GENERALIZATIONS OF MOCK THETA FUNCTIONS AND RADIAL LIMITS

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Abstract. In the last letter to Hardy, Ramanujan introduced seventeen functions defined by \( q \)-series convergent for \( |q| < 1 \) with a complex variable \( q \), and called these functions “mock theta functions”. Subsequently, mock theta functions were widely studied in the literature. In the survey of B. Gordon and R. J. McIntosh, they showed that the odd (resp. even) order mock theta functions are related to the function \( g_3(x, q) \) (resp. \( g_2(x, q) \)). These two functions are usually called “universal mock theta functions”. In 2014, D. R. Hickerson and E. T. Mortenson expressed all the classical mock theta functions and the two universal mock theta functions in terms of Appell–Lerch sums. In this paper, based on some \( q \)-series identities, we find four functions, and express them in terms of Appell–Lerch sums. For example,

\[
1 + (xq^{-1} - x^{-1}) \sum_{n=0}^{\infty} \frac{(-1; q)_n q^n}{(xq^{-1}, x^{-1}; q^2)_n} = 2m(x, q^2, q).
\]

Then we establish some identities related to these functions and the universal mock theta function \( g_2(x, q) \). These relations imply that all the classical mock theta functions can be expressed in terms of these four functions. Furthermore, by means of \( q \)-series identities and some properties of Appell–Lerch sums, we derive four radial limit results related to these functions.

1. Introduction

Throughout the paper, we use the standard \( q \)-series notation [13]. For positive integers \( n \) and \( m \),

\[
(a; q)_0 := 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k),
\]

\[
(a_1, a_2, \ldots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,
\]

\[
(a_1, a_2, \ldots, a_m; q)_{\infty} := (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_m; q)_{\infty}.
\]

Define

\[
j(x; q) := (x; q)_{\infty} (x^{-1}; q^{-1})_{\infty} (q; q)_{\infty},
\]

\[
J_{a,m} := j(q^a; q^m), \quad J_{a,m} := j(-q^a; q^m), \quad J_{m} := J_{m,3m} = \prod_{i=1}^{\infty} (1 - q^{mi}).
\]

The (unilateral) basic hypergeometric series \( r \phi_s \) is defined as

\[
r \phi_s \left( \begin{array}{c} a_1, a_2, \ldots, a_r \end{array} \bigg| \begin{array}{c} b_1, \ldots, b_s \end{array} ; q, x \right) := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left( (-1) n q^{\frac{n(n-1)}{2}} \right)^{1+s-r} x^n.
\]

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Seventeen mock theta functions were presented by Ramanujan [32] in his last letter to Hardy. These functions were assigned orders 3, 5, and 7. In 1936, Watson [34, 35] proved some identities for the third and fifth order mock theta functions. In view of Bailey’s lemma and Bailey pairs, Andrews [1] established Hecke-type double sums for the fifth and seventh order mock theta functions. For example, he showed that

\[ f_0(q) = \frac{1}{J_1} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j (1 - q^{4n+2}) q^{n(5n+1)/2-j^2}, \]

\[ f_1(q) = \frac{1}{J_1} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j (1 - q^{2n+1}) q^{n(5n+3)/2-j^2}, \]

where the fifth order mock theta functions \( f_0(q) \) and \( f_1(q) \) are defined as

\[ f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n} \quad \text{and} \quad f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q; q)_n}. \]

Then based on these Hecke-type double sums, Hickerson [16] proved the following two identities which are known as Mock Theta Conjectures:

\[ f_0(q) = \frac{J_{5,10} J_{2.5}}{J_1} - 2q^2 g(q^2, q^{10}), \]

\[ f_1(q) = \frac{J_{5,10} J_{4.5}}{J_1} - 2q^3 g(q^4, q^{10}), \]

where the function \( g(x, q) \) is given by

\[ g(x, q) = x^{-1} \left( -1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(x; q)_n (x^{-1} q; q)_n} \right) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x; x^{-1} q; q)_{n+1}}. \]

Since then, mock theta functions have received a great deal of attention. For the development of the classical mock theta functions, one can see [3, 14] and the references therein. Gordon and McIntosh [14] stated that the odd order mock theta functions can be represented by \( g_3(x, q) \) which is \( g(x, q) \) and the even order mock theta functions can be expressed in terms of \( g_2(x, q) \) where

\[ g_2(x, q) = \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{(n^2+n)/2}}{(x; x^{-1} q; q)_{n+1}}. \]

They also obtained the following relation between \( g_3(x, q) \) and \( g_2(x, q) \) [14, Equation (6.1)]:

\[ g_3(x^4, q^4) = \frac{q g_2(x^6 q, q^6)}{x^2} + \frac{x^2 g_2(x^6 q^{-1}, q^6)}{q} = \frac{x^2 (q^2; q^2)^3}{(q^{12}; q^{12})_\infty (x^6 q^2; q^2)_\infty (x^{12} q^6; q^6)_{12}} j(x^4; q^2) j(x^4; q^6) j(x^6 q^{-1}; q^2). \]

We usually call \( g_2(x, q) \) and \( g_3(x, q) \) the universal mock theta functions due to the fact that they are at the core of representing classical mock theta functions. In addition to these two functions, McIntosh [28] also considered

\[ N(x, q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x q, x^{-1} q; q)_n}, \quad K_1(x, q) = \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{(n+1)^2}}{(x q, x^{-1} q; q^2)_{n+1}}, \]

\[ K(x, q) = \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n^2}}{(x q^2, x^{-1} q^2; q^2)_n}, \quad K_2(x, q) = \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{(n^2+n)/2}}{(x q, x^{-1} q; q)_n}. \]

Notice that some of these functions have combinatorial interpretations. For example, \( N(x, q) \) is the generating function of partitions of \( n \) with rank \( m \) [9], \( K(x, q) \) is the generating function
of partitions of $n$ with distinct odd parts with $M_2$-rank $m$ [5,25], and $K_2(x,q)$ is the generating
function of overpartitions of $n$ with rank $m$ [23]. A relation between $K(x,q)$ and $K_1(x,q)$ was
given by Ramanujan [2, Entry 12.3.2]. Utilizing Appell–Lerch sums [20,21], McIntosh [28]
provided various linear relations for these functions. For example, he showed that
\[
\frac{1+x}{1-x} K_2(x,q) = 1 + 2xg_2(x,q).
\]
For more on universal mock theta functions, one can see [6,8,12,19,29].

In [17], Hickerson and Mortenson gave the following definition of Appell–Lerch sums.

**Definition 1.1.** Let $x$, $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ with neither $z$ nor $xz$ an integral power of $q$. Then
\[
m(x,q,z) = \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1 - q^{r+1} x z}.
\]

Changing $r$ to $r+1$ in the above series gives another useful form of $m(x,q,z)$:
\[
m(x,q,z) = \frac{-z}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r+1}{2}} z^r}{1 - q^{r} x z}.
\]

In view of the definition of Appell-Lerch sums, Hickerson and Mortenson [17] presented that
\[
g(x,q) = -x^{-1} m(x^{-3} q^2, q^2, x^2) - x^{-2} m(x^{-3} q^3, x^2),
g_2(x,q) = -x^{-1} m(x^{-2} q, q^2, x).
\]
Moreover, they built some relations between Hecke-type double sums and Appell–Lerch sums,
and established explicit representations of all the classical mock theta functions by means
of Appell–Lerch sums. Subsequently, some new families of mock theta functions which are
expressed in terms of Appell–Lerch sums were established in [15,24,26,27].

In [30], according to the equations given by Ramanujan [33], Mortenson stated the fol-
lowing series in terms of Appell–Lerch sums, and built some mixed mock modular bilateral
$q$-hypergeometric series.

**Proposition 1.2.** [30, Proposition 2.6] We have
\[
(1 + x^{-1}) \sum_{n=0}^{\infty} \frac{(-q; q)_{2n} q^{n+1}}{(xq, x^{-1}q; q^2)_{n+1}} = -m(x, q^2, q),
\]
\[
\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n^2}}{(-x; q^2)_{n+1} (-x^{-1} q^2; q^2)_n} = m(x, q, -1) + \frac{J_{1,2}^2}{2j(-x;q)},
\]
\[
= 2m(x,q,-1) - m(x, q, \sqrt{-x^{-1} q}) = m(-x^{-2} q^4, q^{-1}) - x q^{-2} m(-x^2 q^{-1}, q^4, -q)
\]
\[
\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n^2}}{(-x; q)_{n+1} (-x^{-1} q; q)_{n}} = m(x, q, -1),
\]
\[
(1 + x^{-1}) \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n+1}^2}{(-x q, x^{-1} q; q^2)_{n+1}} = m(x, q, -1) - \frac{J_{1,2}^2}{2j(-x;q)},
\]
\[
\sum_{n=0}^{\infty} \frac{(q^2; q^4)_n (-1)^n q^{2n^2}}{(-x; q^4)_{n+1} (-x^{-1} q^4; q^4)_n} = m(x, q^2, q) + \frac{J_{1,4}^2 (-x q^2; q^4)}{J(-x; q^4) j(xq; q^2)}.
\]
Notice that the symbol $\sum^*$ in (1.2) denotes convergence problems. The series on the left-hand side of (1.2) is a divergent series. Then based on the following identity [2, Equation (12.3.6)]

$$\lim_{\alpha \to 1} \sum_{n=0}^{\infty} \frac{(\alpha q; q)_n(q; q^2)_n(-\alpha)^n}{(q, -\alphaqx, -\alphaq/x; q)_n} = \frac{1}{J_{0,1}} \sum_{n=-\infty}^{\infty} \frac{(1 + x)(1 + x^{-1})q^{(n+1)/2}}{(1 + xq^n)(1 + x^{-1}q^n)},$$

this divergent series can be replaced by the right-hand side of (1.4).

Recall that given any mock theta function $f(q)$, for every root of unity $\zeta$, when $q$ tends to $\zeta$ radially, there exists a theta function $\theta_{\zeta}(q)$ such that $f(q) - \theta_{\zeta}(q)$ is bounded. Moreover, there is no single theta function which works for all $\zeta$. In light of the bilateral $q$-series obtained in [30], Mortenson [31] established some radial limits for the mock theta functions considered by Zudilin [36] and also derived some new radial limit results. For example, Mortenson derived the following theorem.

**Theorem 1.3.** [31, Theorem 5.1] If $\zeta$ is a primitive even order $2k$ root of unity, $k$ odd, then, as $q$ approaches $\zeta$ radially within the unit disk, we have that

$$\lim_{q \to \zeta} \left(2 \sum_{n=0}^{\infty} \frac{(q; q^2)^n(-1)^nq^{(n+1)^2}}{(-q; q^2)_n} + \frac{J^2_{1,2}}{J_{0,1}}\right) = -2 \sum_{n=0}^{(k-1)/2} \frac{(-\zeta; \zeta^2)^n \zeta^{2n+1}}{(\zeta; \zeta^2)_n}.$$

For more on radial limits, one can see [4, 7, 10, 11, 18].

Inspired by Mortenson’s work [30,31], we find the following four functions:

$$A(x, q) := 1 + (xq^{-1} - x^{-1}) \sum_{n=0}^{\infty} \frac{(-1; q)_nq^n}{(xq^{-1}, x^{-1}q; q^2)_n},$$

$$B(x, q) := 1 + (x - x^{-1}) \sum_{n=0}^{\infty} \frac{(q; q^2)_n(-q)^n}{(x, x^{-1}; q)_n},$$

$$C(x, q) := \frac{1}{x + 1} + \frac{x - 1}{x + 1} \sum_{n=0}^{\infty} \frac{(-q; q)_nq^{(n^2-n)/2}}{(x, x^{-1}; q)_n},$$

$$D(x, q) := -\frac{(1 + x)q^2}{1 - x^2q^2} - \frac{(1 + xq^2)(1 + x^{-1})}{1 + qx} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n(-q^2; q^2)_nq^{n+1}}{(xq; q^2)_n(x^{-1}; q^2)_n}.$$  \hspace{1cm} (1.6)

Then we establish the Appell–Lerch sum representations of these functions.

**Theorem 1.4.** We have

$$A(x, q) = 2m(x, q^2, q).$$

**Theorem 1.5.** We have

$$B(x, q) = 2m(-x, q, -1).$$

**Theorem 1.6.** We have

$$C(x, q) = m(x^2q, q^2, q) + \frac{xJ^2_{1,4}}{j(x^2; q^2)}.$$  \hspace{1cm} (1.6)

**Theorem 1.7.** We have

$$D(x, q) = m(x, q^2, q).$$
Moreover, we derive the following relations.

**Theorem 1.8.** We have

\[
A(x, q) - B(-x, q^2) = -\frac{J^2_{1,2}j(-xq; q^2)}{j(-x; q^2)j(xq; q^2)}, \quad (1.7)
\]

\[
A(x^2q, q) - 2C(x, q) = -\frac{2xJ^2_{1,4}}{j(x^2; q^2)}, \quad (1.8)
\]

\[
A(x, q) - 2D(x, q) = 0, \quad (1.9)
\]

\[
A(x^{-2}q, q) + 2xg_2(x, q) = \frac{2xJ^2_{1,4}}{j(x^2; q^2)}, \quad (1.10)
\]

\[
B(-x^{-2}q^2) + 2xg_2(x, q) = \frac{J^2_{2,4}j(-x; q)}{j(x; q)j(-x^2q; q^2)}, \quad (1.11)
\]

\[
C(x^{-1}, q) + xg_2(x, q) = 0, \quad (1.12)
\]

\[
D(x^{-2}q, q) + xg_2(x, q) = \frac{xJ^2_{1,4}}{j(x^2; q^2)}. \quad (1.13)
\]

Equations (1.10)-(1.13) imply that all the classical mock theta functions can be expressed by \(A(x, q), B(x, q), C(x, q),\) or \(D(x, q),\) respectively.

Finally, in view of an identity given by Liu [22, Theorem 6] and the properties of Appell–Lerch sums, we obtain the following radial limits.

**Theorem 1.9.** If \(\zeta\) is a primitive odd order \(2k + 1\) root of unity, then when \(q\) tends to \(\zeta\) radially within the unit disk, we have

\[
\lim_{q \to \zeta} \left( A(q^3, q^2) - 1 - \frac{T_{2,4}^2}{J_{1,4}} \right) = \zeta(1 - \zeta^2) \sum_{n=0}^{k} \frac{(\zeta^4; \zeta^8)_{n}(-1)^n\zeta^{4n}}{(-\zeta; \zeta^2)_{2n+2}}. \quad (1.14)
\]

**Theorem 1.10.** If \(\zeta\) is a primitive even order \(4k\) root of unity, then when \(q\) tends to \(\zeta\) radially within the unit disk, we have

\[
\lim_{q \to \zeta} \left( B(-1, q^2) - 1 - \frac{T_{2,4}^3}{2J_{1,4}^2} \right) = \frac{1 - \zeta^2}{\zeta} \sum_{n=0}^{k} \frac{(-1; \zeta)_{2n}\zeta^n}{(\zeta, \zeta^{-1}; \zeta^2)_{n+1}}. \quad (1.15)
\]

**Theorem 1.11.** If \(\zeta\) is a primitive odd order \(2k + 1\) root of unity, then when \(q\) tends to \(\zeta\) radially within the unit disk, we have

\[
\lim_{q \to \zeta} \left( C(q, q^2) - q - \frac{T_{2,4}^2T_{2,8}}{J_{2,4}J_{4,8}} \right) = \sum_{n=0}^{k} \frac{(\zeta^4; \zeta^8)_{n}(-1)^n\zeta^{4n^2}}{(-\zeta^4; \zeta^8)_{n+1}(-\zeta^4; \zeta^8)_{n}}. \quad (1.16)
\]

**Theorem 1.12.** If \(\zeta\) is a primitive odd order \(2k + 1\) root of unity, then when \(q\) tends to \(\zeta\) radially within the unit disk, we have

\[
\lim_{q \to \zeta} \left( D(1, q) + \frac{2q^2}{1-q^2} + 4q \frac{J_{1,4}^8}{J_{2}^8} \right) = \frac{2\zeta(1 + \zeta^2)^2}{1 - \zeta^2} \sum_{n=0}^{k} \frac{(\zeta, -\zeta^4; \zeta^2)_{n}(-1)^n\zeta^n}{(-\zeta, -\zeta^3; \zeta^2)_{n+1}}. \quad (1.17)
\]

This paper is organized as follows. In Section 2, we state some preliminaries. In Section 3, the proofs of Theorems 1.4-1.8 are established. In Section 4, we prove Theorems 1.9-1.12.
2. Preliminaries

The following identities are frequently used in this paper.

\[ j(x; q) = j(x^{-1}q; q), \]
\[ j(qx; q) = -x^{-1}j(x; q), \]
\[ j(x; q) J_{1,4} = j(x; q^2) j(xq; q^2). \]

In order to prove the main theorems, the following results are needed.

**Lemma 2.1.** [33, p.1] If \( z \) is not an integral power of \( q \), then

\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1 - zq^n} = \frac{J_1^3}{j(z; q)}. \tag{2.1}
\]

**Proposition 2.2.** [17, Proposition 3.1] For generic \( x, z \in \mathbb{C}^* \),

\[
m(x, q, z) = m(x, q, zq), \tag{2.2}
\]
\[
m(x, q, z) = x^{-1}m(x^{-1}, q, z^{-1}), \tag{2.3}
\]
\[
m(xq, q, z) = 1 - xm(x, q, z). \tag{2.4}
\]

Following [17], the term “generic” means that the parameters do not cause poles in the Appell–Lerch sums or in the quotients of theta functions.

**Corollary 2.3.** [17, Corollary 3.2] We have

\[ m(-1, q^2, q) = 0. \tag{2.5} \]

**Lemma 2.4.** [17, Theorem 3.3] For generic \( x, z_0, z_1 \in \mathbb{C}^* \),

\[
m(x, q, z_1) - m(x, q, z_0) = \frac{z_0 J_1^3 j(z_1/z_0; q) j(xz_0z_1; q)}{j(z_0; q) j(z_1; q) j(xz_0; q) j(xz_1; q)}. \tag{2.6}
\]

**Lemma 2.5.** [13, Appendix (III.17)] The Watson–Whipple transformation formula is given by

\[
s_7^\phi \left( \begin{array}{llllll}
a, & a^{1/2}q, & a^{-1/2}q, & b, & c, & d, & e, & f, & a^2q^2 \end{array} ; q, aq/bcdef \right) \\
\end{array} 
\right) 
= \frac{(aq, aq/de, aq/df, aq/ef; q)^\infty}{(aq/d, aq/e, aq/f, aq/def; q)^\infty} \phi_3 \left( \begin{array}{llllll}
aq/bc, & d, & c, & f, & aq/b, & aq/c, & def/a \end{array} ; q, q \right). 
\]

Letting \( f \to \infty \) in the above identity yields

\[
\sum_{n=0}^{\infty} \frac{1 - aq^{2n}}{1 - a} \frac{(aq, aq/de, aq/df, aq/ef; q)^\infty}{(aq/d, aq/e, aq/f, aq/def; q)^\infty} \phi_3 \left( \begin{array}{llllll}
aq/bc, & d, & c, & f, & aq/b, & aq/c, & def/a \end{array} ; q, q \right).
\]

**Lemma 2.6.** [22, Theorem 6] We have

\[
d \sum_{n=0}^{\infty} \frac{(q/bc, acdf; q)_n (bd)^n}{(ad, df; q)_{n+1}} - c \sum_{n=0}^{\infty} \frac{(q/bd, acdf; q)_n (bc)^n}{(ac, cf; q)_{n+1}} = d \frac{(q/dc, c/d, abcd, acdf, bcdf; q)^\infty}{(ac, ad, cf, df, bc, bd; q)^\infty}. \tag{2.8}
\]
In this section, we establish the Appell–Lerch sums of $A(x, q)$, $B(x, q)$, $C(x, q)$, and $D(x, q)$.

**Proof of Theorem 1.4.** Setting $a = 1$, $b = x^{-1}q$, $c = xq^{-1}$, $d = -1$, $e = -q$, and $q \to q^2$ in (2.7), and then dividing both sides by $(1 - xq^{-1})(1 - x^{-1}q)$, we obtain

\[
\sum_{n=0}^{\infty} \frac{(-1; q)_{2n}q^n}{(xq^{-1}, x^{-1}q; q^2)_{n+1}} = \frac{1}{J_{1, 2}} \left( \frac{1}{(1 - xq^{-1})(1 - x^{-1}q)} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(1 - xq^{2n-1})(1 - x^{-1}q^{2n+1})} \right)
\]

\[
= \frac{1}{J_{1, 2}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(1 - xq^{2n-1})(1 - x^{-1}q^{2n+1})}
\]

\[
= \frac{1}{(xq^{-1} - x^{-1}q)J_{1, 2}} \sum_{n=0}^{\infty} \left( \frac{(-1)^n xq^{n^2+2n-1}}{1 - xq^{2n-1}} - \frac{(-1)^n x^{-1}q^{n^2+2n+1}}{1 - x^{-1}q^{2n+1}} \right)
\]

\[
= -xq^{-2}m(xq^{-2}, q^2, q) \frac{x^{-1}m(x^{-1}, q^2)}{x^{-1} - xq^{-1}},
\]

where the last step follows from (1.1). Then applying (2.2) and (2.3) yields that

\[
m(x^{-1}, q^2, q) = x^{-1}m(x, q^2, q^{-1}) = x^{-1}m(x, q^2, q).
\]

In addition, in view of (2.4), we have

\[
-xq^{-2}m(xq^{-2}, q^2, q) = m(x, q^2, q) - 1.
\]

Finally, substituting (3.2) and (3.3) into (3.1), we complete the proof.

**Proof of Theorem 1.5.** Substituting $a = q$, $b = x^{-1}q$, $c = x$, $d = q^{1/2}$, and $e = -q^{1/2}$ into (2.7), we deduce

\[
\sum_{n=0}^{\infty} \frac{(q; q^2)_n(-q)^n}{(x, x^{-1}q; q)_{n+1}} = \frac{2}{J_{0, 1}} \sum_{n=0}^{\infty} \frac{q^{n^2+3n/2}}{(1 - xq^n)(1 - x^{-1}q^{n+1})}
\]

\[
= \frac{1}{J_{0, 1}} \sum_{n=-\infty}^{\infty} \frac{q^{n^2+3n/2}}{(1 - xq^n)(1 - x^{-1}q^{n+1})}
\]

\[
= \frac{1}{(x^{-1}q - x)J_{0, 1}} \left( \sum_{n=-\infty}^{\infty} \frac{x^{-1}q^{n^2+3n+2}/2}{1 - x^{-1}q^{n+1}} - \sum_{n=-\infty}^{\infty} \frac{xq^{n^2+3n}/2}{1 - xq^n} \right)
\]

\[
= \frac{1}{(x^{-1}q - x)J_{0, 1}} \left( \sum_{n=-\infty}^{\infty} \frac{q^{n^2+n}/2}{1 - xq^n} - \sum_{n=-\infty}^{\infty} \frac{xq^{n^2+3n}/2}{1 - xq^n} \right)
\]

\[
= -\frac{m(-x, q, -1)}{x^{-1}q - x} - \frac{xq^{-1}m(-xq^{-1}, q, -q)}{x^{-1}q - x} \quad \text{(by (1.1))}
\]

\[
= -\frac{m(-x, q, -1)}{x^{-1}q - x} - \frac{xq^{-1}m(-xq^{-1}, q, -1)}{x^{-1}q - x} \quad \text{(by (2.2))}
\]

\[
= \frac{m(-x, q, -1)}{x^{-1}q - x} - \frac{m(-x, q, -1)}{x^{-1}q - x} + \frac{1}{x^{-1}q - x},
\]

where the last step follows from (2.4). Hence, we complete the proof. □
Proof of Theorem 1.6. Letting \( a = 1, b = x, c = x^{-1}, d = -q, \) and \( e \to \infty \) in (2.7), we derive

\[
\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{(n^2-n)/2}}{(x, x^{-1}; q)_{n+1}} = \frac{2}{J_{1,2}} \left( \frac{1}{(1-x)(1-x^{-1})} + \sum_{n=1}^{\infty} \frac{(1+q^n)^2(-1)^n q^{n^2}}{2(1-xq^n)(1-x^{-1}q^n)} \right)
\]

\[
= \frac{1}{2J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(1+q^n)^2(-1)^n q^{n^2}}{(1-xq^n)(1-x^{-1}q^n)}
\]

\[
= \frac{1}{2(1-x)J_{1,2}} \sum_{n=-\infty}^{\infty} \left( \frac{(1+q^n)(-1)^n q^{n^2}}{1-x^{-1}q^n} - \frac{(1+q^n)(-1)^n xq^n}{1-xq^n} \right)
\]

\[
= \frac{-x}{(1-x)J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(1+q^n)(-1)^n q^{n^2}}{1-x^{-2}q^{2n}}
\]

\[
= \frac{-x}{(1-x)J_{1,2}} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2}}{1-x^{-2}q^{2n}} + x \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1-x^{-2}q^{2n}} + (1+x) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1-x^{-2}q^{2n}} \right)
\]

\[
= \frac{-x}{1-x} \left( (m(x^2q, q^2, q^{-1}) - xq^{-1}m(x^2q^{-1}, q^2, q)) - \frac{x(1+x)}{1-x} \frac{J_{1,4}^2}{j(x^2; q^2)} \right)
\]

\[
= \frac{-x}{1-x} \left( (m(x^2q, q^2, q) - x^{-1}m(x^2q, q^2, q)) - \frac{x(1+x)}{1-x} \frac{J_{1,4}^2}{j(x^2; q^2)} \right)
\]

\[
= \frac{-(1+x)}{1-x} m(x^2q, q^2, q) + \frac{1}{1-x} - \frac{x(1+x)}{1-x} \frac{J_{1,4}^2}{j(x^2; q^2)}
\]

where we obtain the last third step by utilizing (1.1) and (2.1), and the penultimate step follows from (2.2) and (2.4). Therefore, we complete the proof. \( \square \)

Proof of Theorem 1.7. Setting \( a = q^4, b = xq^2, c = x^{-1}q^2, d = -q, e = -q^4, \) and \( q \to q^2 \) in (2.7), we have

\[
\sum_{n=0}^{\infty} \frac{(-q, -q^4; q^2)_n q^n}{(xq^2, x^{-1}q^2; q^2)_{n+1}} = \frac{1}{(1+q^2)J_{1,2}} \sum_{n=0}^{\infty} \frac{(1-q^{4n+4})(-1)^n q^{n^2+2n}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+1})(1+q^{2n+3})}
\]

\[
= \frac{1}{2(1+q^2)J_{1,2}} \left( \sum_{n=0}^{\infty} \frac{(1-q^{4n+4})(-1)^n q^{n^2+2n}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+1})(1+q^{2n+3})} \right)
\]

\[
= \frac{1}{2(1+q^2)J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(1-q^{4n+4})(-1)^n q^{n^2+2n}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+1})}
\]
\[-\frac{1}{2(1+q^2)} \left( \sum_{n=-\infty}^{\infty} \frac{(1-q^{4n+4})(-1)^n q^{n^2+4n+3}}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})(1+q^{2n+3})} \right) \]
\[= \frac{1}{(1+q^2)} \sum_{n=-\infty}^{\infty} \left( \frac{(-1)^n q^{n^2+2n}}{(1-x^{-1}q^{2n+2})(1+q^{2n+1})} + \frac{(-1)^n xq^{n^2+4n+2}}{(1-xq^{2n+2})(1+q^{2n+1})} \right) \]
\[= \frac{x^{-1}q}{(1+q^2)(1+x^{-1}q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1-x^{-1}q^{2n+2}} + \frac{1}{(1+q^2)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{2n+1}} \]
\[-\frac{x^2q^3}{(1+q^2)(1+xq)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}}{1-xq^{2n+2}} + \frac{xq^2}{(1+q^2)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}}{1+q^{2n+1}} \]
\[-\frac{(1-x^2)q^3}{(1+q^2)(1+xq)(1-x^{-1}q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}}{1-xq^{2n+2}} \]
\[= -\frac{1}{(1+q^2)(1+x^{-1}q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{2n+1}} \]
\[= \frac{1}{(1+q^2)(1+x^{-1}q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{2n+1}} \]
\[= -\frac{1}{(1+q^2)(1+x^{-1}q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{2n+1}} \]
\[\sum_{n=0}^{\infty} \frac{(-q, q^4; q^2)_n q^n}{(x^2, x^{-1}q^2; q^2)_{n+1}} = -\frac{1}{(1+q^2)(1+xq)(1+x^{-1}q)} m(q^{-1}, q^2, q^3) \]
\[= \frac{1}{(1+q^2)(1+xq)(1+x^{-1}q)} m(-q^{-2}, q^2, q^3), \]
\[m(-q^{-2}, q^2, q^3) = -q^2 + q^2 m(-1, q^2, q^3) = -q^2 + q^2 m(-1, q^2, q). \]

where the last step follows from (1.1). Notice that we use the technique \( n \to -n - 2 \) several times in the above equalities.

Next, based on (2.2) and (2.4), we obtain
\[m(-q^{-2}, q^2, q^3) = -q^2 + q^2 m(-1, q^2, q^3) = -q^2 + q^2 m(-1, q^2, q). \]

Then substituting (3.5) into (3.4) and applying (2.2) and (2.5), we derive
\[\sum_{n=0}^{\infty} \frac{(-q, q^4; q^2)_n q^n}{(x^2, x^{-1}q^2; q^2)_{n+1}} = -\frac{1}{(1+q^2)(1+xq)(1+x^{-1}q)} m(q^{-1}, q^2, q) - \frac{x}{(1+q^2)(1+xq)}. \]

Finally, replacing \( x \) by \( xq \) in the above identity yields
\[\sum_{n=0}^{\infty} \frac{(-q, -q^4; q^2)_n q^n}{(x^2, x^{-1}q^2; q^2)_{n+1}} = -\frac{1}{(1+q^2)(1+xq)(1+x^{-1}q)} m(x, q^2, q) - \frac{xq}{(1+q^2)(1+xq^2)} \]
which implies the theorem. \( \Box \)

**Proof of Theorem 1.8.** In view of Theorems 1.4 and 1.5, we have
\[A(x, q) - B(-x, q^2) = 2m(x, q^2, q) - 2m(x, q^2, -1). \]

Then using (2.6) in the above identity, we complete the proof of (1.7).
The proofs of (1.8)-(1.13) are similar to that of (1.7).

4. Proofs of Theorems 1.9-1.12

In this section, using (2.8) given by Liu [22, Theorem 6] and some properties of Appell–Lerch sums, we establish some radial limits.

Proof of Theorem 1.9. Setting \( a = q, b = q^2, c = -q^2, d = 1, f = q^{-1}, \) and \( q \to q^4 \) in (2.8), we deduce

\[
\sum_{n=0}^{\infty} \frac{(-1; q^2)_2nq^{2n}2n}{(q, q^{-1}; q^4)_{n+1}} + q^2 \sum_{n=0}^{\infty} \frac{(q^4; q^8)_n(-1)^nq^{4n}}{(-q; q^2)_{2n+2}} = -\frac{q}{1 - q^2} J_{2,4}^2.
\]

If \( \zeta \) is a primitive order \( 2k + 1 \) root of unity, then when \( q \) tends to \( \zeta \) radially within the unit disk,

\[
\lim_{q \to \zeta} \left( \sum_{n=0}^{\infty} \frac{(-1; q^2)_2nq^{2n}2n}{(q, q^{-1}; q^4)_{n+1}} + \frac{q}{1 - q^2} J_{2,4}^2 \right) = -\zeta^2 \sum_{n=0}^{k} \frac{(\zeta^4; \zeta^8)_n(-1)^n\zeta^{4n}}{(-\zeta; \zeta^2)_{2n+2}}.
\]

Therefore, combining (1.5) and the above identity, we complete the proof.

Proof of Theorem 1.10. Letting \( a = -1, b = -q^2, c = -q^{-1}, d = 1, f = -q^2, \) and \( q \to q^2 \) in (2.8), we derive

\[
\sum_{n=0}^{\infty} \frac{(q^2; q^4)_n(-1)^nnq^{2n}}{(-1, -q^2; q^2)_{n+1}} + q^{-1} \sum_{n=0}^{\infty} \frac{(-1; q^2)_2nq^n}{(q, q^{-1}; q^2)_{n+1}} = -\frac{J_{1,2}^3}{2(1 - q^2)J_{4,4}^2}.
\]

Namely,

\[
B(-1, q^2) - 1 - \frac{J_{1,2}^3}{2J_{4,4}^2} = \frac{1 - q^2}{q} \sum_{n=0}^{\infty} \frac{(-1; q^2)_2nq^n}{(q, q^{-1}; q^2)_{n+1}}.
\]

Hence, if \( \zeta \) is a primitive order \( 4k \) root of unity, then we obtain the theorem by letting \( q \) tend to \( \zeta \) radially within the unit disk.

Proof of Theorem 1.11. In view of Theorem 1.6, we deduce

\[
C(q, q^2) = m(q^4, q^4, q^2) + \frac{J_{2,4}^2}{J_{2,4}^2}.
\] (4.1)

Then applying (1.3) with \( x = q^4 \) and \( q \to q^2 \) yields

\[
\sum_{n=0}^{\infty} \frac{(q^4; q^8)_n(-1)^nnq^{4n^2}}{(-q^4; q^8)_{n+1}(-q^4; q^8)_n} = m(q^4, q^4, q^2) - q^2 \frac{J_{0,8}^2}{J_{2,4}^2 J_{4,4}^2}.
\] (4.2)

Combining (4.1) and (4.2), we derive

\[
\sum_{n=0}^{\infty} \frac{(q^4; q^8)_n(-1)^nnq^{4n^2}}{(-q^4; q^8)_{n+1}(-q^4; q^8)_n} = C(q, q^2) - \frac{J_{2,8}^2}{J_{2,4}^2 J_{4,4}^2} - q^2 \frac{J_{0,8}^2}{J_{2,4}^2 J_{4,4}^2}.
\]

\[
= C(q, q^2) - \frac{J_{2,8}^2 + qJ_{0,8}}{J_{2,4}^2 J_{4,4}^2}.
\]

\[
= C(q, q^2) - \frac{J_{1,4}^2 J_{2,8}}{J_{2,4}^2 J_{4,4}^2}.
\]
where we use the following identity [17] with $x = -q$, $y = -q^3$, and $q \to q^4$ to obtain the third equality.

$$j(x; q)j(y; q) = j(-xy; q^2)j(-qx y^{-1}; q^2) - xj(-qxy; q^2)j(-x^{-1}y; q^2).$$

Thus, if $\zeta$ is a primitive order $2k + 1$ root of unity, then when $q$ tends to $\zeta$ radially within the unit disk, we prove the theorem.

**Proof of Theorem 1.12.** Substituting $a = q^3$, $b = q$, $c = -1$, $d = 1$, $f = q$, and $q \to q^2$ in (2.8) yields

$$\sum_{n=0}^{\infty} \frac{(-q, -q^4; q^2)_n q^n}{(q, q^3; q^2)_{n+1}} + \frac{\sum_{n=0}^{\infty} (q, -q^4; q^2)_n (-1)^n q^n}{(q, q^3; q^2)_{n+1}} = \frac{2(1-q^2) J_4^8}{(1+q^2)^2 J_2^4}. \quad (4.3)$$

In addition, in view of (1.6), we have

$$D(1, q) = -\frac{2q^2}{1-q^2} - \frac{2q(1+q^2)^2}{1-q^2} \sum_{n=0}^{\infty} \frac{(-q, -q^4; q^2)_n q^n}{(q, q^3; q^2)_{n+1}}. \quad (4.4)$$

Then combining (4.3) and (4.4), we derive

$$D(1, q) + \frac{2q^2}{1-q^2} + \frac{4q J_2^8}{J_2^4} = \frac{2q(1+q^2)^2}{1-q^2} \sum_{n=0}^{\infty} \frac{(q, -q^4; q^2)_n (-1)^n q^n}{(q, q^3; q^2)_{n+1}}.$$}

Thus, if $\zeta$ is a primitive order $2k + 1$ root of unity, then letting $q$ tend to $\zeta$ radially within the unit disk, we complete the proof.

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