q^{n^2+2n+1}}{1-x^{-1}q^{2n+1}}\right)\\
&=\frac{-xq^{-2}m(xq^{-2},q^2,q)+x^{-1}m(x^{-1},q^2,q)}{xq^{-1}-x^{-1}q},
\end{aligned}\tag{3.1}$$

where the last step follows from (1.1). Then applying (2.2) and (2.3) yields that

$$m(x^{-1},q^2,q)=xm(x,q^2,q^{-1})=xm(x,q^2,q).\tag{3.2}$$

In addition, in view of (2.4), we have

$$-xq^{-2}m(xq^{-2},q^2,q)=m(x,q^2,q)-1.\tag{3.3}$$

Finally, substituting (3.2) and (3.3) into (3.1), we prove the theorem. □

Proof of Theorem 1.5. Substituting $a=q$, $b=x^{-1}q$, $c=x$, $d=q^{1/2}$, and $e=-q^{1/2}$ into (2.7), we deduce

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(x,x^{-1}q;q)_{n+1}}&=\frac{2}{\overline{J}_{0,1}}\sum_{n=0}^{\infty}\frac{q^{(n^2+3n)/2}}{(1-xq^n)(1-x^{-1}q^{n+1})}\\
&=\frac{1}{\overline{J}_{0,1}}\sum_{n=-\infty}^{\infty}\frac{q^{(n^2+3n)/2}}{(1-xq^n)(1-x^{-1}q^{n+1})}\\
&=\frac{1}{(x^{-1}q-x)\overline{J}_{0,1}}\left(\sum_{n=-\infty}^{\infty}\frac{x^{-1}q^{(n^2+3n+2)/2}}{1-x^{-1}q^{n+1}}-\sum_{n=-\infty}^{\infty}\frac{xq^{(n^2+3n)/2}}{1-xq^n}\right)\\
&=\frac{1}{(x^{-1}q-x)\overline{J}_{0,1}}\left(-\sum_{n=-\infty}^{\infty}\frac{q^{(n^2+n)/2}}{1-xq^n}-\sum_{n=-\infty}^{\infty}\frac{xq^{(n^2+3n)/2}}{1-xq^n}\right)\\
&=-\frac{m(-x,q,-1)}{x^{-1}q-x}-\frac{xq^{-1}m(-xq^{-1},q,-q)}{x^{-1}q-x}\qquad (by\ (1.1))\\
&=-\frac{m(-x,q,-1)}{x^{-1}q-x}-\frac{xq^{-1}m(-xq^{-1},q,-1)}{x^{-1}q-x}\qquad (by\ (2.2))\\
&=-\frac{m(-x,q,-1)}{x^{-1}q-x}-\frac{m(-x,q,-1)}{x^{-1}q-x}+\frac{1}{x^{-1}q-x},
\end{aligned}$$

where the last step follows from (2.4). Hence, we complete the proof. □

Proof of Theorem 1.6. Letting $a=1$, $b=x$, $c=x^{-1}$, $d=-q$, and $e\to\infty$ in (2.7), we derive

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{(n^2-n)/2}}{(x,x^{-1};q)_{n+1}}\\
&=\frac{2}{J_{1,2}}\left(\frac{1}{(1-x)(1-x^{-1})}+\sum_{n=1}^{\infty}\frac{(1+q^n)^2(-1)^n q^{n^2}}{2(1-xq^n)(1-x^{-1}q^n)}\right)\\
&=\frac{1}{2J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1+q^n)^2(-1)^n q^{n^2}}{(1-xq^n)(1-x^{-1}q^n)}\\
&=\frac{1}{2(1-x)J_{1,2}}\sum_{n=-\infty}^{\infty}\left(\frac{(1+q^n)(-1)^n q^{n^2}}{1-x^{-1}q^n}-\frac{(1+q^n)(-1)^n xq^{n^2}}{1-xq^n}\right)\\
&=\frac{-x}{(1-x)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1+q^n)(-1)^n q^{n^2}}{1-xq^n}\\
&=\frac{-x}{(1-x)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1+q^n)(1+xq^n)(-1)^n q^{n^2}}{1-x^2q^{2n}}\\
&=\frac{-x}{(1-x)J_{1,2}}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2}}{1-x^2q^{2n}}+x\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1-x^2q^{2n}}+(1+x)\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-x^2q^{2n}}\right)\\
&=\frac{-x}{1-x}\left(m(x^2q,q^2,q^{-1})-xq^{-1}m(x^2q^{-1},q^2,q)\right)-\frac{x(1+x)}{1-x}\frac{\overline{J}_{1,4}^2}{j(x^2;q^2)}\\
&=\frac{-x}{1-x}\left(m(x^2q,q^2,q)-x^{-1}+x^{-1}m(x^2q,q^2,q)\right)-\frac{x(1+x)}{1-x}\frac{\overline{J}_{1,4}^2}{j(x^2;q^2)}\\
&=\frac{-(1+x)}{1-x}m(x^2q,q^2,q)+\frac{1}{1-x}-\frac{x(1+x)}{1-x}\frac{\overline{J}_{1,4}^2}{j(x^2;q^2)},
\end{aligned}$$

where we obtain the last third step by utilizing (1.1) and (2.1), and the penultimate step follows from (2.2) and (2.4). Therefore, we complete the proof. □

Proof of Theorem 1.7. Setting $a=q^4$, $b=xq^2$, $c=x^{-1}q^2$, $d=-q$, $e=-q^4$, and $q\to q^2$ in (2.7), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q,-q^4;q^2)_n q^n}{(xq^2,x^{-1}q^2;q^2)_{n+1}}\\
&=\frac{1}{(1+q^2)J_{1,2}}\sum_{n=0}^{\infty}\frac{(1-q^{4n+4})^2(-1)^n q^{n^2+2n}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+1})(1+q^{2n+3})}\\
&=\frac{1}{2(1+q^2)J_{1,2}}\left(\sum_{n=0}^{\infty}+\sum_{n=-\infty}^{-2}\right)\frac{(1-q^{4n+4})^2(-1)^n q^{n^2+2n}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+1})(1+q^{2n+3})}\\
&=\frac{1}{2(1+q^2)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{4n+4})^2(-1)^n q^{n^2+2n}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+1})(1+q^{2n+3})}\\
&=\frac{1}{2(1+q^2)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{4n+4})(-1)^n q^{n^2+2n}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+1})}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2(1+q^2)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{4n+4})(-1)^nq^{n^2+4n+3}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+3})}\\
&=\frac{1}{(1+q^2)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{4n+4})(-1)^nq^{n^2+2n}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+1})}\\
&=\frac{1}{(1+q^2)J_{1,2}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^nq^{n^2+2n}}{(1-x^{-1}q^{2n+2})(1+q^{2n+1})}+\frac{(-1)^nxq^{n^2+4n+2}}{(1-xq^{2n+2})(1+q^{2n+1})}\right)\\
&=\frac{x^{-1}q}{(1+q^2)(1+x^{-1}q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1-x^{-1}q^{2n+2}}+\frac{1}{(1+q^2)(1+x^{-1}q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{2n+1}}\\
&\quad+\frac{x^2q^3}{(1+q^2)(1+xq)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1-xq^{2n+2}}+\frac{xq^2}{(1+q^2)(1+xq)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1+q^{2n+1}}\\
&=-\frac{q^3}{(1+q^2)(1+x^{-1}q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1-xq^{2n+2}}+\frac{1}{(1+q^2)(1+x^{-1}q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{2n+1}}\\
&\quad+\frac{x^2q^3}{(1+q^2)(1+xq)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1-xq^{2n+2}}+\frac{xq^2}{(1+q^2)(1+xq)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1+q^{2n+1}}\\
&=-\frac{(1-x^2)q^3}{(1+q^2)(1+xq)(1+x^{-1}q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1-xq^{2n+2}}\\
&\quad+\frac{1}{(1+q^2)(1+x^{-1}q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{2n+1}}+\frac{xq^2}{(1+q^2)(1+xq)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1+q^{2n+1}}\\
&=-\frac{(1-x^2)q^{-1}}{(1+q^2)(1+xq)(1+x^{-1}q)}m(xq^{-1},q^2,q^3)-\frac{q^{-1}}{(1+q^2)(1+x^{-1}q)}m(-1,q^2,q)\\
&\quad+\frac{xq^{-2}}{(1+q^2)(1+xq)}m(-q^{-2},q^2,q^3), \qquad (3.4)
\end{aligned}
$$

where the last step follows from (1.1). Notice that we use the technique $n\to -n-2$ several times in the above equalities.

Next, based on (2.2) and (2.4), we obtain

$$m(-q^{-2},q^2,q^3)=-q^2+q^2m(-1,q^2,q^3)=-q^2+q^2m(-1,q^2,q). \qquad (3.5)$$

Then substituting (3.5) into (3.4) and applying (2.2) and (2.5), we derive

$$\sum_{n=0}^{\infty}\frac{(-q,-q^4;q^2)_nq^n}{(xq^2,x^{-1}q^2;q^2)_{n+1}}=-\frac{(1-x^2)q^{-1}}{(1+q^2)(1+xq)(1+x^{-1}q)}m(xq^{-1},q^2,q)-\frac{x}{(1+q^2)(1+xq)}.$$

Finally, replacing x by xq in the above identity yields

$$\sum_{n=0}^{\infty}\frac{(-q,-q^4;q^2)_nq^n}{(xq^3,x^{-1}q;q^2)_{n+1}}=-\frac{(1-x^2q^2)q^{-1}}{(1+q^2)(1+xq^2)(1+x^{-1})}m(x,q^2,q)-\frac{xq}{(1+q^2)(1+xq^2)}$$

which implies the theorem. □

Proof of Theorem 1.8. In view of Theorems 1.4 and 1.5, we have

$$A(x,q)-B(-x,q^2)=2m(x,q^2,q)-2m(x,q^2,-1).$$

Then using (2.6) in the above identity, we complete the proof of (1.7).

The proofs of (1.8)-(1.13) are similar to that of (1.7). □

4. Proofs of Theorems 1.9-1.12

In this section, using (2.8) given by Liu [22, Theorem 6] and some properties of Appell–Lerch sums, we establish some radial limits.

Proof of Theorem 1.9. Setting $a = q$, $b = q^2$, $c = -q^2$, $d = 1$, $f = q^{-1}$, and $q \to q^4$ in (2.8), we deduce

$$\sum_{n=0}^{\infty} \frac{(-1;q^2)_{2n} q^{2n}}{(q, q^{-1}; q^4)_{n+1}} + q^2 \sum_{n=0}^{\infty} \frac{(q^4;q^8)_n (-1)^n q^{4n}}{(-q;q^2)_{2n+2}} = -\frac{q}{1-q^2} \frac{\overline{J}_{2,4}^2}{J_{1,4}}.$$

If ζ is a primitive order $2k+1$ root of unity, then when q tends to ζ radially within the unit disk,

$$\lim_{q\to\zeta} \left(\sum_{n=0}^{\infty} \frac{(-1;q^2)_{2n} q^{2n}}{(q, q^{-1}; q^4)_{n+1}} + \frac{q}{1-q^2} \frac{\overline{J}_{2,4}^2}{J_{1,4}} \right) = -\zeta^2 \sum_{n=0}^{k} \frac{(\zeta^4;\zeta^8)_n (-1)^n \zeta^{4n}}{(-\zeta;\zeta^2)_{2n+2}}.$$

Therefore, combining (1.5) and the above identity, we complete the proof. □

Proof of Theorem 1.10. Letting $a = -1$, $b = -q^2$, $c = -q^{-1}$, $d = 1$, $f = -q^2$, and $q \to q^2$ in (2.8), we derive

$$\sum_{n=0}^{\infty} \frac{(q^2;q^4)_n (-1)^n q^{2n}}{(-1, -q^2; q^2)_{n+1}} + q^{-1} \sum_{n=0}^{\infty} \frac{(-1;q)_{2n} q^n}{(q, q^{-1}; q^2)_{n+1}} = -\frac{\overline{J}_{1,2}^3}{2(1-q^2) J_{1,4}^2}.$$

Namely,

$$B(-1, q^2) - 1 - \frac{\overline{J}_{1,2}^3}{2J_{1,4}^2} = \frac{1-q^2}{q} \sum_{n=0}^{\infty} \frac{(-1;q)_{2n} q^n}{(q, q^{-1}; q^2)_{n+1}}.$$

Hence, if ζ is a primitive order $4k$ root of unity, then we obtain the theorem by letting q tend to ζ radially within the unit disk. □

Proof of Theorem 1.11. In view of Theorem 1.6, we deduce

$$C(q, q^2) = m(q^4, q^4, q^2) + q \frac{\overline{J}_{2,8}^2}{J_{2,4}}. \tag{4.1}$$

Then applying (1.3) with $x = q^4$ and $q \to q^2$ yields

$$\sum_{n=0}^{\infty} \frac{(q^4;q^8)_n (-1)^n q^{4n^2}}{(-q^4;q^8)_{n+1} (-q^4;q^8)_n} = m(q^4, q^4, q^2) - q^2 \frac{\overline{J}_{0,8} \overline{J}_{2,8}^2}{J_{2,4} \overline{J}_{4,8}}. \tag{4.2}$$

Combining (4.1) and (4.2), we derive

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q^4;q^8)_n (-1)^n q^{4n^2}}{(-q^4;q^8)_{n+1} (-q^4;q^8)_n} &= C(q, q^2) - q \frac{\overline{J}_{2,8}^2}{J_{2,4}} - q^2 \frac{\overline{J}_{0,8} \overline{J}_{2,8}^2}{J_{2,4} \overline{J}_{4,8}} \\ &= C(q, q^2) - q \frac{\overline{J}_{2,8}^2 (\overline{J}_{4,8} + q \overline{J}_{0,8})}{J_{2,4} \overline{J}_{4,8}} \\ &= C(q, q^2) - q \frac{\overline{J}_{1,4}^2 \overline{J}_{2,8}}{J_{2,4} \overline{J}_{4,8}}, \end{aligned}$$

where we use the following identity [17] with $x = -q$, $y = -q^3$, and $q \to q^4$ to obtain the third equality.

$$j(x;q)j(y;q) = j(-xy;q^2)j(-qx^{-1}y;q^2) - xj(-qxy;q^2)j(-x^{-1}y;q^2).$$

Thus, if ζ is a primitive order $2k+1$ root of unity, then when q tends to ζ radially within the unit disk, we prove the theorem. $\square$

Proof of Theorem 1.12. Substituting $a = q^3$, $b = q$, $c = -1$, $d = 1$, $f = q$, and $q \to q^2$ in (2.8) yields

$$\sum_{n=0}^{\infty} \frac{(-q,-q^4;q^2)_n q^n}{(q,q^3;q^2)_{n+1}} + \sum_{n=0}^{\infty} \frac{(q,-q^4;q^2)_n (-1)^n q^n}{(-q,-q^3;q^2)_{n+1}} = \frac{2(1-q^2)J_4^8}{(1+q^2)^2 J_2^7}. \tag{4.3}$$

In addition, in view of (1.6), we have

$$D(1,q) = -\frac{2q^2}{1-q^2} - \frac{2q(1+q^2)^2}{1-q^2} \sum_{n=0}^{\infty} \frac{(-q,-q^4;q^2)_n q^n}{(q,q^3;q^2)_{n+1}}. \tag{4.4}$$

Then combining (4.3) and (4.4), we derive

$$D(1,q) + \frac{2q^2}{1-q^2} + \frac{4qJ_4^8}{J_2^7} = \frac{2q(1+q^2)^2}{1-q^2} \sum_{n=0}^{\infty} \frac{(q,-q^4;q^2)_n (-1)^n q^n}{(-q,-q^3;q^2)_{n+1}}.$$

Thus, if ζ is a primitive order $2k+1$ root of unity, then letting q tend to ζ radially within the unit disk, we complete the proof. $\square$

Acknowledgements: This work was supported by the National Natural Science Foundation of China (Grant Nos. 12171255 and 12001309), the Foreign Youth Exchange Program of China Association of Science and Technology, the Natural Science Foundation Youth Fund of Qinghai Normal University, the Natural Science Foundation Youth Fund of Qinghai (Grant No. 2022-ZJ-972Q), and Natural Science Foundation of Tianjin (Grant No. 19JCYBJC30100).

References

[1] G. E. Andrews, *The fifth and seventh order mock theta functions*, Trans. Amer. Math. Soc., 293 (1986), pp. 113–134.
[2] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part I*, Springer, New York, 2005.
[3] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part V*, Springer, New York, 2018.
[4] J. Bajpai, S. Kimport, J. Liang, D. Ma, and J. Ricci, *Bilateral series and Ramanujan's radial limits*, Proc. Amer. Math. Soc., 143 (2015), pp. 479–492.
[5] A. Berkovich and F. G. Garvan, *Some observations on Dyson's new symmetries of partitions*, J. Combin. Theory Ser. A, 100 (2002), pp. 61–93.
[6] K. Bringmann, A. Folsom, and R. C. Rhoades, *Partial theta functions and mock modular forms as q-hypergeometric series*, Ramanujan J., 29 (2012), pp. 295–310.
[7] K. Bringmann and L. Rolen, *Radial limits of mock theta functions*, Res. Math. Sci., 2 (2015), Art. 17, 18 pp.
[8] G. Carroll, J. Corbett, A. Folsom, and E. Thieu, *Universal mock theta functions as quantum Jacobi forms*, Res. Math. Sci., 6 (2019), Paper No. 6, 15 pp.
[9] F. J. Dyson, *Some guesses in the theory of partitions*, Eureka, 8 (1944), pp. 10–15.
[10] A. Folsom, K. Ono, and R. C. Rhoades, *Mock theta functions and quantum modular forms*, Forum Math. Pi, 1 (2013), e2, 27 pp.
[11] A. Folsom, K. Ono, and R. C. Rhoades, *Ramanujan's radial limits*, in Ramanujan 125, Contemp. Math. 627, Amer. Math. Soc., Providence, RI, 2014, pp. 91–102.
[12] F. G. Garvan, *Universal mock theta functions and two-variable Hecke-Rogers identities*, Ramanujan J., 36 (2015), pp. 267–296.

[13] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Second Ed., Cambridge University Press, Cambridge, 2004.

[14] B. Gordon and R. J. McIntosh, *A survey of classical mock theta functions*, in "Partitions, q-series, and modular forms", Dev. Math., 23 (2012), pp. 95–144.

[15] N. S. S. Gu and J. Liu, *Families of multisums as mock theta functions*, Adv. in Appl. Math., 79 (2016), pp. 98–124.

[16] D. Hickerson, *A proof of the mock theta conjectures*, Invent. Math., 94 (1988), pp. 639–660.

[17] D. R. Hickerson and E. T. Mortenson, *Hecke-type double sums, Appell–Lerch sums, and mock theta functions, I*, Proc. Lond. Math. Soc. (3), 109 (2014), pp. 382–422.

[18] M.-J. Jang and S. Löbrich, *Radial limits of the universal mock theta function g_3*, Proc. Amer. Math. Soc., 145 (2017), pp. 925–935.

[19] S.-Y. Kang, *Mock Jacobi forms in basic hypergeometric series*, Compos. Math., 145 (2009), pp. 553–565.

[20] M. Lerch, *Poznámky k theorii funkcí elliptických*, Rozpravy České Akademie Císaře Františka Josefa pro vědy, slovesnost a umění v praze, 24 (1892), pp. 465–480.

[21] M. Lerch, *Nová analogie řady theta a některé zvláštní hypergeometrické řady Heineovy*, Rozpravy, 3 (1893), pp. 1–10.

[22] Z.-G. Liu, *Some operator identities and q-series transformation formulas*, Discrete Math., 265 (2003), pp. 119–139.

[23] J. Lovejoy, *Rank and conjugation for the Frobenius representation of an overpartition*, Ann. Comb., 9 (2005), pp. 321–334.

[24] J. Lovejoy, *Bailey pairs and indefinite quadratic forms*, J. Math. Anal. Appl., 410 (2014), pp. 1002–1013.

[25] J. Lovejoy and R. Osburn, *M_2-rank differences for partitions without repeated odd parts*, J. Théor. Nombres Bordeaux, 21 (2009), pp. 313–334.

[26] J. Lovejoy and R. Osburn, *The Bailey chain and mock theta functions*, Adv. Math., 238 (2013), pp. 442–458.

[27] J. Lovejoy and R. Osburn, *q-hypergeometric double sums as mock theta functions*, Pacific J. Math., 264 (2013), pp. 151–162.

[28] R. J. McIntosh, *The H and K family of mock theta functions*, Canad. J. Math., 64 (2012), pp. 935–960.

[29] R. J. McIntosh, *Some identities for Appell–Lerch sums and a universal mock theta function*, Ramanujan J., 45 (2018), pp. 767–779.

[30] E. T. Mortenson, *On the dual nature of partial theta functions and Appell–Lerch sums*, Adv. Math., 264 (2014), pp. 236–260.

[31] E. Mortenson, *Ramanujan's radial limits and mixed mock modular bilateral q-hypergeometric series*, Proc. Edinb. Math. Soc., 59 (2016), pp. 787–799.

[32] S. Ramanujan, *Collected Papers*, Cambridge Univ. Press, 1927; Reprinted, Chelsea, New York, 1962.

[33] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa Publishing House, New Delhi, 1988.

[34] G. N. Watson, *The final problem: an account of the mock theta functions*, J. London Math. Soc., 11 (1936), pp. 55–80.

[35] G. N. Watson, *The mock theta functions (2)*, Proc. London Math. Soc. (2), 42 (1936), pp. 274–304.

[36] W. Zudilin, *On three theorems of Folsom, Ono and Rhoades*, Proc. Amer. Math. Soc., 143 (2015), pp. 1471–1476.

(S.-P. Cui) School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, P.R. China; Academy of Plateau Science and Sustainability, Xining, Qinghai 810008, P.R. China

E-mail address: jiayoucui@163.com

(N. S. S. Gu) Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P.R. China

E-mail address: gu@nankai.edu.cn

(C.-Y. Su) College of Mathematical Science, Tianjin Normal University, Tianjin 300387, P.R. China

E-mail address: cyangsu@163.com