Extremal numbers of disjoint triangles in r-partite graphs

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Abstract

For two graphs G and F, the extremal number of F in G, denoted by $\mathrm{ex}(G, F)$, is the maximum number of edges in a spanning subgraph of G not containing F as a subgraph. Determining $\mathrm{ex}(K_n, F)$ for a given graph F is a classical extremal problem in graph theory. In 1962, Erdős determined $\mathrm{ex}(K_n, kK_3)$, which generalized Mantel's Theorem. On the other hand, in 1974, Bollobás, Erdős, and Straus determined $\mathrm{ex}(K_{n_1,n_2,\dots,n_r}, K_t)$, which extended Turán's Theorem to complete multipartite graphs. In this paper, we determine $\mathrm{ex}(K_{n_1,n_2,\dots,n_r}, kK_3)$ for $r \geq 4$ and $10k - 4 \leq n_1 + 4k \leq n_2 \leq n_3 \leq \cdots \leq n_r$.
Keywords: extremal numbers; multipartite graphs; disjoint triangles

1 Introduction

All graphs considered in this paper are finite and simple. We follow [12] for undefined notation and terminology. Let $G = G(V(G), E(G))$ be a graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. Denote $e(G) = |E(G)|$. For any subset $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S and let $G - S = G[V(G) \setminus S]$. Given two disjoint vertex sets V_1 and V_2, the *join* $V_1 \vee V_2$ is the edge set obtained by joining each vertex of V_1 to each vertex of V_2. Furthermore, given two graphs G and H, the *join* $G \vee H$ is the graph obtained from the disjoint union of G and H by joining each vertex of G to each vertex of H. Let r, t, k be three positive integers. For convenience, we write $[r] = \{1, 2, 3, ..., r\}$ in the context. Denote by K_t the complete graph on t vertices and kK_t the disjoint union of k copies of K_t. Moreover, for r positive integers $n_1, n_2, \dots, n_r$, we use $K_{n_1,n_2,\dots,n_r}$ to denote the complete r-partite graph with parts of sizes $n_1, n_2, \dots, n_r$.

Let $\mathrm{ex}(G, H) = \max\{e(G') : |V(G')| = |V(G)|,\ G' \subseteq G, \text{ and } H \nsubseteq G'\}$ and call it the *extremal number* of H in G. Let n and t be two integers with $n \geq t$. The classical Turán problem considers the case $G = K_n$, i.e., determining the value of $\mathrm{ex}(K_n, H)$ for a

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given H. For instance, the well-known Mantel's Theorem [8] and Turán's Theorem [11] determined $\mathrm{ex}(K_n, K_3)$ and $\mathrm{ex}(K_n, K_t)$, respectively. Turán's Theorem is widely considered to be the first extremal theorem on graphs and initiated the study of extremal graph theory. Let $T_r(n)$ be a balanced complete r-partite graph on n vertices, that is, any two parts of sizes differ by at most one. In 1959, Erdős and Gallai [6] determined $\mathrm{ex}(K_n, kK_2)$ for any positive integers n and k. Later, Erdős [5] proved $\mathrm{ex}(K_n, kK_3) = e(K_{k-1} \vee T_2(n-k+1))$ for $n > 400(k-1)^2$ and Moon [9] proved that for $n > \frac{9k}{2} + 4$. Moreover, Moon [9] (only when $n-k+1$ is divisible by p) and Simonovits [10] showed that $K_{k-1} \vee T_{r-1}(n-k+1)$ is the unique extremal graph containing no copy of kK_r for sufficiently large n. The extremal problem on multipartite graphs was first considered by Bollobás, Erdős, and Straus [1]. They determined $\mathrm{ex}(K_{n_1,n_2,\ldots,n_r}, K_t)$ for $r \geq t \geq 2$. Later, Chen, Li, and Tu [2] determined $\mathrm{ex}(K_{n_1,n_2}, kK_2)$. Recently, De Silva, Heysse and Young [4] proved $\mathrm{ex}(K_{n_1,n_2,\ldots,n_r}, kK_2) = (k-1)(n_2 + n_3 + \cdots + n_r)$ for $n_1 \leq n_2 \leq \cdots \leq n_r$.

To make it easier to state the results in this paper, we define a function f. Let $x_1, x_2, \ldots, x_r$ be positive integers and $t \geq 2$ be an integer. Note that $\{x_1, x_2, \ldots, x_r\}$ is a multiset. For any subset $P \subseteq [r]$, let $x_P := \sum_{i \in P} x_i$. Now, we define the desired function f as follows. For $t = 2$, let $f_2(\{x_1, x_2, \ldots, x_r\}) = 0$. For $t \geq 3$, let $f_t(\{x_1, x_2, \ldots, x_r\}) = \max_{\mathcal{P}} \{\sum_{1 \leq i < j \leq t-1} x_{P_i} \cdot x_{P_j}\}$, where $\mathcal{P} = (P_1, P_2, \ldots, P_{t-1})$ is a partition of $[r]$ into $t-1$ nonempty parts. For convenience, we simply write $f_t(x_1, x_2, \ldots, x_r)$ for $f_t(\{x_1, x_2, \ldots, x_r\})$. Notice that $f_r(x_1, x_2, \ldots, x_r) = (\sum_{1 \leq i < j \leq r} x_i x_j) - x_1 x_2$ if $x_1 \leq x_2 \leq \cdots \leq x_r$.

Theorem 1.1 [1] *Let $n_1, n_2, \ldots, n_r, t$ be positive integers with $r \geq t \geq 2$. Then $ex(K_{n_1,n_2,\ldots,n_r}, K_t) = f_t(n_1, n_2, \ldots, n_r)$.*

By replacing the forbidden graph K_t with kK_t, De Silva et al. [3] considered a special case $t = r$ and raised the problem about $\mathrm{ex}(K_{n_1,n_2,\ldots,n_r}, kK_t)$ for $r > t$.

Theorem 1.2 [3] *For any integers $r \geq 2$ and $1 \leq k \leq n_1 \leq n_2 \leq \cdots \leq n_r$, we have*

$$\begin{aligned} ex(K_{n_1,n_2,\ldots,n_r}, kK_r) &= \sum_{1 \leq i < j \leq r} n_i n_j - n_1 n_2 + (k-1) n_2 \\ &= (k-1)(n - n_1) + f_r(n_1 - (k-1), n_2, \ldots, n_r), \end{aligned}$$

where $n = \sum n_i$.

Problem 1.3 [3] *Determine $ex(K_{n_1,n_2,\ldots,n_r}, kK_t)$ for $r > t$.*

Han and Zhao [7] determined $\mathrm{ex}(K_{n_1,n_2,n_3,n_4}, kK_3)$ for a sufficiently large integer n_1 with $n_1 \leq n_2 \leq n_3 \leq n_4$.

Theorem 1.4 [7] *Let $n_1, \dots, n_4$ be sufficiently large and $n_1 \le n_2 \le n_3 \le n_4$. For any integer $k \ge 1$, we have*

$$\begin{aligned} ex(K_{n_1,n_2,n_3,n_4}, kK_3) &= (k-1)(n-n_1) + f_3(n_1-(k-1), n_2, n_3, n_4) \\ &= \begin{cases} n_4(n_1+n_2+n_3)+(k-1)(n_2+n_3), & \text{if } n_4 > n_2+n_3; \\ (n_1+n_4)(n_2+n_3)+(k-1)n_4, & \text{if } n_4 \le n_2+n_3. \end{cases} \end{aligned}$$

In addition, Han and Zhao [7] proposed a conjecture on $\mathrm{ex}(K_{n_1,n_2,\dots,n_r}, kK_t)$.

Conjecture 1.5 [7] *Given three integers k, r, and t with $r > t \ge 3$ and $k \ge 2$, let $n_1, \dots, n_r$ be sufficiently large. For $I \subseteq [r]$, write $m_I := \min_{i\in I} n_i$. Given a partition $\mathcal{P}$ of $[r]$, let $n_{\mathcal{P}} := \max_{I\in\mathcal{P}}\{n_I - m_I\}$. Then*

$$ex(K_{n_1,n_2,\dots,n_r}, kK_t) = \max_{\mathcal{P}} \left\{ (k-1)n_{\mathcal{P}} + \sum_{I\neq I'\in\mathcal{P}} n_I \cdot n_{I'} \right\},$$

where the maximum is taken over all partitions $\mathcal{P}$ of $[r]$ into $t-1$ parts.

Actually, we can obtain an equivalent statement of Conjecture 1.5 as follows, equivalence will be proved in Section 2.

Conjecture 1.6 *Given three integers k, r, and t with $r > t \ge 3$ and $k \ge 2$, let $n_1 \le n_2 \cdots \le n_r$ be integers with $\sum_{i=1}^{r} n_i = n$. If n_1 is sufficiently large, then $ex(K_{n_1,n_2,\dots,n_r}, kK_t) = (k-1)(n-n_1) + f_t(n_1-(k-1), n_2, \dots, n_r)$.*

To support Conjecture 1.6, we provide a simple construction, which gives a lower bound for $\mathrm{ex}(K_{n_1,n_2,\dots,n_r}, kK_t)$.

Theorem 1.7 *Let k and t be two integers with $k \ge 2$ and $t \ge 3$. Let $n_1, n_2, \dots, n_r$ be r integers with $n_1 \le \min_{i\ge 2}\{n_i\}$, $n_1 \ge k$, $r \ge t$, and $\sum_{i=1}^{r} n_i = n$. Then $ex(K_{n_1,n_2,\dots,n_r}, kK_t) \ge (k-1)(n-n_1) + f_t(n_1-(k-1), n_2, \dots, n_r)$.*

Furthermore, with Theorem 1.7, we confirm Conjecture 1.6 for $t = 3$, $r \ge 4$, and $n_1 + 4k \le n_2$.

Theorem 1.8 *Let $r, n_1, n_2, \dots, n_r$ be integers with $r \ge 4$, $10k-4 \le n_1 + 4k \le \min_{i\ge 2}\{n_i\}$ and $\sum_{i=1}^{r} n_i = n$. Then $ex(K_{n_1,n_2,\dots,n_r}, kK_3) = (k-1)(n-n_1) + f_3(n_1-(k-1), n_2, \dots, n_r)$.*

In Conjecture 1.6, it is required that "n_1 is sufficiently large". But how large it can be? By aforementioned results and Theorem 1.8, we conjecture that the lower bound for n_1 is k.

Conjecture 1.9 *Given three integers k, r, and t with $r \geq t \geq 2$ and $k \geq 1$, let $n_1, n_2, \ldots, n_r$ be integers with $n_1 \leq \min_{i\geq 2}\{n_i\}$ and $\sum_{i=1}^r n_i = n$. If $n_1 \geq k$, then $ex(K_{n_1,n_2,\ldots,n_r}, kK_t) = (k-1)(n-n_1) + f_t(n_1-(k-1), n_2, \ldots, n_r)$.*

The rest of this paper is organized as follows. In Section 2, we first prove the equivalence of Conjectures 1.5 and 1.6. Later, some basic properties of $f_3(n_1, n_2, \ldots, n_r)$ are provided, which will be used frequently in the proof of Theorem 1.8. The proofs of Theorems 1.7 and 1.8 are presented in Sections 3 and 4, respectively.

2 Equivalence and Properties of $f_3(n_1, n_2, \ldots, n_r)$

Proposition 2.1 *Conjecture 1.5 and Conjecture 1.6 are equivalent.*

Proof. Let $r > t \geq 3$, $k \geq 2$, $n_1 \leq n_2 \cdots \leq n_r$. Our goal is to prove

$$\max_{\mathcal{P}} \left\{ (k-1)n_{\mathcal{P}} + \sum_{I \neq I' \in \mathcal{P}} n_I \cdot n_{I'} \right\} = (k-1)(n-n_1) + f_t(n_1-(k-1), n_2, \ldots, n_r). \quad (1)$$

First we show the $\geq$ direction of (1). Let $x_1 = n_1 - (k-1)$, $x_i = n_i$ for any $i \geq 2$, and $\mathcal{P}_0 = (P_1, P_2, \ldots, P_{t-1})$ be a partition of $[r]$ maximizing $f_t(x_1, x_2, \ldots, x_r)$. Assume $1 \in P_1$. Hence

$$\begin{aligned}
&(k-1)(n-n_1) + f_t(n_1-(k-1), n_2, \ldots, n_r) \\
&= (k-1)(n_{P_1} - n_1) + (k-1)(n - n_{P_1}) + \sum_{i \neq j \in [t-1]} x_{P_i} \cdot x_{P_j} \\
&\leq (k-1)n_{\mathcal{P}_0} + \sum_{i \neq j \in [t-1]} n_{P_i} \cdot n_{P_j} \\
&\leq \max_{\mathcal{P}} \left\{ (k-1)n_{\mathcal{P}} + \sum_{I \neq I' \in \mathcal{P}} n_I \cdot n_{I'} \right\}.
\end{aligned}$$

Now we prove the $\leq$ direction of (1). For any given partition $\mathcal{P} = (P_1, P_2, \ldots, P_{t-1})$ of $[r]$, let $\ell \in [t-1]$ such that $n_{\mathcal{P}} = n_{P_\ell} - m_{P_\ell}$ and $\alpha \in P_\ell$ such that $n_\alpha = m_{P_\ell}$.

Notice that

$$\begin{aligned}
&(k-1)n_{\mathcal{P}} + \sum_{I \neq I' \in \mathcal{P}} n_I \cdot n_{I'} \\
&= (k-1)(n_{P_\ell} - m_{P_\ell}) + n_{P_\ell} \cdot (n - n_{P_\ell}) + \sum_{I \neq I' \in \mathcal{P} \setminus \{P_\ell\}} n_I \cdot n_{I'} \\
&= (k-1)(n - m_{P_\ell}) + (n_{P_\ell} - (k-1)) \cdot (n - n_{P_\ell}) + \sum_{I \neq I' \in \mathcal{P} \setminus \{P_\ell\}} n_I \cdot n_{I'} \\
&\leq (k-1)(n - n_\alpha) + f_t(\{n_1, \cdots, n_r\} \cup \{n_\alpha - (k-1)\} \setminus \{n_\alpha\}).
\end{aligned}$$

Next we assume that $\mathcal{P}$ be the partition maximizing the value $(k-1)n_{\mathcal{P}} + \sum_{I \neq I' \in \mathcal{P}} n_I \cdot n_{I'}$. It remains to show

$$\begin{aligned} &(k-1)(n-n_\alpha) + f_t(\{n_1, \cdots, n_r\} \cup \{n_\alpha - (k-1)\} \setminus \{n_\alpha\}) \\ &\leq (k-1)(n-n_1) + f_t(\{n_1, \cdots, n_r\} \cup \{n_1 - (k-1)\} \setminus \{n_1\}). \end{aligned} \tag{2}$$

If $n_\alpha = n_1$, then we are done. We may assume that $n_\alpha > n_1$. We can see $1 \notin P_\ell$. We may assume $1 \in P_1$ and $\ell \neq 1$. Now note that $m_{P_1} = n_1$ and $m_{P_\ell} = n_\alpha$. Then $n_{P_\ell} - n_\alpha > n_{P_1} - n_1$ by $\ell \neq 1$. Since $n_\alpha > n_1$ and $n_{P_\ell} - n_\alpha > n_{P_1} - n_1$, we can switch 1 and α in the partition $\mathcal{P}$ and thus increases $(k-1)n_{\mathcal{P}} + \sum_{I \neq I' \in \mathcal{P}} n_I \cdot n_{I'}$, a contradiction. Thus the Inequation (2) holds.

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Next we provided some properties of the function $f_3(n_1, n_2, \ldots, n_r)$.

Proposition 2.2 *Let $n_1, n_2, \ldots, n_r, \mu$ be positive integers with $\mu \in [r]$ and $n_\mu - 1 \geq n_1 + 1$. Then $f_3(\{n_1, n_2, \ldots, n_r\} \setminus \{n_1, n_\mu\} \cup \{n_1 + 1, n_\mu - 1\}) \leq f_3(n_1, n_2, \ldots, n_r) + n_\mu - (n_1 + 1)$.*

Proof. Let $x_1 = n_1 + 1$, $x_\mu = n_\mu - 1$, and $x_j = n_j$ for any $j \in [r] \setminus \{1, \mu\}$. Then $f_3(\{n_1, n_2, \ldots, n_r\} \setminus \{n_1, n_\mu\} \cup \{n_1 + 1, n_\mu - 1\}) = f_3(x_1, x_2, \ldots, x_r)$. Let $\mathcal{P} = (P_1, P_2)$ be the partition of $[r]$ maximizing $f_3(x_1, x_2, ..., x_r)$. Note that $\sum_{j=1}^r x_j = \sum_{j=1}^r n_j$. If $\{1, \mu\} \subseteq P_1$ or $\{1, \mu\} \subseteq P_2$, then $x_{P_1} = n_{P_1}$ and $x_{P_2} = n_{P_2}$. Thus, $f_3(x_1, x_2, \ldots, x_r) = x_{P_1} \cdot x_{P_2} = n_{P_1} \cdot n_{P_2} \leq f_3(n_1, n_2, \ldots, n_r)$. Hence, without loss of generality, we may assume that $1 \in P_1$ and $\mu \in P_2$. Observe that $x_{P_1} = n_{P_1} + 1$, $x_{P_2} = n_{P_2} - 1$, $x_{P_1 \setminus \{1\}} = n_{P_1 \setminus \{1\}}$, and $x_{P_2 \setminus \{\mu\}} = n_{P_2 \setminus \{\mu\}}$. If $x_{P_1 \setminus \{1\}} \geq x_{P_2 \setminus \{\mu\}}$, then $n_{P_1 \setminus \{1\}} \geq n_{P_2 \setminus \{\mu\}}$, and we have

$$\begin{aligned} f_3(x_1, x_2, \ldots, x_r) =& x_{P_1} \cdot x_{P_2} \\ =& (n_{P_1} + 1) \cdot (n_{P_2} - 1) \\ =& n_{P_1} \cdot n_{P_2} - n_{P_1} + n_{P_2} - 1 \\ \leq& f_3(n_1, n_2, \ldots, n_r) - (n_{P_1 \setminus \{1\}} + n_1) + (n_{P_2 \setminus \{\mu\}} + n_\mu) - 1 \\ \leq& f_3(n_1, n_2, \ldots, n_r) + n_\mu - (n_1 + 1). \end{aligned}$$

Now $x_{P_1 \setminus \{1\}} < x_{P_2 \setminus \{\mu\}}$, equivalently, $n_{P_1 \setminus \{1\}} < n_{P_2 \setminus \{\mu\}}$. Recall that $x_1 = n_1 + 1 \leq n_\mu - 1 = x_\mu$. If $n_1 + 1 < n_\mu - 1$, then $|(x_{P_1 \setminus \{1\}} + x_\mu) - (x_{P_2 \setminus \{\mu\}} + x_1)| < |(x_{P_1 \setminus \{1\}} + x_1) - (x_{P_2 \setminus \{\mu\}} + x_\mu)| = |x_{P_1} - x_{P_2}|$, which contradicts the choice of $\mathcal{P}$ that minimizes $|x_{P_1} - x_{P_2}|$, i.e., maximizes $x_{P_1} \cdot x_{P_2}$. Thus, we obtain $n_1 + 1 = n_\mu - 1$. This implies

$x_{P_1} \cdot x_{P_2} = (x_{P_1\setminus\{1\}} + x_\mu) \cdot (x_{P_2\setminus\{\mu\}} + x_1)$. Then,

$$\begin{aligned} f_3(x_1, x_2, \ldots, x_r) =& x_{P_1} \cdot x_{P_2} \\ =& (x_{P_1\setminus\{1\}} + x_\mu) \cdot (x_{P_2\setminus\{\mu\}} + x_1) \\ =& (n_{P_1\setminus\{1\}} + n_\mu - 1) \cdot (n_{P_2\setminus\{\mu\}} + n_1 + 1) \\ =& (n_{P_1\setminus\{1\}} + n_\mu) \cdot (n_{P_2\setminus\{\mu\}} + n_1) + (n_{P_1\setminus\{1\}} + n_\mu) - (n_{P_2\setminus\{\mu\}} + n_1) - 1 \\ <& f_3(n_1, n_2, \ldots, n_r) + n_\mu - (n_1 + 1). \end{aligned}$$

■

Proposition 2.3 *Let $n_1, n_2, \ldots, n_r$ be r integers with $n_1 + 2 \leq \min_{i\geq 2}\{n_i\}$. For two indices $i, j \in [r]$ with $i \neq j$, let $\mathcal{P} = (P_1, P_2)$ be the partition attaining $f_3(\{n_1, \ldots, n_r\} \setminus \{n_i, n_j\} \cup \{n_i - 1, n_j - 1\})$. Then*

$$\begin{aligned} f_3(\{n_1, \ldots, n_r\} \setminus \{n_i, n_j\} \cup \{n_i - 1, n_j - 1\}) \leq& f_3(n_1, n_2, \ldots, n_r) - \sum_{m=1}^{r} n_m \\ &+ max\{n_1 + 2, n_i - n_1 + 1\} \end{aligned}$$

Proof. Let $x_i = n_i - 1$, $x_j = n_j - 1$, and $x_\ell = n_\ell$ for any $\ell \in [r] \setminus \{i, j\}$. Then we know $f_3(\{n_1, \ldots, n_r\} \setminus \{n_i, n_j\} \cup \{n_i - 1, n_j - 1\}) = f_3(x_1, x_2, \ldots, x_r)$ and the partition $\mathcal{P}$ attains $f_3(x_1, x_2, \ldots, x_r)$. Moreover, for any $\ell \in [r] \setminus \{1\}$, we have $x_\ell \geq n_\ell - 1 > n_1 \geq x_1$ since $n_\ell \geq n_1 + 2$.

If $\{i, j\} \not\subseteq P_\zeta$ for any $\zeta \in [2]$, we have $x_{P_\zeta} = n_{P_\zeta} - 1$ and $x_{P_{3-\zeta}} = n_{P_{3-\zeta}} - 1$. Thus

$$\begin{aligned} f_3(x_1, x_2, \ldots, x_r) =& x_{P_\zeta} \cdot x_{P_{3-\zeta}} = (n_{P_\zeta} - 1) \cdot (n_{P_{3-\zeta}} - 1) \\ =& n_{P_\zeta} \cdot n_{P_{3-\zeta}} - n_{P_\zeta} - n_{P_{3-\zeta}} + 1 \\ \leq& f_3(n_1, n_2, \ldots, n_r) + 1 - \sum_{m=1}^{r} n_m. \end{aligned}$$

Assume $\{i, j\} \subseteq P_\zeta$ for some $\zeta \in [2]$. Notice that $x_{P_\zeta} = n_{P_\zeta} - 2$ and $x_{P_{3-\zeta}} = n_{P_{3-\zeta}}$. Thus,

$$\begin{aligned} f_3(x_1, x_2, \ldots, x_r) =& x_{P_\zeta} \cdot x_{P_{3-\zeta}} = (n_{P_\zeta} - 2) \cdot n_{P_{3-\zeta}} \\ =& n_{P_\zeta} \cdot n_{P_{3-\zeta}} - 2n_{P_{3-\zeta}} \\ \leq& f_3(n_1, n_2, \ldots, n_r) - 2n_{P_{3-\zeta}}. \end{aligned} \tag{3}$$

If $1 \in P_\zeta$, we have $x_{P_{3-\zeta}} \geq x_{P_\zeta} - x_1$. Otherwise $x_{P_\zeta} \cdot x_{P_{3-\zeta}} < x_{P_\zeta} \cdot x_{P_{3-\zeta}} + x_1 \cdot (x_{P_\zeta} - x_1 - x_{P_{3-\zeta}}) = (x_{P_\zeta} - x_1) \cdot (x_{P_{3-\zeta}} + x_1)$, which contradicts to the choice of $\mathcal{P}$. With $x_{P_\zeta} + x_{P_{3-\zeta}} = x_{\mathcal{P}}$, we know

$$n_{P_{3-\zeta}} = x_{P_{3-\zeta}} \geq \frac{x_{\mathcal{P}} - x_1}{2} \geq \frac{\sum_{m=1}^{r} n_m - 2 - n_1}{2}.$$

Thus, by the inequality (3),

$$\begin{aligned} f_3(x_1,x_2,\ldots,x_r) \leq & f_3(n_1,n_2,\ldots,n_r) - 2n_{P_{3-\zeta}} \\ \leq & f_3(n_1,n_2,\ldots,n_r) + n_1 + 2 - \sum_{m=1}^{r} n_m. \end{aligned}$$

It remains to consider the case $1 \in P_{3-\zeta}$. Recall $x_i > x_1$ and $i \in P_\zeta$, we have $x_{P_{3-\zeta}} - x_1 \geq x_{P_\zeta} - x_i$. Otherwise $x_{P_\zeta} \cdot x_{P_{3-\zeta}} < x_{P_\zeta} \cdot x_{P_{3-\zeta}} + (x_i - x_1) \cdot (x_{P_\zeta} - x_i - x_{P_{3-\zeta}} + x_1) = (x_{P_\zeta} - x_i + x_1) \cdot (x_{P_{3-\zeta}} - x_1 + x_i)$, a contradiction to the choice of $\mathcal{P}$ again. With $x_{P_\zeta} + x_{P_{3-\zeta}} = x_{\mathcal{P}}$, we know

$$n_{P_{3-\zeta}} = x_{P_{3-\zeta}} \geq \frac{x_{\mathcal{P}} + x_1 - x_i}{2} = \frac{\sum_{m=1}^{r} n_m - 1 + n_1 - n_i}{2}.$$

Thus, by the inequality (3),

$$\begin{aligned} f_3(x_1,x_2,\ldots,x_r) \leq & f_3(n_1,n_2,\ldots,n_r) - 2n_{P_{3-\zeta}} \\ \leq & f_3(n_1,n_2,\ldots,n_r) - n_1 + n_i + 1 - \sum_{m=1}^{r} n_m. \end{aligned}$$

■

3 Proof of Theorem 1.7

We shall provide a construction to complete the proof. Let $x_1 = n_1 - (k-1)$ and $x_i = n_i$ for any $i \neq 1$. Then $f_t(n_1 - (k-1), n_2, \ldots, n_r) = f_t(x_1, x_2, \ldots, x_r)$. Let $\mathcal{P} = (P_1, P_2, \ldots, P_{t-1})$ be the partition of $[r]$ that attains $f_t(x_1, x_2, \ldots, x_r)$. Assume that V_0 is a set of $(k-1)$ vertices and V_i is a set of x_i vertices for any $i \in [r]$ such that V_0, V_1, ..., V_r are pairwise disjoint. According to the partition $\mathcal{P}$ of $[r]$, we set $V_{P_\ell} = \cup_{i \in P_\ell} V_i$ for each $\ell \in [t-1]$. So $V_{[r]} = \cup_{\ell \in [t-1]} V_{P_\ell}$ and $|V_{P_\ell}| = x_{P_\ell}$. Now we construct the graph G as follows. Let $V(G) = V_0 \cup V_{[r]}$ and

$$E(G) = \{V_0 \vee (\cup_{i=2}^{r} V_i)\} \cup \{\cup_{1 \leq \ell < j \leq t-1} (V_{P_\ell} \vee V_{P_j})\}.$$

We see $|V(G)| = (k-1) + \sum_{i=1}^{r} x_i = \sum_{i=1}^{r} n_i = n$, and

$$\begin{aligned} |E(G)| = & (k-1)(n-n_1) + \sum_{1 \leq \ell < j \leq t-1} x_{P_\ell} \cdot x_{P_j} \\ = & (k-1)(n-n_1) + f_t(n_1 - (k-1), n_2, \ldots, n_r). \end{aligned}$$

We can see G is a subgraph of $K_{n_1,n_2,\ldots,n_r}$ and $G - V_0$ is a complete $(t-1)$-partite graph. So any copy of K_t in G must contain at least one vertex of V_0. Since $|V_0| = k-1$, G contains no copy of kK_t. Thus $\mathrm{ex}(K_{n_1,n_2,\ldots,n_r}, kK_t) \geq (k-1)(n-n_1) + f_t(n_1 - (k-1), n_2, \ldots, n_r)$. This completes the proof of Theorem 1.7. ■

4 Proof of Theorem 1.8

Recall that, in this section, $r, n_1, n_2, \ldots, n_r$ are integers with $r \geq 4$, $10k - 4 \leq n_1 + 4k \leq \min_{i\geq 2}\{n_i\}$, and $\sum_{i=1}^{r} n_i = n$. By Theorem 1.7, it suffices to prove

$$ex(K_{n_1,n_2,\ldots,n_r}, kK_3) \leq (k-1)(n-n_1) + f_3(n_1 - (k-1), n_2, \ldots, n_r). \tag{4}$$

By way of contradiction, we suppose that k is the minimum positive integer such that $\text{ex}(K_{n_1,n_2,\ldots,n_r}, kK_3) > (k-1)(n-n_1) + f_3(n_1-(k-1), n_2, \ldots, n_r)$. Thus, there exists a graph G, as a spanning subgraph of $K_{n_1,n_2,\ldots,n_r}$, containing no copy of kK_3 and satisfying

$$e(G) = ex(K_{n_1,n_2,\ldots,n_r}, kK_3) > (k-1)(n-n_1) + f_3(n_1-(k-1), n_2, \ldots, n_r). \tag{5}$$

Moreover, we use $V_1, V_2, \ldots, V_r$ to denote the parts of G , where $|V_i| = n_i$ for each $i \in [r]$. It is worth noting that the inequality (4) holds for $k = 1$ by Theorem 1.1. Hence, we may assume $k \geq 2$.

Claim 1 *$G - \{v\}$ contains at least one copy of $(k-1)K_3$ for each $v \in V(G)$.*

Proof. By contradiction, suppose that there is a vertex v in $V(G)$ such that $G - \{v\}$ contains no copy of $(k-1)K_3$. Suppose that $v \in V_\ell$ for some $\ell \in [r]$. Let $n'_\ell = n_\ell - 1$, $n'_j = n_j$ for any $j \neq \ell$, and $n' = n - 1$. Then $10(k-1) - 4 \leq n'_1 + 4(k-1) \leq \min_{i\geq 2}\{n'_i\}$. Moreover, $G - \{v\} \subseteq K_{n'_1,n'_2,\ldots,n'_r}$ by the construction. Because $G - \{v\}$ contains no copy of $(k-1)K_3$, we obtain $e(G - \{v\}) \leq ex(K_{n'_1,n'_2,\ldots,n'_r}, (k-1)K_3)$. By the minimality of k, we have

$$\begin{aligned} e(G) =& e(G - \{v\}) + d_G(v) \\ \leq& ex(K_{n'_1,n'_2,\ldots,n'_r}, (k-1)K_3) + d_G(v) \\ \leq& (k-2)(n' - n'_1) + f_3(n'_1 - (k-2), n'_2, n'_3, \ldots, n'_r) + d_G(v). \end{aligned} \tag{6}$$

If $v \in V_1$, then $n'_1 = n_1 - 1$ and $d_G(v) = |N_G(v)| \leq |V(G)\backslash V_1| = n - n_1$ since $G \subseteq K_{n_1,n_2,\ldots,n_r}$. By the inequality (6), we have

$$\begin{aligned} e(G) \leq& (k-2)(n-1-n_1+1) + f_3(n_1 - 1 - (k-2), n_2, n_3, \ldots, n_r) + n - n_1 \\ =& (k-1)(n-n_1) + f_3(n_1 - (k-1), n_2, n_3, \ldots, n_r), \end{aligned}$$

which contradicts the inequality (5). Hence, we may assume $v \notin V_1$. Then $d_G(v) \leq n - n_\ell$, $n'_\ell = n_\ell - 1$, and $n'_i = n_i$ for any $i \neq \ell$. Note that $n_\ell - 1 \geq n_1 + 4k - 1 \geq n_1 - k + 2$. By the inequality (6) and Proposition 2.2, we have

$$\begin{aligned} e(G) \leq& (k-2)(n-1-n_1) + f_3(\{n_1-(k-2), n_2, n_3, \ldots, n_r\} \setminus \{n_\ell\} \cup \{n_\ell - 1\}) + n - n_\ell \\ \leq& (k-2)(n-1-n_1) + (f_3(n_1-(k-1), n_2, n_3, \ldots, n_r) + n_\ell - (n_1 - k + 2)) + n - n_\ell \\ =& (k-1)(n-n_1) - k + 2 + n_1 + f_3(n_1-(k-1), n_2, n_3, \ldots, n_r) - (n_1 - k + 2) \\ =& (k-1)(n-n_1) + f_3(n_1-(k-1), n_2, n_3, \ldots, n_r), \end{aligned}$$

which is a contradiction to the inequality (5). This completes the proof of Claim 1. ■

Since $k \geq 2$, the graph G contains at least one triangle by Claim 1. Arbitrarily choose a triangle K_3^* in G with vertices u_1, u_2, u_3. Assume that $u_1 \in V_\alpha$, $u_2 \in V_\eta$, and $u_3 \in V_\xi$, where α, η, ξ are three distinct integers in $[r]$. Moreover, let $S = \{\alpha, \eta, \xi\}$, $n'_\ell = n_\ell - 1$ for any $\ell \in S$, $n'_j = n_j$ for any $j \in [r] \setminus S$, and $n' = \sum_{i=1}^r n'_i$. Notice that $n' = \sum_{i=1}^r n_i - 3 = n - 3$ and $G - \{u_1, u_2, u_3\} \subseteq K_{n'_1, n'_2, \ldots, n'_r}$.

Claim 2 $e(G) \leq (k-2)(n'-n'_1) + f_3(n'_1-(k-2), n'_2, \ldots, n'_r) + \sum_{1\leq i<j\leq 3} |N_G(u_i) \cap N_G(u_j)| + n - 3$.

Proof. Note that $G - \{u_1, u_2, u_3\}$ contains no copy of $(k-1)K_3$ since G contains no copy of kK_3. Moreover, we have $10(k-1) - 4 \leq n'_1 + 4(k-1) \leq \min_{i\geq 2}\{n'_i\}$. By the minimality of k, we know $e(G - \{u_1, u_2, u_3\}) \leq ex(K_{n'_1, n'_2, \ldots, n'_r}, (k-1)K_3) \leq (k-2)(n' - n'_1) + f_3(n'_1 - (k-2), n'_2, \ldots, n'_r)$. Thus,

$$\begin{aligned} e(G) =& e(G - \{u_1, u_2, u_3\}) + \sum_{i=1}^{3} |N_G(u_i)| - 3 \\ \leq& (k-2)(n' - n'_1) + f_3(n'_1 - (k-2), n'_2, \ldots, n'_r) + \sum_{i=1}^{3} |N_G(u_i)| - 3. \qquad (7) \end{aligned}$$

By the Principle of Inclusion-Exclusion,

$$\begin{aligned} \sum_{i=1}^{3} |N_G(u_i)| =& |\bigcup_{i=1}^{3} N_G(u_i)| + \sum_{1\leq i<j\leq 3} |N_G(u_i)) \cap N_G(u_j)| - |\bigcap_{i=1}^{3} N_G(u_i)| \\ \leq& |V(G)| + \sum_{1\leq i<j\leq 3} |N_G(u_i) \cap N_G(u_j)| \\ =& n + \sum_{1\leq i<j\leq 3} |N_G(u_i) \cap N_G(u_j)|. \qquad (8) \end{aligned}$$

Combining inequalities (7) and (8), we see

$$e(G) \leq (k-2)(n' - n'_1) + f_3(n'_1 - (k-2), n'_2, \ldots, n'_r) + \sum_{1\leq i<j\leq 3} |N_G(u_i) \cap N_G(u_j)| + n - 3.$$

■

We finish our proof in the following two cases.

Case 1. $|N_G(u_i) \cap N_G(u_j)| \leq 6(k-1)$ **for every two distinct vertices** u_i **and** u_j **in** $\{u_1, u_2, u_3\}$**.**

Recall that $G - \{u_1, u_2, u_3\} \subseteq K_{n'_1, n'_2, \ldots, n'_r}$. We shall consider the two cases when $1 \in \{\alpha, \eta, \xi\}$ and $1 \notin \{\alpha, \eta, \xi\}$. First, assume $1 \in \{\alpha, \eta, \xi\}$. Without loss of generality, say

$1=\alpha$. Then $n_1'=n_1-1$ and $\{n_2',n_3',\dots,n_r'\}=\{n_2,n_3,\dots,n_r\}\setminus\{n_\eta,n_\xi\}\cup\{n_\eta-1,n_\xi-1\}$. By Claim 2, we have

$$\begin{aligned}e(G)\le&(k-2)(n-3-n_1+1)+18(k-1)+n-3\\&+f_3(\{n_1-1-k+2,n_2,n_3,\dots,n_r\}\setminus\{n_\eta,n_\xi\}\cup\{n_\eta-1,n_\xi-1\})\\=&(k-1)(n-n_1)-2(k-2)-(n-n_1)+18(k-1)+n-3\\&+f_3(\{n_1-k+1,n_2,n_3,\dots,n_r\}\setminus\{n_\eta,n_\xi\}\cup\{n_\eta-1,n_\xi-1\})\\=&(k-1)(n-n_1)+n_1+16(k-1)-1\\&+f_3(\{n_1-k+1,n_2,n_3,\dots,n_r\}\setminus\{n_\eta,n_\xi\}\cup\{n_\eta-1,n_\xi-1\}).\end{aligned}$$

Now, we assume $1\notin\{\alpha,\eta,\xi\}$. Then $n_i'=n_i-1>n_1$ for any $i\in\{\alpha,\eta,\xi\}$, and $n_j'=n_j$ for any $j\in[r]\setminus\{\alpha,\eta,\xi\}$. By Claim 2,

$$\begin{aligned}e(G)\le&(k-2)(n-3-n_1)+18(k-1)+n-3\\&+f_3(\{n_1-k+2,n_2,n_3,\dots,n_r\}\setminus\{n_\alpha,n_\eta,n_\xi\}\cup\{n_\alpha-1,n_\eta-1,n_\xi-1\})\\=&(k-1)(n-n_1)+15(k-1)+n_1\\&+f_3(\{n_1-k+2,n_2,n_3,\dots,n_r\}\setminus\{n_\alpha,n_\eta,n_\xi\}\cup\{n_\alpha-1,n_\eta-1,n_\xi-1\}).\end{aligned}\tag{9}$$

Note that $n_\alpha-1\ge n_1-k+2$. By Proposition 2.2,

$$\begin{aligned}&f_3(\{n_1-k+2,n_2,n_3,\dots,n_r\}\setminus\{n_\alpha,n_\eta,n_\xi\}\cup\{n_\alpha-1,n_\eta-1,n_\xi-1\})\\&\quad\le f_3(\{n_1-k+1,n_2,n_3,\dots,n_r\}\setminus\{n_\eta,n_\xi\}\cup\{n_\eta-1,n_\xi-1\})+n_\alpha-(n_1-k+2).\end{aligned}\tag{10}$$

Thus, combining inequalities (9) and (10),

$$\begin{aligned}e(G)\le&(k-1)(n-n_1)+15(k-1)+n_1+n_\alpha-(n_1-k+2)\\&+f_3(\{n_1-k+1,n_2,n_3,\dots,n_r\}\setminus\{n_\eta,n_\xi\}\cup\{n_\eta-1,n_\xi-1\})\\=&(k-1)(n-n_1)+16(k-1)+n_\alpha-1\\&+f_3(\{n_1-k+1,n_2,n_3,\dots,n_r\}\setminus\{n_\eta,n_\xi\}\cup\{n_\eta-1,n_\xi-1\}).\end{aligned}\tag{11}$$

In both cases ($1\in\{\alpha,\eta,\xi\}$ and $1\notin\{\alpha,\eta,\xi\}$), we have inequality (11) holds. Since $n_1-k+1+2\le\min_{i\ge2}\{n_i\}$, by Proposition 2.3 and the inequality (11),

$$\begin{aligned}e(G)\le&(k-1)(n-n_1)+n_\alpha+16(k-1)-1+f_3(n_1-k+1,n_2,n_3,\dots,n_r)-(n-k+1)\\&+\max\{n_1-(k-1)+2,n_\eta-n_1+(k-1)+1\}\\=&(k-1)(n-n_1)+f_3(n_1-k+1,n_2,n_3,\dots,n_r)\\&+17k-18-n+\max\{n_\alpha+n_1-k+3,n_\eta+n_\alpha-n_1+k\}.\end{aligned}$$

Since $n - n_\alpha - n_1 \geq (r-2)n_2 \geq 2n_2 \geq 20k - 8$ and $n + n_1 - n_\eta - n_\alpha \geq (r-1)n_1 \geq 3n_1 \geq 18k - 12$, we have

$$e(G) \leq (k-1)(n-n_1) + f_3(n_1 - k + 1, n_2, n_3, \ldots, n_r),$$

which contradicts the inequality (5).

Case 2. There exist two distinct vertices u_i and u_j in $\{u_1, u_2, u_3\}$ such that $|N_G(u_i) \cap N_G(u_j)| > 6(k-1)$.

Without loss of generality, we may assume $|N_G(u_2) \cap N_G(u_3)| \geq 6(k-1) + 1$. Then we have the following claim.

Claim 3 *For any vertex set $A \subseteq V(G)$, if $|N_G(u_2) \cap N_G(u_3) \cap A| \geq 6(k-1)+1$, then there is a vertex u_0 with $u_0 \in A \cap N_G(u_2) \cap N_G(u_3)$ such that $|N_G(u_0) \cap N_G(u_i)| \leq 3(k-1)$ for any $i \in \{2,3\}$.*

Proof. By Claim 1, denote by $\mathcal{T}_i$ a copy of $(k-1)K_3$ in $G \setminus \{u_i\}$ for any $i \in \{2,3\}$. Since $|N_G(u_2) \cap N_G(u_3) \cap A| \geq 6(k-1) + 1$ and $|\mathcal{T}_1| = |\mathcal{T}_2| = 3(k-1)$, there is a vertex, say u_0, such that $u_0 \in (N_G(u_2) \cap N_G(u_3) \cap A) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$. If $|N_G(u_0) \cap N_G(u_i)| \geq 3(k-1) + 1$ for some $i \in \{2,3\}$, then there exists a vertex $w_i \in (N_G(u_0) \cap N_G(u_i)) \setminus \mathcal{T}_i$. Notice that $\{w_i, u_0, u_i\} \cap \mathcal{T}_i = \emptyset$ and $u_0 u_i w_i u_0$ is a triangle. So the disjoint union of $u_0 u_i w_i u_0$ and $\mathcal{T}_i$ is a copy of kK_3 in G, a contradiction to the choice of G. Thus, $|N_G(u_0) \cap N_G(u_i)| \leq 3(k-1)$ for any $i \in \{2,3\}$. ■

By setting $A = V(G)$ in Claim 3, there exists a vertex u_0 with $u_0 \in N_G(u_2) \cap N_G(u_3)$ such that $|N_G(u_0) \cap N_G(u_i)| \leq 3(k-1)$ for any $i \in \{2,3\}$. Note that $u_0 u_2 u_3 u_0$ is a triangle in G. Recall $u_2 \in V_\eta$ and $u_3 \in V_\xi$. Assume $u_0 \in V_{\alpha'}$.

According to the arbitrariness of the choice of K_3^* in G, we know that Claims 2 and 3 are applicable to the triangle $u_0 u_2 u_3 u_0$ by replacing K_3^* with $u_0 u_2 u_3 u_0$ and replacing S with $\{\alpha', \eta, \xi\}$. The remaining proof is divided into two subcases.

Subcase 1. $1 \in \{\alpha', \eta, \xi\}$**.**

Note that $n_1' = n_1 - 1$ and $n' = n - 3$. Moreover, we have $|N_G(u_2) \cap N_G(u_3)| \leq n - n_\eta - n_\xi$ by $u_2 \in V_\eta$ and $u_3 \in V_\xi$. Let $\{1, s, \ell\} = \{\alpha', \eta, \xi\}$. Due to $\{s, \ell\} \cap \{\eta, \xi\} \neq \emptyset$, without loss of generality, we may assume $s \in \{\eta, \xi\}$. Since $|N_G(u_0) \cap N_G(u_i)| \leq 3(k-1)$ for any $i \in \{2,3\}$, by Claim 2, we know

$$
\begin{aligned}
e(G) \leq & (k-2)(n-3-n_1+1) + (6(k-1) + |N_G(u_2) \cap N_G(u_3)|) + n - 3 \\
& + f_3(\{n_1-k+1, n_2, n_3, \ldots, n_r\} \setminus \{n_s, n_\ell\} \cup \{n_s-1, n_\ell-1\}) \\
\leq & (k-1)(n-n_1) + 4k - 5 + n + n_1 - n_\eta - n_\xi \\
& + f_3(\{n_1-k+1, n_2, n_3, \ldots, n_r\} \setminus \{n_s, n_\ell\} \cup \{n_s-1, n_\ell-1\}) \\
\leq & (k-1)(n-n_1) + 4k - 5 + n + n_1 - n_\eta - n_\xi + f_3(\{n_1-k+1, n_2, n_3, \ldots, n_r\}) \\
& -(n-(k-1)) + \max\{(n_1-k+1)+2, n_s-(n_1-k+1)+1\} \\
\leq & (k-1)(n-n_1) + f_3(\{n_1-k+1, n_2, n_3, \ldots, n_r\}) \\
& +5k - 6 + n_1 - n_\eta - n_\xi + \max\{n_1-k+3, n_s-n_1+k+2\} \\
\leq & (k-1)(n-n_1) + f_3(\{n_1-k+1, n_2, n_3, \ldots, n_r\}),
\end{aligned}
$$

where the third inequality holds by Proposition 2.3 and the last inequality holds by $4k-3+2n_1 \leq n_\eta + n_\xi$ and $6k-4+n_s \leq n_\eta + n_\xi$.

Subcase 2. $1 \notin \{\alpha', \eta, \xi\}$**.**

By setting $A = V_1$ in Claim 3, if $|N(u_2) \cap N(u_3) \cap V_1| \geq 6(k-1)+1$, then there exists a vertex u_0' with $u_0' \in N_G(u_2) \cap N_G(u_3) \cap V_1$ such that $|N_G(u_0') \cap N_G(u_i)| \leq 3(k-1)$ for any $i \in \{2,3\}$. Note that $u_0'u_2u_3u_0'$ is also a triangle in G. This cases reduces to Subcase 1. Hence, we may assume $|N(u_2) \cap N(u_3) \cap V_1| \leq 6(k-1)$. Then

$$
\begin{aligned}
|N_G(u_2) \cap N_G(u_3)| &= |N_G(u_2) \cap N_G(u_3) \cap (V(G) \setminus V_1)| + |N_G(u_2) \cap N_G(u_3) \cap V_1| \\
&\leq n - n_\eta - n_\xi - n_1 + 6(k-1).
\end{aligned}
$$

Recall that $n' = n-3$, $n_1' = n_1$, and $|N_G(u_0) \cap N_G(u_i)| \leq 3(k-1)$ for any $i \in \{2,3\}$. By Claim 2,

$$
\begin{aligned}
e(G) \leq & (k-2)(n-3-n_1) + (6(k-1) + n - n_\eta - n_\xi - n_1 + 6(k-1)) + n - 3 \\
& + f_3(\{n_1-k+2, n_2, n_3, \ldots, n_r\} \setminus \{n_{\alpha'}, n_\eta, n_\xi\} \cup \{n_{\alpha'}-1, n_\eta-1, n_\xi-1\}) \\
= & (k-1)(n-n_1) + 9k - 9 + n - n_\eta - n_\xi \\
& + f_3(\{n_1-k+2, n_2, n_3, \ldots, n_r\} \setminus \{n_{\alpha'}, n_\eta, n_\xi\} \cup \{n_{\alpha'}-1, n_\eta-1, n_\xi-1\}). \qquad (12)
\end{aligned}
$$

Since $n_1 + 4k \leq \min_{i \geq 2}\{n_i\}$, we know $n_1 - k + 3 \leq \min_{i \geq 2}\{n_i\}$. By Proposition 2.2 and the inequality (12),

$$
\begin{aligned}
e(G) \leq & (k-1)(n-n_1) + 9k - 9 + n - n_\eta - n_\xi + n_\xi - (n_1-k+2) \\
& + f_3(\{n_1-k+1, n_2, n_3, \ldots, n_r\} \setminus \{n_{\alpha'}, n_\eta\} \cup \{n_{\alpha'}-1, n_\eta-1\}) \\
= & (k-1)(n-n_1) + 10k - 11 + n - n_\eta - n_1 \\
& + f_3(\{n_1-k+1, n_2, n_3, \ldots, n_r\} \setminus \{n_{\alpha'}, n_\eta\} \cup \{n_{\alpha'}-1, n_\eta-1\}). \qquad (13)
\end{aligned}
$$

Proposition 2.3 implies that

$$f_3(\{n_1-k+1, n_2, n_3, \ldots, n_r\} \setminus \{n_{\alpha'}, n_\eta\} \cup \{n_{\alpha'}-1, n_\eta-1\})$$
$$\leq f_3(n_1-k+1, n_2, \ldots, n_r) + \max\{n_1-(k-1)+2, n_\eta-(n_1-k+1)+1\} - n + k - 1. \quad (14)$$

Since $n_1 \geq 6k-4$ and $n_\eta \geq n_1+4k \geq 10k-4$, combining inequalities (13) and (14), we have

$$\begin{aligned} e(G) \leq & (k-1)(n-n_1) + 10k - 11 + n - n_\eta - n_1 \\ & + f_3(n_1-k+1, n_2, \ldots, n_r) + \max\{n_1-k+3, n_\eta-n_1+k\} - n + k - 1. \\ = & (k-1)(n-n_1) + f_3(n_1-k+1, n_2, \ldots, n_r) + 11k - 12 \\ & - \max\{n_\eta+k-3, 2n_1-k\} \\ \leq & (k-1)(n-n_1) + f_3(n_1-k+1, n_2, \ldots, n_r), \end{aligned}$$

which contradicts to the inequality (5). This completes the proof of the inequality (4), and so Theorem 1.8 follows.

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