# Sharp bounds for the generalized connectivity $\kappa_3(G)$ *

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**Abstract**

Let $G$ be a nontrivial connected graph of order $n$ and let $k$ be an integer with $2 \leq k \leq n$. For a set $S$ of $k$ vertices of $G$, let $\kappa(S)$ denote the maximum number $\ell$ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in $G$ such that $V(T_i) \cap V(T_j) = S$ for every pair $i, j$ of distinct integers with $1 \leq i, j \leq \ell$. A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in $G$ with this property is called an internally disjoint set of trees connecting $S$. Chartrand et al. generalized the concept of connectivity as follows: The *k-connectivity*, denoted by $\kappa_k(G)$, of $G$ is defined by $\kappa_k(G)$ =min$\{\kappa(S)\}$, where the minimum is taken over all $k$-subsets $S$ of $V(G)$. Thus $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of $G$.

For general $k$, the investigation of $\kappa_k(G)$ is very difficult. We therefore focus on the investigation on $\kappa_3(G)$ in this paper. We study the relation between the connectivity and the 3-connectivity of a graph. First we give sharp upper and lower bounds of $\kappa_3(G)$ for general graphs $G$, and construct two kinds of graphs which attain the upper and lower bound, respectively. We then show that if $G$ is a connected planar graph, then $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$, and give some classes of graphs which attain the bounds. In the end we give an algorithm to determine $\kappa_3(G)$ for general graphs $G$. This algorithm runs in a polynomial time for graphs with a fixed value of connectivity, which implies that the problem of determining $\kappa_3(G)$ for graphs with a small minimum degree or connectivity can be solved in polynomial time, in particular, the problem whether $\kappa(G) = \kappa_3(G)$ for a planar graph $G$ can be solved in polynomial time.

**Keywords:** connectivity, $k$-connectivity, internally disjoint trees (paths), path-boundle

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# 1 Introduction

We follow the terminology and notation of [2] and all graphs considered here are always simple. As usual, the union of two graphs $G$ and $H$ is the graph, denoted by $G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. By starting with a disjoint union of two graphs $G$ and $H$ and adding edges joining every vertex of $G$ to every vertex of $H$, we obtain the *join* of $G$ and $H$, and denote it by $G \bigvee H$. Let $U$ be a set of vertices. Then, $G - U$ is the graph obtained from $G$ by deleting all the vertices in $V(G) \cap U$ together with their incident edges. A path $P = x_0x_1 \ldots x_k$ is called an *$x_0x_k$-path*, denoted by $x_0Px_k$. For the $x_0x_k$-path $P$, we denote three special subpaths of $P$ by $\hat{x}_0P\hat{x}_k := x_1 \ldots x_{k-1}$, $\hat{x}_0Px_k := x_1 \ldots x_k$ and $x_0P\hat{x}_k := x_0 \ldots x_{k-1}$. For $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$, an *$XY$-linkage* is defined as a set of $k$ vertex-disjoint paths $x_iP_iy_i$, $1 \leq i \leq k$. The *connectivity* $\kappa(G)$ of a graph $G$ is defined as the minimum cardinality of a set $Q$ of vertices of $G$ such that $G - Q$ is disconnected or trivial. A well-known theorem of Whitney [6] provides an equivalent definition of connectivity. For each 2-subset $S = \{u, v\}$ of vertices of $G$, let $\kappa(S)$ denote the maximum number of internally disjoint $uv$-paths in $G$. Then $\kappa(G)$ =min$\{\kappa(S)\}$, where the minimum is taken over all 2-subsets $S$ of $V(G)$.

In [3], the authors generalized the concept of connectivity. Let $G$ be a nontrivial connected graph of order $n$ and let $k$ be an integer with $2 \leq k \leq n$. For a set $S$ of $k$ vertices of $G$, let $\kappa(S)$ denote the maximum number $\ell$ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in $G$ such that $V(T_i) \cap V(T_j) = S$ for every pair $i, j$ of distinct integers with $1 \leq i, j \leq \ell$ (Note that the trees are vertex-disjoint in $G \backslash S$). A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in $G$ with this property is called an *internally disjoint set of trees connecting $S$*. The *$k$-connectivity*, denoted by $\kappa_k(G)$, of $G$ is then defined by $\kappa_k(G)$ =min$\{\kappa(S)\}$, where the minimum is taken over all $k$-subsets $S$ of $V(G)$. Thus, $\kappa_2(G) = \kappa(G)$.

Chartrand et al. in [3] proved that if $G$ is the complete 3-partite graph $K_{3,4,5}$, then $\kappa_3(G) = 6$. They also gave a general result for the complete graph $K_n$:

**Theorem 1.1.** *For every two integers $n$ and $k$ with $2 \leq k \leq n$,*

$$\kappa_k(K_n) = n - \lceil k/2 \rceil.$$

For general $k$, the investigation of $\kappa_k(G)$ is very difficult. Therefore, in this paper we will focus on the investigation of $\kappa_3(G)$. We study the relation between the 2-connectivity and the 3-connectivity of a graph. First, we give sharp upper and lower bounds of $\kappa_3(G)$ for general graphs $G$, and construct two kinds of graphs which attain the upper and lower bound, respectively. Then, we study the 3-connectivity for planar graphs. We will show that if $G$ is a connected planar graph, then $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$, and give some classes of graphs which attain the bounds. In the end, we give an algorithm to determine $\kappa_3(G)$ for general graph $G$. This algorithm runs in a polynomial time for graphs with a fixed value of connectivity, which implies that the problem of determining $\kappa_3(G)$ for graphs with a small minimum degree or connectivity can be solved in polynomial time, in particular, the problem whether $\kappa(G) = \kappa_3(G)$ for a planar graph $G$ can be solved in polynomial time.

## 2 Upper and lower bounds

Before we give the main results, there is an easy observation:

**Observation 2.1.** *If $G'$ is a spanning subgraph of $G$, then $\kappa_k(G') \leq \kappa_k(G)$ for $2 \leq k \leq n$.*

Now we give an upper bound of $\kappa_3(G)$.

**Theorem 2.2.** *Let $G$ be a connected graph with $n$ vertices. Then $\kappa_3(G) \leq \kappa(G)$. Moreover, the upper bound is sharp.*

*Proof.* We prove the theorem by three cases on $\kappa(G)$.

*Case 1:* $\kappa(G) = n - 1$.

Then $G$ must be a complete graph $K_n$. By Theorem 1.1, we know $\kappa_3(K_n) = n - \lceil \frac{3}{2} \rceil = n - 2$. So $\kappa_3(G) = n - 2 \leq \kappa(G) = n - 1$.

*Case 2:* $\kappa(G) = n - 2$.

Let $Q$ be an $(n-2)$-vertex cut of $G$. Here and in what follows, by a *$k$-vertex cut* we mean a vertex cut that has $k$ vertices. Assume $V(G) - Q = \{u, v\}$. Then $u$ and $v$ are two nonadjacent vertices and both of them are adjacent to all the vertices in $Q$. Then $G$ must have a spanning supergraph $G' = K_n - uv$ (i.e., $G$ is a spanning subgraph of $G'$). It is easy to check that $\kappa_3(G') = n-2$. By Observation 2.1, we get $\kappa_3(G) \leq \kappa_3(G') = n-2 = \kappa(G)$.

*Case 3:* $1 \leq \kappa(G) \leq n - 3$.

Let $Q$ be a $\kappa(G)$-vertex cut of $G$. Then $G - Q$ has at least 2 components. Since $|Q| \leq n - 3$, we can choose $S = \{v_1, v_2, v_3\}$, such that $v_1, v_2, v_3 \in V(G) \backslash Q$ and two of the three vertices are in different components. Then we know that any tree connecting $S$ must contain a vertex in $Q$. By the definition of $\kappa(S)$, we get $\kappa(S) \leq |Q|$. So $\kappa_3(G) \leq \kappa(S) \leq |Q| = \kappa(G)$.

From the above, we conclude that $\kappa_3(G) \leq \kappa(G)$.

Furthermore, for any two integers $k \geq 1$ and $n \geq k + 2$, consider the graph $G = K_k \bigvee (n-k)K_1$. Then, obviously $\kappa(G) = k$, and it is not difficult to check that $\kappa_3(G) = k$.

For example, for $k \geq 1$ and $n = k + 2$, $G = K_k \bigvee 2K_1 = K_k \bigvee \{w_1, w_2\}$. Let $S = \{v_1, v_2, v_3\}$ be a 3-subset of vertices of $G$.

If $S \subseteq V(K_k)$ (see Figure 1(a)), since $\kappa_3(K_k) = k - 2$ by Theorem 1.1, then the $k - 2$ trees together with $T_1 = w_1v_1 \cup w_1v_2 \cup w_1v_3$ and $T_2 = w_2v_1 \cup w_2v_2 \cup w_2v_3$ form $k$ pairwise internally disjoint trees connecting $S$, namely $\kappa(S) \geq k$.

If $v_1 = w_1$ and $v_2, v_3 \in V(K_k)$ (see Figure 1(b)), for any vertex $u \in V(K_k) \backslash \{v_2, v_3\}$, $T' = uw_1 \cup uv_2 \cup uv_3$ is a tree connecting $S$. There are altogether $k - 2$ trees of this form. Let $T_1 = w_1v_2 \cup v_2v_3$ and $T_2 = w_1v_3 \cup v_3w_2 \cup w_2v_2$. So there exist $k$ pairwise internally disjoint trees connecting $S$, namely $\kappa(S) \geq k$.

If $v_1 = w_1$, $v_2 = w_2$ and $v_3 \in V(K_k)$ (see Figure 1(c)), for any vertex $u \in V(K_k) \backslash \{v_3\}$, $T' = uw_1 \cup uw_2 \cup uv_3$ is a tree connecting $S$. Then the $k - 1$ trees together with $T = w_1v_3 \cup v_3w_2$ form $k$ pairwise internally disjoint trees connecting $S$, namely $\kappa(S) \geq k$.

Till now, we have shown $\kappa_3(G) \geq k$ for the special case. The proof of the other cases is similar. So $\kappa_3(G) \geq k$ and we know $\kappa_3(G) \leq \kappa(G) = k$. Therefore, $\kappa_3(G) = \kappa(G)$ and the upper bound is sharp. ∎

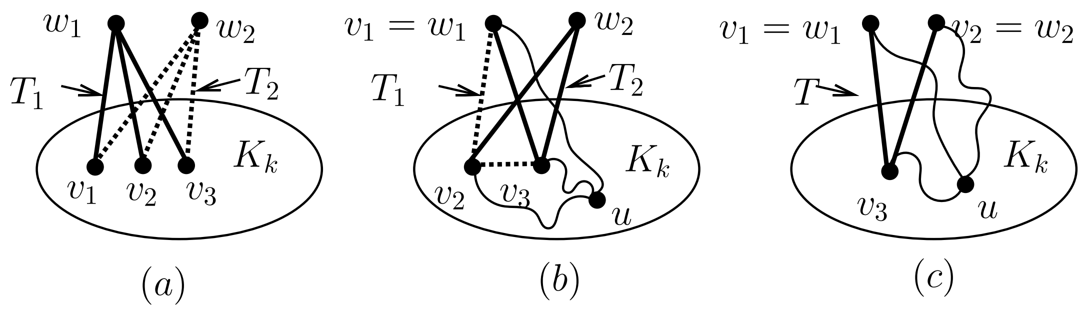


Figure 1: The three cases of $S$ for $G = K_k \bigvee 2K_1$

In the following, we will give a lower bound of $\kappa_3(G)$. Before proceeding, we recall the *Fan Lemma*, which will be used frequently in the sequel.

**Lemma 2.1.** *(The Fan Lemma [2]-p.214) Let $G$ be a $k$-connected graph, $x$ a vertex of $G$, and let $Y \subseteq V - \{x\}$ be a set of at least $k$ vertices of $G$. Then there exists a $k$-fan in $G$ from $x$ to $Y$, namely there exists a family of $k$ internally disjoint $(x, Y)$-paths whose terminal vertices are distinct in $Y$.*

Our lower bound is given as follows:

**Theorem 2.3.** *Let $G$ be a connected graph with $n$ vertices. For every two integers $k$ and $r$ with $k \geq 0$ and $r \in \{0, 1, 2, 3\}$, if $\kappa(G) = 4k + r$, then $\kappa_3(G) \geq 3k + \lceil \frac{r}{2} \rceil$. Moreover, the lower bound is sharp.*

Before proving the theorem, we need some preparations.

**Path-Bundle Transformation**

Denote the connectivity $\kappa(G)$ of $G$ simply by $\kappa$. Let $S = \{v_1, v_2, v_3\}$. For some $1 \leq t \leq \lfloor \frac{\kappa}{2} \rfloor$ and $s \geq t + 1$, a family $\{P_1, P_2, \ldots, P_s\}$ of $s$ $v_1v_2$-paths is called an $(s, t)$-*original-path-bundle connecting* $S$, if $v_3$ is on $t$ paths $P_1, \ldots, P_t$ of them, and the $s$ paths have no internal vertices in common except $v_3$, as shown in Figure 2(a). If there is not only an $(s, t)$-original-path-bundle $\{P_1', P_2', \ldots, P_s'\}$ connecting $S$, but also a family $\{M_1, M_2, \ldots, M_{\kappa-2t}\}$ of $\kappa - 2t$ internally disjoint $(v_3, X)$-paths avoiding the vertices in $V(P_1' \cup \cdots \cup P_t' - \{v_1, v_2, v_3\})$, where $X = V(P_{t+1}' \cup \cdots \cup P_s')$, then we call the family of paths $\{P_1', P_2', \ldots, P_s'\} \cup \{M_1, M_2, \ldots, M_{\kappa-2t}\}$ an $(s, t)$-*reduced-path-bundle connecting* $S$, as shown in Figure 2(b). Moreover, when there is no confusion, we will simply call them an $(s, t)$-*original-path-bundle* and an $(s, t)$-*reduced-path-bundle*, respectively.

We will show that if an $(s, t)$-original-path-bundle connecting $S$ exists, then an $(s, t)$-reduced-path-bundle connecting $S$ must exist. The operation of transforming an $(s, t)$-original-path-bundle to an $(s, t)$-reduced-path-bundle is called the *Path-Bundle Transformation*. See Figure 2.

Notice that when $\kappa$ is even and $t = \frac{\kappa}{2}$, an $(s,t)$-reduced-path-bundle is actually an $(s,t)$-original-path-bundle, since $\kappa - 2t = 0$. So, for the special case we do not need do the Path-Bundle Transformation any more. Later we will see that this special path-bundle structure is very useful.

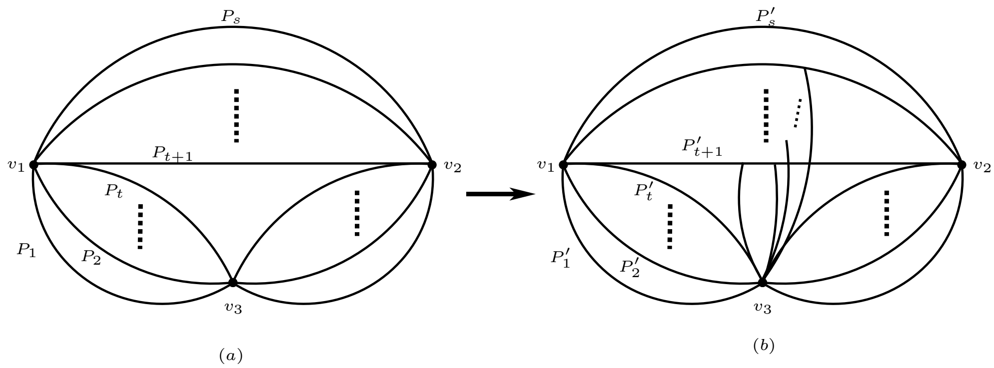


Figure 2: (a) An $(s,t)$-original-path-bundle (b) An $(s,t)$-reduced-path-bundle and the illustration for the “Path-Bundle Transformation”

Now, we mainly describe how to employ the Path-Bundle Transformation and show that why the structure we get is exactly an $(s,t)$-reduced-path-bundle.

Let a family $\{P_1, P_2, \ldots, P_s\}$ of $s$ $v_1v_2$-paths be an $(s,t)$-original-path-bundle in $G$ and let $X = V(P_{t+1} \cup \cdots \cup P_s)$, where $1 \leq t \leq \lfloor \frac{\kappa}{2} \rfloor$ and $s \geq t+1$. Since $G$ is $\kappa$-connected, there is a family of $\kappa$ internally disjoint $(v_3, X)$-paths $\{M_1, M_2, \ldots, M_\kappa\}$. Let $V(M_1 \cup \cdots \cup M_\kappa) \cap V(P_1 \cup \cdots \cup P_t - \{v_1, v_2\}) = N$, where $N \neq \emptyset$, since at least the vertex $v_3$ belongs to $N$. $P_1, \ldots, P_t$ can be regarded as $2t$ paths $v_1P_1v_3, v_1P_2v_3, \ldots, v_1P_tv_3, v_2P_1v_3, \ldots$, $v_2P_tv_3$. Let the vertices in $N \cap V(v_1P_iv_3)$ be kept in the queue $T_i$ according to the order in which they appear on the path $P_i$ from $v_1$ to $v_3$, for $1 \leq i \leq t$. Similarly, let the vertices in $N \cap V(v_2P_iv_3)$ be kept in a queue $T_{i+t}$ according to the order in which they appear on the path $P_i$ from $v_2$ to $v_3$. We may assume that $T_i = \{t_1^i, t_2^i, \ldots, v_3\}$ for $1 \leq i \leq 2t$. (Figure 3 shows the description, in which the crosses indicate the vertices $t_j^i \in N$.)

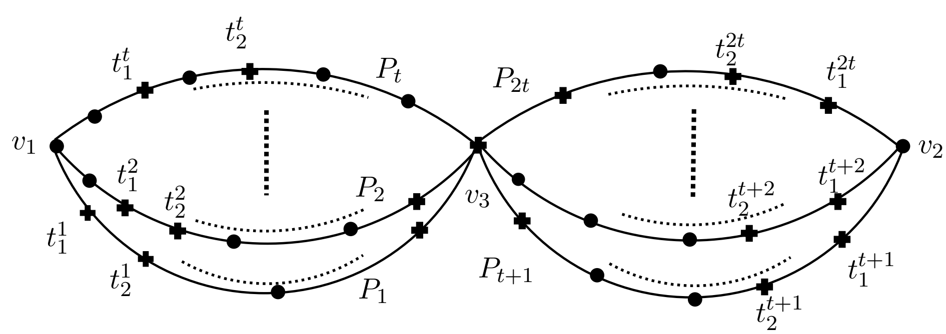


Figure 3: The cross vertices of paths

For each $t_j^i$, there exists a path $M_l$ $(1 \leq l \leq \kappa)$ containing $t_j^i$, since $t_j^i \in N = V(M_1 \cup \cdots \cup M_\kappa) \cap V(P_1 \cup \cdots \cup P_t - \{v_1, v_2\})$.

After giving the necessary notation, let us start the operation. We partition the process into two main stages: Finding and Adjusting.

**Finding stage:** First, we mark the vertex $t_1^i$ in $T_i$ for each $1 \leq i \leq 2t$, and mark the corresponding $(v_3, X)$-path $M_{i_1}$ containing $t_1^i$. If the paths $M_{1_1}, M_{2_1}, \ldots, M_{2t_1}$ are pairwise distinct (here $M_{i_j} \in \{M_1, M_2, \ldots, M_\kappa\}$ denotes the path containing the $j$-th vertex of queue $T_i$), then we find $2t$ marked paths, the Finding stage is completed and we proceed to the Adjusting stage. Otherwise, there are at least two marked vertices on a same marked path. Suppose that $t_1^{i_1}, t_1^{i_2}, \ldots$ are on the same marked path $M_i$. For these vertices $t_1^{i_1}, t_1^{i_2}, \ldots$, keep the mark of the vertex nearest to $v_3$ on the path $M_i$ and unmark the other vertices. Then for each vertex $t_1^i$ which was unmarked just now, mark the next vertex $t_2^i$ in $T_i$ and also mark the corresponding $(v_3, X)$-path $M_{i_2}$ containing $t_2^i$. For example, suppose $t_1^1, t_1^2, t_1^{2t}$ are on the same marked path, namely $M_i = M_{1_1} = M_{2_1} = M_{2t_1}$, and among all of them, suppose $t_1^1$ is the vertex nearest to $v_3$ on the path $M_i$. See Figure 4, where the stars indicate the updated $2t$ marked vertices.

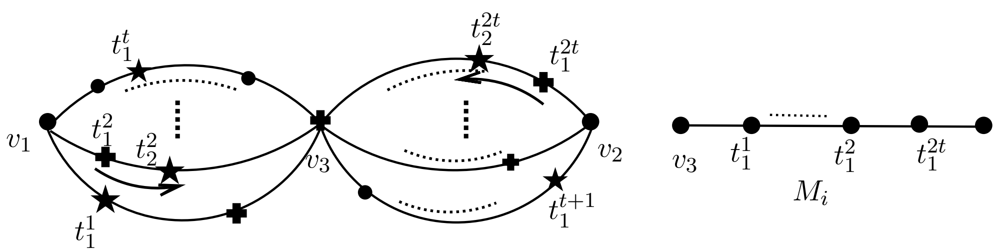


Figure 4: The updated $2t$ marked vertices are $t_1^1$, "$t_2^2$", $t_1^3, \ldots, t_1^{2t-1}$, "$t_2^{2t}$" and the updated $2t$ marked paths are $M_{1_1}$, "$M_{2_2}$", $M_{3_1}, \ldots, M_{2t-1_1}$, "$M_{2t_2}$".

If the updated $2t$ marked paths are pairwise distinct, this is what we want and then we proceed to the Adjusting stage. Otherwise, repeat the operation as before, i.e., there is a marked path containing at least two marked vertices; then keep the mark of the vertex nearest to $v_3$ and unmark the other vertices; for each vertex which was unmarked just now, in the corresponding queue $T_i$ containing it, mark the next vertex and let the $(v_3, X)$-path containing the new marked vertex be marked, until we find $2t$ distinct marked paths. Note that the procedure will terminate, since each $T_i$ has finite elements and contains the special vertex $v_3$. We know that $v_3$ is a vertex of every path $M_1, M_2, \ldots, M_\kappa$ (so $v_3$ can correspond to any $M_i$). Therefore, if for some $T_i$, $v_3$ is marked, then we can choose any of the $(v_3, X)$-paths which have not been marked, to be the corresponding path and mark it.

For the procedure described above, there are some remarks as follows:

**Remark 2.4.** *After the Finding stage, there are only* $2t$ *marked vertices* $q_1, q_2, \ldots, q_{2t}$ *and the* $2t$ *final marked vertices must be in* $T_1, T_2, \ldots, T_{2t}$*, respectively since, at first we chose* $2t$ *marked vertices which came from the* $2t$ *queues, respectively. Then once we unmark a vertex, we find the next vertex and mark it in the same queue. So there are always* $2t$ *marked vertices which are in the* $2t$ *queues, respectively. Without loss of generality, suppose* $q_i \in T_i$ *for* $1 \leq i \leq 2t$*. Then* $q_i = q_j$ *for* $i \neq j$*, if and only if* $q_i = q_j = v_3$*.*

**Remark 2.5.** *Once a path $M_i$ is marked, it will always be a marked path from then on. Although at some step we unmark some vertices on $M_i$, the mark of the vertex nearest to $v_3$ on $M_i$ is kept at this step. So $M_i$ is still marked. Moreover, the final $2t$ distinct marked paths are exactly the paths corresponding to the final $2t$ marked vertices, respectively. Without loss of generality, let the $2t$ distinct marked paths be $M_1, M_2, \ldots, M_{2t}$ and $q_i \in V(M_i)$.*

**Remark 2.6.** *If both $q_i$ and $v$ are vertices on $M_i$, $q_i$ is one of the final marked vertices and $v$ was ever marked and unmarked at some step, then $q_i$ is closer to $v_3$ than $v$ on $M_i$, since at some step the vertex whose mark is kept is always nearest to $v_3$ on $M_i$. So $v \notin V(v_3M_iq_i)$. For example, in Figure 4, if the updated $2t$ marked paths $M_{1_1}, M_{2_2}, M_{3_1}, \ldots, M_{2t-1_1}, M_{2t_2}$ are distinct, then $t_1^1, t_2^2, t_1^3, \ldots, t_1^{2t-1}, t_2^{2t}$ are the $2t$ final marked vertices. We know that $t_1^1, t_1^2$ and $t_1^{2t}$ are all on $M_{1_1}$, $t_1^1$ is the final marked vertex and that $t_1^2$ and $t_1^{2t}$ are the vertices which were ever marked and unmarked at some step. Obviously $t_1^1$ is closer to $v_3$ than both $t_1^2$ and $t_1^{2t}$ on $M_{1_1}$.*

**Adjusting stage:** Now we find $2t$ marked paths $M_1, M_2, \ldots, M_{2t}$, each of which contains a final marked vertex $q_i$ such that $q_i \in T_i$, namely $q_i, q_{i+t} \in P_i$, for $1 \le i \le t$. Then we use the $2t$ paths $M_1, M_2, \ldots, M_{2t}$ to adjust paths $P_1, \ldots, P_t$. Let $P_i' = v_1P_iq_iM_iv_3M_{i+t}q_{i+t}P_iv_2$ for $1 \le i \le t$ and $P_j' = P_j$ for $t+1 \le j \le s$ and there are $\kappa - 2t$ internally disjoint $(v_3, X)$-paths $M_{2t+1}, M_{2t+2}, \ldots, M_\kappa$ left that have not been marked so far. See Figure 5. Note that when $q_i = v_3$ and $q_{i+t} = v_3$, we have $P_i' = P_i$ for $1 \le i \le t$.

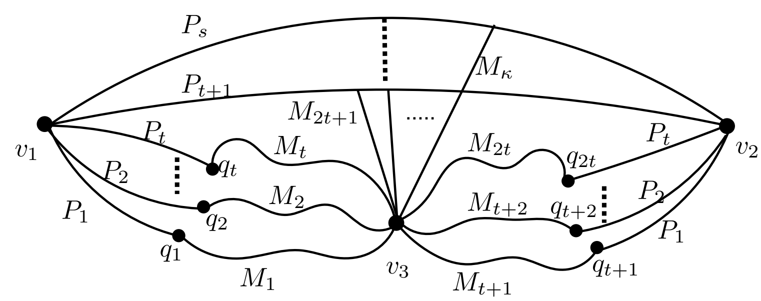


Figure 5: An $(s,t)$-reduced-path-bundle in the Adjusting stage

When this operation is completed, the family of walks $\{P_1', P_2', \ldots, P_s'\} \cup \{M_{2t+1}, M_{2t+2}, \ldots, M_\kappa\}$ is exactly an $(s,t)$-reduced-path-bundle, which can be verified by the following three claims. Keep in mind that paths $P_1, P_2, \ldots, P_s$ have no internal vertices in common except $v_3$, and that $\{M_1, M_2, \ldots, M_\kappa\}$ is a family of $\kappa$ internally disjoint $(v_3, X)$-paths, where $X = V(P_{t+1} \cup \cdots \cup P_s)$. For an $x_0x_k$-path $P = x_0x_1 \ldots x_k$, recall that $x_0Px_k := P$, $\hat{x}_0P\hat{x}_k := x_1 \ldots x_{k-1}$, $\hat{x}_0Px_k := x_1 \ldots x_k$ and $x_0P\hat{x}_k := x_0 \ldots x_{k-1}$.

**Claim 1:** The $t$ walks $P_1', \ldots, P_t'$ from $v_1$ to $v_2$ are paths and have no internal vertices in common expect $v_3$.

*Proof.* It follows from the following three arguments:

(1) For $1 \le i \ne j \le 2t$, since $V(\hat{v}_3M_iq_i) \subset V(M_i)$, $V(\hat{v}_3M_jq_j) \subset V(M_j)$ and $M_i, M_j$ are internally disjoint, then $V(\hat{v}_3M_iq_i) \cap V(\hat{v}_3M_jq_j) = \emptyset$.

(2) Now we show that $V(\hat{q}_i P_{i-kt} \hat{v}_{k+1}) \cap V(\hat{v}_3 M_j \hat{q}_j) = \emptyset$ for $k = 0$ or $1$ and $1 \leq i, j \leq 2t$. Let $v$ be a vertex in $V(\hat{q}_i P_{i-kt} \hat{v}_{k+1})$. If $v$ is on $M_j$, obviously, $v \in N$ and hence $v \in T_i$. Since $v$ is ordered in front of $q_i$ in the queue $T_i$ and $q_i$ is marked, we know that $v$ was marked before. $q_j$ is the final marked vertex on $M_j$ so $q_j$ is closer to $v_3$ than $v$ on $M_j$ by Remark 2.6. It follows that $v \notin V(\hat{v}_3 M_j \hat{q}_j)$. If $v$ is not on $M_j$, it is certainly not on $v_3 M_j q_j$. Therefore, we conclude that $V(\hat{q}_i P_{i-kt} \hat{v}_{k+1}) \cap V(\hat{v}_3 M_j \hat{q}_j) = \emptyset$.

(3) It is easy to see that $V(\hat{q}_i P_{i-k_1 t} \hat{v}_{k_1+1}) \cap V(\hat{q}_j P_{j-k_2 t} \hat{v}_{k_2+1}) = \emptyset$ for $1 \leq i \neq j \leq 2t$ and $k_1, k_2 = 0$ or $1$. ∎

**Claim 2:** $P'_i$ and $P'_j$ are internally disjoint paths for $1 \leq i \leq t$ and $t+1 \leq j \leq s$.

*Proof.* $P'_i = v_1 P_i q_i M_i v_3 M_{i+t} q_{i+t} P_i v_2$ for $1 \leq i \leq t$. Since $V(q_i M_i v_3) \subset V(M_i)$ and $V(v_3 M_{i+t} q_{i+t}) \subset V(M_{i+t})$, obviously $V(q_i M_i v_3 M_{i+t} q_{i+t}) \cap X = \emptyset$. It is easy to see that $V(\hat{v}_1 P_i q_i \cup q_{i+t} P_i \hat{v}_2) \cap X = \emptyset$ for $1 \leq i \leq t$. It follows that $V(P'_i - \{v_1, v_2\}) \cap X = \emptyset$. Since $P'_j = P_j$ for $t+1 \leq j \leq s$, we conclude that $V(P'_i - \{v_1, v_2\}) \cap V(P'_j) = \emptyset$ for $1 \leq i \leq t$ and $t+1 \leq j \leq s$. ∎

Since $P'_{t+1}, \ldots, P'_s$ are internally disjoint paths, by the above two claims, we can get that $\{P'_1, P'_2, \ldots, P'_s\}$ is an $(s,t)$-original-path-bundle.

**Claim 3:** The set of the remaining unmarked paths $\{M_{2t+1}, \ldots, M_\kappa\}$ is a family of $\kappa - 2t$ internally disjoint $(v_3, X')$-paths, where $X' = V(P'_{t+1} \cup \cdots \cup P'_s)$. Moreover, these remaining paths avoid the vertices in $V(P'_1 \cup \cdots \cup P'_t - \{v_1, v_2, v_3\})$.

*Proof.* It is clear that $\{M_{2t+1}, \ldots, M_\kappa\}$ is a family of $\kappa - 2t$ internally disjoint $(v_3, X')$-paths, since $\{M_{2t+1}, \ldots, M_\kappa\}$ is a family of internally disjoint $(v_3, X)$-paths and $X = X'$.

If there exists a vertex $v$ in $V(M_i) \cap V(P'_j - \{v_1, v_2, v_3\})$ for $2t+1 \leq i \leq \kappa$ and $1 \leq j \leq t$, then $v$ must be in $V(M_i) \cap V(\hat{q}_{j+kt} P_j \hat{v}_{k+1})$ for $k = 0$ or $1$. Then we know that $v$ was marked at some step and so was $M_i$. But, by Remark 2.5 once $M_i$ is marked, it will always be a marked path from then on, a contradiction. Therefore $V(M_i) \cap V(P'_j - \{v_1, v_2, v_3\}) = \emptyset$ for $2t+1 \leq i \leq \kappa$ and $1 \leq j \leq t$. The proof is complete. ∎

Now, by Claim 3 we know that the $(s,t)$-original-path-bundle $\{P'_1, P'_2, \ldots, P'_s\}$ together with the family of $\kappa - 2t$ internally disjoint $(v_3, X')$-paths $\{M_{2t+1}, \ldots, M_\kappa\}$ form an $(s,t)$-reduced-path-bundle.

We conclude that if $G$ contains an $(s,t)$-original-path-bundle, then from a family of $\kappa$ internally disjoint paths, by employing the Path-Bundle Transformation to them we can find an $(s,t)$-reduced-path-bundle.

Notice that, for an $(s,t)$-original-path-bundle $\{P_1, P_2, \ldots, P_s\}$ and $X = V(P_{t+1} \cup \cdots \cup P_s)$, if $|X| \geq \kappa$, then there exists a $\kappa$-fan $\{M_1, M_2, \ldots, M_\kappa\}$ from $v_3$ to $X$ by the Fan Lemma. Then, for this $\kappa$-fan we employ the Path-Bundle Transformation. In this case, for the $(s,t)$-reduced-path-bundle we get that the family of $\kappa - 2t$ internally disjoint $(v_3, X')$-paths $\{M_{2t+1}, \ldots, M_\kappa\}$ is actually a $(\kappa - 2t)$-fan from $v_3$ to $X'$, namely the terminal vertices of them are distinct. So it is easy to see that either the terminal vertices of $M_{2t+1}, \ldots, M_\kappa$ are on $\kappa - 2t$ distinct paths of $P'_{t+1}, \ldots, P'_s$ (in this case, $s - t \geq \kappa - 2t$), or there is a pair of distinct terminal vertices on a single path.

Let $N(v_1) = \{u | (v_1, u) \in E(G)\}$ and $N[v_1] = N(v_1) \cup \{v_1\}$. For the case that $|X| < \kappa$, since $G$ is $\kappa$-connected, we can get $|N[v_1]| \geq \kappa + 1$. Let $T = N[v_1] \cup X - \{v_3\}$. Then $|T| \geq \kappa$ and hence there is a $\kappa$-fan $\{M'_1, M'_2, \dots, M'_\kappa\}$ from $v_3$ to $T$. For $1 \leq i \leq \kappa$, let $y'_i$ be the terminal vertex of $M'_i$. If $y'_i \notin X$, then it must be in $N(v_1)$ and let $M_i = M'_i \cup y'_i v_1$. Otherwise, let $M_i = M'_i$. Now we get a family of $\kappa$ internally disjoint $(v_3, X)$-paths $\{M_1, M_2, \dots, M_\kappa\}$, and if there are two terminal vertices $y_i, y_j$ with $y_i = y_j$ for $i \neq j$, then $y_i = y_j = v_1$. Then, for this family of paths we employ the Path-Bundle Transformation. In this case, for the $(s,t)$-reduced-path-bundle we get that if two terminal vertices of $M_{2t+1}, \dots, M_\kappa$ are the same, then they must be $v_1$. Notice that $v_1$ and $v_2$ can be regarded as vertices on any of the paths $P'_{t+1}, \dots, P'_s$. Therefore, either the terminal vertices of $M_{2t+1}, \dots, M_\kappa$ are on $\kappa - 2t$ distinct paths of $P'_{t+1}, \dots, P'_s$ (in this case, $s - t \geq \kappa - 2t$), or there are two distinct terminal vertices (other than both $v_1$ and $v_2$) on a single path.

So we get the following conclusion.

**Remark 2.7.** *If an $(s,t)$-original-path-bundle $\{P_1, P_2, \dots, P_s\}$ exists, then employing the Path-Bundle Transformation, we can get an $(s,t)$-reduced-path-bundle $\{P'_1, P'_2, \dots, P'_s\} \cup \{M_1, M_2, \dots, M_{\kappa-2t}\}$ such that $P'_i = P_i$ for $t+1 \leq i \leq s$, and either the terminal vertices of $M_1, M_2 \dots, M_{\kappa-2t}$ are on $\kappa-2t$ distinct paths of $P'_{t+1}, \dots, P'_s$ (in this case, $s-t \geq \kappa-2t$), or there are at least two distinct terminal vertices, which are not $v_1$ or $v_2$, on a single path.*

**Two types of special $(s,t)$-reduced-path-bundles**

Now we point out two types of special $(s,t)$-reduced-path-bundles, which will be two crucial structures in the proof of our Theorem 2.3. We require that both of them satisfy $s = \kappa$.

**$(\kappa, t)$-reduced-path-bundle of type I:** If $\{P'_1, P'_2, \dots, P'_\kappa\} \cup \{M_1, M_2, \dots, M_{\kappa-2t}\}$ is a $(\kappa,t)$-reduced-path-bundle such that $1 \leq t \leq \lceil \frac{\kappa}{2} \rceil - 1$ and the terminal vertices $y_1, y_2, \dots, y_{\kappa-2t}$ of $M_1, M_2 \dots, M_{\kappa-2t}$ are on $\kappa - 2t$ distinct paths of $P'_{t+1}, \dots, P'_\kappa$, then the $(\kappa,t)$-reduced-path-bundle is called a $(\kappa,t)$-reduced-path-bundle of type I, as shown in Figure 6.

We will see that if $G$ contains a $(\kappa,t)$-reduced-path-bundle of type I connecting $S$, then $G$ contains $\kappa - \lceil \frac{t}{2} \rceil$ pairwise internally disjoint trees connecting $S$. Without loss of generality, we may assume that the terminal vertex $y_i$ is in $V(P'_{i+2t})$ for $1 \leq i \leq \kappa - 2t$. See Figure 6. Now let $T_1 = v_3 P'_1 v_1 P'_{t+1} v_2$, $T_2 = v_3 P'_1 v_2 P'_{t+2} v_1$,. . ., $T_{2i-1} = v_3 P'_i v_1 P'_{t+2i-1} v_2$, $T_{2i} = v_3 P'_i v_2 P'_{t+2i} v_1$, ... and $T_t = v_3 P'_{\lceil \frac{t}{2} \rceil} v_1 P'_{2t} v_2$ if $t$ is odd and $T_t = v_3 P'_{\lceil \frac{t}{2} \rceil} v_2 P'_{2t} v_1$ if $t$ is even. Let $T'_j = P'_{2t+j} \cup M_j$ for $1 \leq j \leq \kappa - 2t$, and let $\bar{T}_l = P'_l$ for $\lceil \frac{t}{2} \rceil + 1 \leq l \leq t$. So there are $t + (\kappa - 2t) + (t - \lceil \frac{t}{2} \rceil) = \kappa - \lceil \frac{t}{2} \rceil$ trees connecting $S$. Moreover, it is obvious that these trees are pairwise internally disjoint. It follows that $\kappa(S) \geq \kappa - \lceil \frac{t}{2} \rceil$ in this case, where $1 \leq t \leq \lceil \frac{\kappa}{2} \rceil - 1$.

**$(\kappa, t)$-reduced-path-bundle of type II:** When $\kappa$ is even and $t = \frac{\kappa}{2}$, then a $(\kappa,t)$-reduced-path-bundle $\{P'_1, P'_2, \dots, P'_\kappa\}$ is called a $(\kappa,t)$-reduced-path-bundle of type II. Obviously, it is also a $(\kappa, \frac{\kappa}{2})$-original-path-bundle, as shown in Figure 7.

Once again, we will see that if $G$ contains a $(\kappa,t)$-reduced-path-bundle of type II connecting $S$, then $G$ contains $\kappa - \lceil \frac{t}{2} \rceil$ pairwise internally disjoint trees connecting $S$. See Figure 7. Let $T_{2i-1} = v_3 P'_i v_1 P'_{t+2i-1} v_2$ for $1 \leq i \leq \lceil \frac{t}{2} \rceil$ and $T_{2i} = v_3 P'_i v_2 P'_{t+2i} v_1$ for $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$. Let $T'_j = P'_j$ for $\lceil \frac{t}{2} \rceil + 1 \leq j \leq t$. So there are $t + (t - \lceil \frac{t}{2} \rceil) = \kappa - \lceil \frac{t}{2} \rceil$ trees connecting $S$. Moreover, these trees are obviously pairwise internally disjoint. It follows

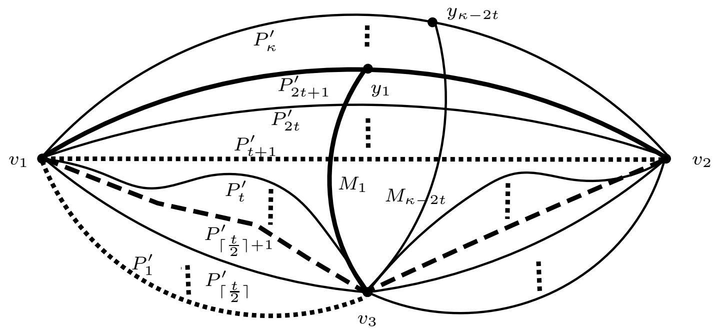


Figure 6: A $(\kappa, t)$-reduced-path-bundle of type I

that $\kappa(S) \geq \kappa - \lceil \frac{t}{2} \rceil$ in this case, where $\kappa$ is even and $t = \frac{\kappa}{2}$.

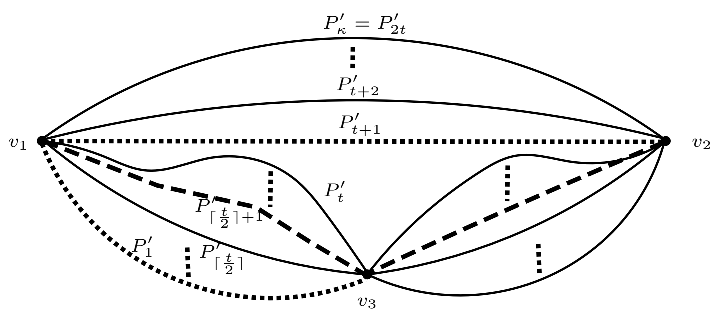


Figure 7: A $(\kappa, t)$-reduced-path-bundle of type II

*Proof of Theorem 2.3.* At first, we prove that the theorem is true for the case that $\kappa(G) = 4k$, where $k \in \mathbb{N}$. The other cases can be verified similarly.

*Case 1:* $\kappa(G) = 4k$ for $k \in \mathbb{N}$. This case is trivial when $k = 0$. So we may assume that $k \in \mathbb{N}^+$. We will show that $\kappa(S) \geq 3k$, where $S$ consists of any 3 vertices of $G$. Then it follows that $\kappa_3(G) \geq 3k$.

Suppose $S = \{v_1, v_2, v_3\}$. Since $G$ is $\kappa$-connected, there are $\kappa$ pairwise internally disjoint $v_1v_2$-paths $P_1, P_2, \ldots, P_\kappa$. Let $X = V(P_1 \cup \cdots \cup P_\kappa)$.

Suppose $v_3$ is not in $X$. Obviously, $|X| \geq \kappa$ and so by the Fan Lemma there exists a $\kappa$-fan $\{M_1, M_2, \ldots, M_\kappa\}$ from $v_3$ to $X$. If the terminal vertices $y_1, y_2, \ldots, y_\kappa$ of $M_1, M_2, \ldots, M_\kappa$ can be regarded as on the $\kappa$ paths $P_1, P_2, \ldots, P_\kappa$, respectively (keep in mind that if the terminal vertex is $v_1$ or $v_2$, it can be regarded as a vertex contained in any of the paths $P_1, P_2, \ldots, P_\kappa$), then we may let $y_i \in P_i$ and let $T_i = M_i \cup P_i$ for $1 \leq i \leq \kappa$. So we find $\kappa(G) = 4k > 3k$ pairwise internally disjoint trees connecting $S$, as shown in Figure 8(a). Otherwise, there are at least two terminal vertices on a single $v_1v_2$-path and without loss of generality, let $y_1, y_2 \in V(P_1)$ be such that $y_1$ is closer to $v_1$ than $y_2$ on $P_1$. Then $G$ has $\kappa$ pairwise internally disjoint $v_1v_2$-paths $P'_1 = v_1P_1y_1M_1v_3M_2y_2P_1v_2, P_2, \ldots, P_\kappa$ and $v_3$ is on $P'_1$, as shown in Figure 8(b).

Suppose $v_3$ is in $X$, and we know that it must be in the interior of one of the paths

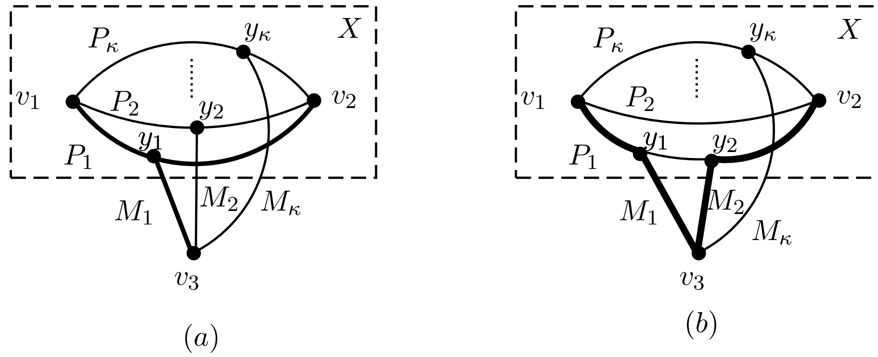


Figure 8: The case that $v_3$ is not in $X$

$P_1$, $P_2$, ..., $P_\kappa$.

Now, in any case, there exists a $(\kappa, 1)$-original-path-bundle $\{P_1, P_2, \ldots, P_\kappa\}$ connecting $S$. Then we know that a $(\kappa, 1)$-reduced-path-bundle $\{P_1', P_2', \ldots, P_\kappa'\} \cup \{M_1, M_2, \ldots, M_{\kappa-2}\}$ must exist. Moreover, either the terminal vertices $y_1, y_2, \ldots, y_{\kappa-2}$ of $M_1, M_2 \ldots, M_{\kappa-2}$ are on $\kappa - 2$ distinct paths of $P_2', \ldots, P_\kappa'$, or there are two distinct terminal vertices, which are not $v_1$ or $v_2$, on a same path by Remark 2.7. For the former case, we get a $(\kappa, 1)$-reduced-path-bundle of type I connecting $S$, which means that $\kappa(S) \geq \kappa - \lceil \frac{1}{2} \rceil = \kappa - 1 = 4k - 1 \geq 3k$, as required. While for the latter case, we may assume that $y_1, y_2 \in V(P_2')$ and $y_1$ is closer to $v_1$ than $y_2$ on $P_2'$. Let $P_2'' = v_1 P_2' y_1 M_1 v_3 M_2 y_2 P_2' v_2$. Note that by the definition of a $(\kappa, 1)$-reduced-path-bundle, for $i = 1$ or 2, $V(M_i) \cap V(P_1') = v_3$ and $V(M_i) \cap X = y_i$, where $X = V(P_2' \cup \ldots \cup P_\kappa')$. Then it is easy to see that $\{P_1', P_2'', P_3', \ldots, P_\kappa'\}$ is a $(\kappa, 2)$-original-path-bundle.

Repeat the operation, namely if there is a $(\kappa, t)$-original-path-bundle $\{P_1, P_2, \ldots, P_\kappa\}$ connecting $S$ for $1 \leq t \leq \lceil \frac{\kappa}{2} \rceil - 1$, then employing the Path-Bundle Transformation, we can obtain a $(\kappa, t)$-reduced-path-bundle $\{P_1', P_2', \ldots, P_\kappa'\} \cup \{M_1, M_2, \ldots, M_{\kappa-2t}\}$ such that either it is a $(\kappa, t)$-reduced-path-bundle of type I or there are two distinct terminal vertices of $M_1, M_2, \ldots, M_{\kappa-2t}$ on a same path. If it is exactly a $(\kappa, t)$-reduced-path-bundle of type I, then $\kappa(S) \geq \kappa - \lceil \frac{t}{2} \rceil \geq 3k$ for $1 \leq t \leq \lceil \frac{\kappa}{2} \rceil - 1$, as required. Otherwise, we may assume that the terminal vertices $y_1, y_2$ of $M_1, M_2$ are both on $P_{t+1}'$. Let $P_{t+1}'' = v_1 P_{t+1}' y_1 M_1 v_3 M_2 y_2 P_{t+1}' v_2$. Then we obtain a $(\kappa, t+1)$-original-path-bundle $\{P_1', \ldots, P_t', P_{t+1}'', P_{t+2}', \ldots, P_\kappa'\}$.

The procedure will terminate when either we get a $(\kappa, t)$-reduced-path-bundle of type I connecting $S$ for $1 \leq t \leq \lceil \frac{\kappa}{2} \rceil - 1$, which means $\kappa(S) \geq 3k$, or we get a $(\kappa, \frac{\kappa}{2})$-original-path-bundle which is actually a $(\kappa, t)$-reduced-path-bundle of type II. We know that if $G$ contains a $(\kappa, t)$-reduced-path-bundle of type II connecting $S$, then $\kappa(S) \geq \kappa - \lceil \frac{t}{2} \rceil = \kappa - \lceil \frac{\kappa}{4} \rceil = 3k$, as required. The proof is complete.

*Case 2:* $\kappa(G) = 4k + 1$ for $k \in \mathbb{N}$. It is obvious that $\kappa_3(G) \geq 1$ when $\kappa(G) = 1$. So we may assume $k > 0$. By a method similar to the previous case, we can show that $\kappa(S) \geq 3k+1$, where $S$ consists of any 3 vertices of $G$. Then it follows that $\kappa_3(G) \geq 3k+1$. But this case is a little bit different from Case 1, since $\kappa$ is odd.

It is clear that, once $G$ contains a $(\kappa, t)$-reduced-path-bundle of type I connecting $S$

for $1 \leq t \leq \lceil \frac{\kappa}{2} \rceil - 1$, then $\kappa(S) \geq \kappa - \lceil \frac{t}{2} \rceil \geq 3k+1$ and the proof is complete.

Let $\{P_1, P_2, \ldots, P_\kappa\}$ be a $(\kappa, t)$-original-path-bundle, where $t = \lceil \frac{\kappa}{2} \rceil - 1 = 2k$. Then employing the Path-Bundle Transformation, we can obtain a $(\kappa, 2k)$-reduced-path-bundle $\{P_1', P_2', \ldots, P_\kappa'\} \cup \{M_1\}$. Since there is only one terminal vertex, the path-bundle is exactly a $(\kappa, 2k)$-reduced-path-bundle of type I. So for Case 2, the procedure can certainly terminate with $\kappa(S) \geq 3k+1$, as required.

*Case 3:* $\kappa(G) = 4k+2$ for $k \in \mathbb{N}$. It is obvious that $G$ is $(4k+1)$-connected and so by Case 2, $\kappa_3(G) \geq 3k+1$.

*Case 4:* $\kappa(G) = 4k+3$ for $k \in \mathbb{N}$. $\kappa - \lceil \frac{t}{2} \rceil \geq 3k+2$ for $1 \leq t \leq \lceil \frac{\kappa}{2} \rceil - 1$. The method used above still works for this case.

From the above, for every two integers $k$ and $r$ with $k \geq 0$ and $r \in \{0, 1, 2, 3\}$, if $\kappa(G) = 4k + r$, we can show that $\kappa(S) \geq 3k + \lceil \frac{r}{2} \rceil$, where $S$ consists of any 3 vertices of $G$. It follows that $\kappa_3(G) \geq 3k + \lceil \frac{r}{2} \rceil$.

Next, we will give graphs attaining the lower bound.

For $\kappa(G) = 4k + 2i$ with $i = 0$ or $1$, we construct a graph $G$ as follows (see Figure 9): Let $Q = Y_1 \cup Y_2$ be a vertex cut of $G$, where $Q$ is a clique and $|Y_1| = |Y_2| = 2k + i$. $G - Q$ has 2 components $C_1, C_2$. $C_1 = \{v_3\}$ and $v_3$ is adjacent to every vertex in $Q$; $C_2 = \{v_1\} \cup \{v_2\} \cup X$, $|X| = 2k + i$, the subgraph induced by $X$ is an empty graph, every vertex in $X$ is adjacent to every vertex in $Q \cup \{v_1, v_2\}$, $v_1$ is adjacent to every vertex in $Y_1$ and $v_2$ is adjacent to every vertex in $Y_2$. It can be easily checked that $\kappa(G) = 4k + 2i$.

Let $S = \{v_1, v_2, v_3\}$ and let $\{T_1, T_2, \ldots, T_l\}$ be a set of internally disjoint trees connecting $S$. For each $T_i$, there must be a $v_1v_3$-path containing a vertex in $Q$ and $(T_i \cap Q) \cap (T_j \cap Q) = \emptyset$ for $1 \leq i < j \leq l$. If $T_i$ contains only one vertex in $Q$, then $v_3$ is a leaf of $T_i$ which means that $T_i - v_3$ is still a tree connecting $v_1$ and $v_2$. But we can see that every vertex in $Q$ is adjacent to only one of $v_1$ and $v_2$. So $T_i - v_3$ must contain a vertex in $X$. Therefore, there are at most $|X|$ trees in $\{T_1, T_2, \ldots, T_l\}$ containing only one vertex of $Q$ and the others contain at least two vertices of $Q$. Then we can get that $l \leq |X| + \lfloor \frac{|Q| - |X|}{2} \rfloor = 2k + i + \lfloor \frac{2k+i}{2} \rfloor = 3k + i$ and $\kappa_3(G) \leq \kappa(S) = l \leq 3k + i$. On the other hand, we know that $\kappa_3(G) \geq 3k + i$. It follows that $\kappa_3(G) = 3k + i$, which means that $G$ attains the lower bound.

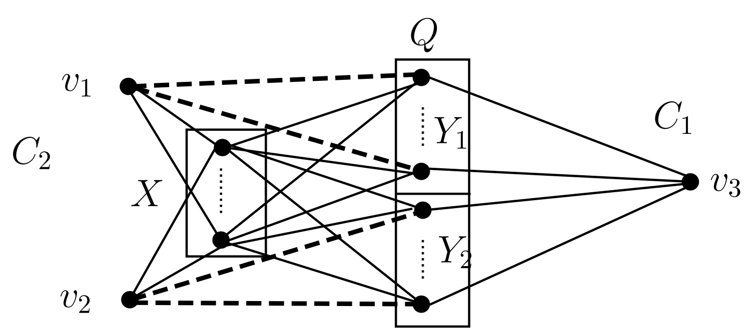


Figure 9: For $\kappa(G) = 4k + 2i$ with $i = 0$ or $1$, the graph attaining the lower bound

For $\kappa(G) = 4k + 2i + 1$ with $i = 0$ or $1$, we construct a graph $G$ as follows: Let

$Q = Y_1 \cup Y_2 \cup \{y_0\}$ be a vertex cut of $G$, where $Q$ is a clique and $|Y_1| = |Y_2| = 2k + i$. $G - Q$ has 2 components $C_1,\ C_2$. $C_1 = \{v_3\}$ and $v_3$ is adjacent to every vertex in $Q$; $C_2 = \{v_1\} \cup \{v_2\} \cup X$, $|X| = 2k + i$, the subgraph induced by $X$ is an empty graph, every vertex in $X$ is adjacent to every vertex in $Q \cup \{v_1, v_2\}$, $v_1$ is adjacent to every vertex in $Y_1$, $v_2$ is adjacent to every vertex in $Y_2$, and both $v_1$ and $v_2$ are adjacent to $y_0$. It can be checked similarly that $\kappa(G) = 4k + 2i + 1$ and $\kappa_3(G) = 3k + i + 1$, which means that $G$ attains the lower bound. ∎

# 3 Bounds for planar graphs

In this section we will study $\kappa_3(G)$ for planar graphs. More precisely, we will give bounds of $\kappa_3(G)$ for planar graphs and some graphs that attain the bounds.

First, we give the following lemma:

**Lemma 3.1.** *Let $G$ be a connected graph with minimum degree $\delta$. Then $\kappa_3(G) \leq \delta$. In particular, if there are two adjacent vertices of degree $\delta$, then $\kappa_3(G) \leq \delta - 1$.*

*Proof.* We know that $\kappa(G) \leq \delta$ ([2]-p.217) and $\kappa_3(G) \leq \kappa(G)$ by Theorem 2.2. So $\kappa_3(G) \leq \delta$.

By contradiction, suppose that there are two adjacent vertices $v_1$ and $v_2$ of degree $\delta$ and $\kappa_3(G) = \delta$. Besides $v_1$ and $v_2$, we choose a vertex $v_3$ in $V(G - \{v_1, v_2\})$ to get a set $S = \{v_1, v_2, v_3\}$. Suppose $T_1, T_2, \ldots, T_\delta$ are $\delta$ pairwise internally disjoint trees connecting $S$. Obviously, the $\delta$ edges incident with $v_1$ must be contained in $T_1, T_2, \ldots, T_\delta$, respectively, and so are the $\delta$ edges incident with $v_2$. Without loss of generality, we may assume that the edge $v_1v_2$ is contained in $T_1$. But since $T_1$ is a tree connecting $\{v_1, v_2, v_3\}$, it must contain another edge incident with $v_1$ or $v_2$, a contradiction. It follows that $\kappa_3(G) \leq \delta - 1$. ∎

By the well-known Kuratowski's Theorem [4], a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$. We will use the theorem to prove the following lemma:

**Lemma 3.2.** *For a connected planar graph $G$ with $\kappa_3(G) = k$, there are no 3 vertices of degree $k$ in $G$, where $k \geq 3$.*

*Proof.* By contradiction, let $v_1, v_2$ and $v_3$ be 3 vertices of degree $k$. Because $\kappa_3(G) = k$, there exist $k$ pairwise internally disjoint trees $T_1, T_2, \ldots, T_k$ connecting $S = \{v_1, v_2, v_3\}$. Obviously, for any $i \in \{1, 2, 3\}$, the $k$ edges incident with $v_i$ are contained in $T_1, T_2, \ldots, T_k$, respectively. Therefore, for any tree $T_i$ $(1 \leq i \leq k)$, $d_{T_i}(v_1) = d_{T_i}(v_2) = d_{T_i}(v_3) = 1$, that is, $v_1, v_2$ and $v_3$ are leaves of $T_i$. It can be checked that, for every tree $T_i$, there exists a vertex $t_i$ such that $T_i$ is a 3-fan from $t_i$ to $S$ (see Figure 10). Since $k \geq 3$, $T_1, T_2$ and $T_3$ exist. But $T_1 \cup T_2 \cup T_3$ is a subdivision of $K_{3,3}$, a contradiction. ∎

A $k$-connected graph $G$ is *minimally $k$-connected* if the graph $G - e$ is not $k$-connected for any edge $e$ of $G$, that is, if no edge can be deleted. In the following, we list some known results which will be used later.

The next lemma is from lines 12 and 13 on page 21 of [1].

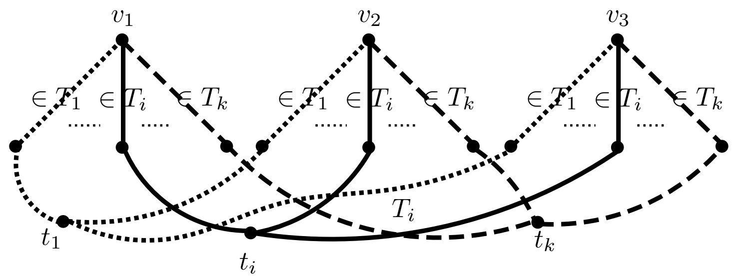


Figure 10: A 3-fan $T_i$ from $t_i$ to $\{v_1, v_2, v_3\}$ and a subdivision of $K_{3,3}$

**Lemma 3.3.** *For $k = 2, 3$ each cycle of a minimally $k$-connected graph contains at least two vertices of degree $k$.*

**Lemma 3.4.** *([1]-p.24) Let $G$ be a minimally $k$-connected graph and let $U$ be the set of vertices of degree $k$. Then $G - U$ is a (possibly empty) forest.*

**Lemma 3.5.** *([1]-p.25) A minimally $k$-connected graph of order $n$ has at least $\frac{(k-1)n+2}{2k-1}$ vertices of degree $k$.*

**Lemma 3.6.** *([2]-p.213) Let $G$ be a $k$-connected graph and let $H$ be a graph obtained from $G$ by adding a new vertex $y$ and joining it to at least $k$ vertices of $G$. Then $H$ is also $k$-connected.*

**Lemma 3.7.** *([2]-p.260) Let $G$ be a planar graph on at least three vertices. Then $|E(G)| \leq 3|V(G)| - 6$.*

**Lemma 3.8.** *([2]-p.260) Every planar graph $G$ has a vertex of degree at most 5, i.e., $\delta(G) \leq 5$.*

Since $\kappa(G) \leq \delta$ ([2]-p.217), by Lemma 3.8 we only need to consider planar graphs $G$ with connectivity $\kappa(G)$ at most 5. From Theorem 2.3, it can be deduced that for any graph (not necessarily planar) if $\kappa(G) = 1$, $\kappa_3(G) \geq 1$; if $\kappa(G) = 2$, $\kappa_3(G) \geq 1$; if $\kappa(G) = 3$, $\kappa_3(G) \geq 2$; if $\kappa(G) = 4$, $\kappa_3(G) \geq 3$; and if $\kappa(G) = 5$, $\kappa_3(G) \geq 4$. While from Theorem 2.2, we know that $\kappa_3(G) \leq \kappa(G)$, and so we get $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$, for $1 \leq \kappa(G) \leq 5$. Therefore, the following theorem is obvious.

**Theorem 3.1.** *If $G$ is a connected planar graph, then $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$.*

Next we study the 3-connectivity of four kinds of graphs.

**Lemma 3.9.** *If $\kappa(G) \geq 3$, then $\kappa_3(G - e) \geq 2$ for any edge $e \in E(G)$.*

*Proof.* It suffices to show that for any minimally 3-connected graph $G$ and any edge $e \in E(G)$, $\kappa_3(G - e) \geq 2$. Let $v_1, v_2$ and $v_3$ be any 3 vertices of $G - e$, which are also 3 vertices of $G$.

*Case 1:* In $G - e$, there exists a 2-subset $S$ of $\{v_1, v_2, v_3\}$ such that $\kappa(S) \geq 3$.

Without loss of generality, we may assume that $\kappa_{G-e}(\{v_1, v_2\}) \geq 3$ and $P_1, P_2, P_3$ are three internally disjoint $v_1v_2$-paths of $G - e$.

*Subcase 1.1:* $v_3 \in V(P_1 \cup P_2 \cup P_3)$. Without loss of generality, we may let $v_3 \in V(P_1)$. See Figure 11(a). Then, in $G - e$ there are two internally disjoint trees connecting $\{v_1, v_2, v_3\}$, that is, $T_1 = v_3P_1v_1P_2v_2$ and $T_2 = v_3P_1v_2P_3v_1$.

*Subcase 1.2:* $v_3 \notin V(P_1 \cup P_2 \cup P_3)$. Let $X = V(P_1 \cup P_2 \cup P_3)$. Since $G - e$ is 2-connected, $v_3$ is not in $X$ and $|X| \geq 2$, then there exists a 2-fan $\{M_1, M_2\}$ from $v_3$ to $X$ by the Fan Lemma. Let $y_1$ and $y_2$ be the two terminal vertices of $M_1$ and $M_2$, respectively.

If $y_1$ and $y_2$ are on two of the three $v_1v_2$-paths, without loss of generality, we may let $y_1 \in V(P_1)$ and $y_2 \in V(P_2)$. See Figure 11(b). Then in $G - e$ there are two internally disjoint trees connecting $\{v_1, v_2, v_3\}$, that is, $T_1 = P_1 \cup M_1$ and $T_2 = P_2 \cup M_2$.

If $y_1$ and $y_2$ are on a same path, without loss of generality, we may let $y_1, y_2 \in V(P_1)$ and let $y_1$ be closer to $v_1$ than $y_2$ on $P_1$. See Figure 11(c). Then, in $G - e$ there are two internally disjoint trees connecting $\{v_1, v_2, v_3\}$, that is, $T_1 = v_3M_1y_1P_1v_1P_2v_2$ and $T_2 = v_3M_2y_2P_1v_2P_3v_1$.

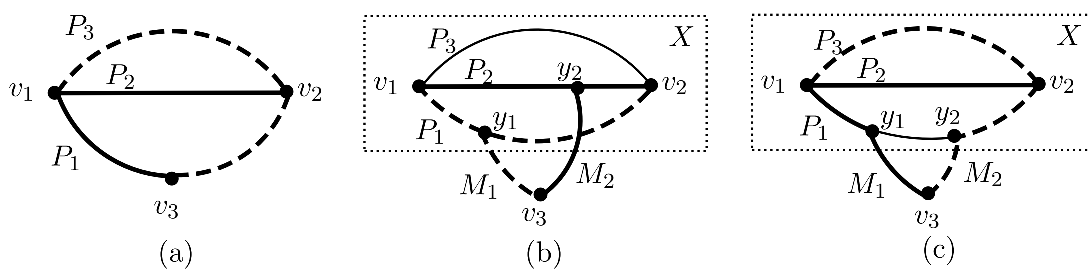


Figure 11: The three graphs for Case 1 with $P_1, P_2, P_3 \subseteq G - e$

*Case 2:* In $G - e$, for each 2-subset $S$ of $\{v_1, v_2, v_3\}$, $\kappa(S) = 2$.

For each 2-subset $S$ of $\{v_1, v_2, v_3\}$, since $G$ is 3-connected, then $\kappa_G(S) \geq 3$. Let $P_1, P_2$ and $P_3$ be three internally disjoint $v_1v_2$-paths of $G$. But since $\kappa_{G-e}(\{v_1, v_2\}) = 2$, it is obvious that the edge $e \in E(P_1 \cup P_2 \cup P_3)$. Without loss of generality, we may assume $e \in E(P_3)$. There exist three subcases: $v_3 \in V(P_1 \cup P_2)$, $v_3 \in V(P_3)$ and $v_3 \notin V(P_1 \cup P_2 \cup P_3)$.

*Subcase 2.1:* $v_3 \in V(P_1 \cup P_2)$. We may assume $v_3 \in V(P_1)$. Since $\kappa(G) = 3$, it is easy to see that $\{P_1, P_2, P_3\}$ is a $(3, 1)$-original-path-bundle connecting $\{v_1, v_2, v_3\}$ in $G$. Then employing the Path-Bundle Transformation, we can get a $(3, 1)$-reduced-path-bundle $\{P_1', P_2', P_3'\} \cup \{M_1\}$ of $G$ such that $P_2' = P_2$ and $P_3' = P_3$, by Remark 2.7. Since $e \in E(P_3)$, then $e \in E(P_3')$. Let $y$ be the terminal vertex of $M_1$. See Figure 12.

If $y \in V(P_2')$, then $T_1 = P_1'$ and $T_2 = P_2' \cup M_1$ are two internally disjoint trees connecting $\{v_1, v_2, v_3\}$ in $G - e$, as shown in Figure 12(a).

If $y \in V(P_3')$ and without loss of generality, let $e \in E(yP_3'v_2)$, then $T_1 = P_1'$ and $T_2 = v_2P_2'v_1P_3'yM_1v_3$ are two internally disjoint trees connecting $\{v_1, v_2, v_3\}$ in $G - e$, as shown in Figure 12(b).

*Subcase 2.2:* $v_3 \in V(P_3)$. Without loss of generality, we may assume $e \in E(v_3P_3v_2)$.

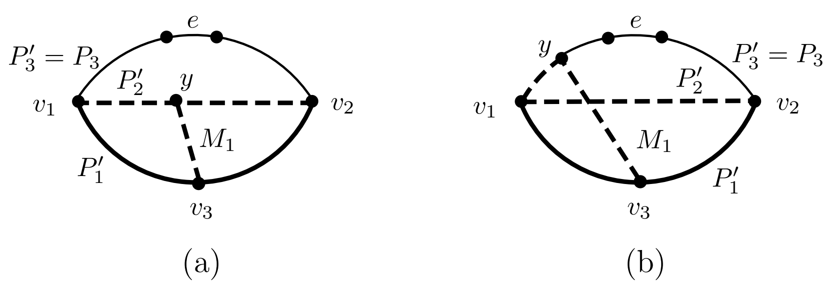


Figure 12: The two graphs for Subcase 2.1

Let $X = V(P_1 \cup P_2)$. Since $G - e$ is 2-connected, $v_3$ is not in $X$ and $|X| \geq 2$, then there exists a 2-fan $\{M_1, M_2\}$ from $v_3$ to $X$ by the Fan Lemma. Let $V(M_1 \cup M_2) \cap V(\hat{v}_1 P_3 v_3) = N$ (in Figure 13, the crosses indicate the vertices of $N$), where $N \neq \emptyset$, since at least the vertex $v_3$ belongs to $N$. Among all vertices in $N$, let $w$ be the vertex nearest to $v_1$ on $P_3$, namely, $V(\hat{v}_1 P_3 \hat{w}) \cap N = V(\hat{v}_1 P_3 \hat{w}) \cap V(M_1 \cup M_2) = \emptyset$. We may let $w \in V(M_1)$ and let the terminal vertex of $M_2$ be on $P_2$. See Figure 13. Then $T_1 = M_2 \cup P_2$ and $T_2 = v_3 M_1 w P_3 v_1 P_1 v_2$ are two trees connecting $\{v_1, v_2, v_3\}$ in $G - e$. Since $\{M_1, M_2\}$ is a 2-fan from $v_3$ to $X$, $V(M_1) \cap V(M_2) = \{v_3\}$, $V(v_3 M_1 w) \cap V(P_2) = \emptyset$ and $V(P_1) \cap V(M_2) = \emptyset$. Moreover, $V(\hat{w} P_3 \hat{v}_1) \cap V(M_2) = \emptyset$. It is easy to check that $T_1$ and $T_2$ are internally disjoint.

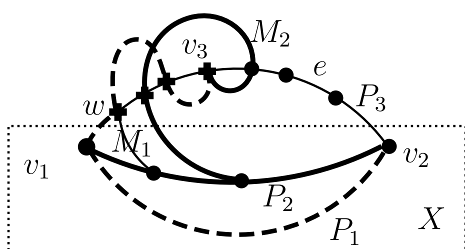


Figure 13: The graph for Subcase 2.2

*Subcase 2.3:* $v_3 \notin V(P_1 \cup P_2 \cup P_3)$. Let $X = V(P_1 \cup P_2 \cup P_3)$, and then $v_3 \notin X$. Since $G$ is 3-connected and $|X| \geq 3$, there exists a 3-fan $\{M_1, M_2, M_3\}$ from $v_3$ to $X$ by the Fan Lemma. Since $e \in E(P_3)$, then $e \notin E(M_1 \cup M_2 \cup M_3)$. Therefore, the 3-fan $\{M_1, M_2, M_3\}$ is still contained in $G - e$. Let $y_1, y_2$ and $y_3$ be the terminal vertices of $M_1, M_2$ and $M_3$, respectively.

If there are two vertices $y_{i_1}$ and $y_{i_2}$ on two distinct paths $P_{j_1}$ and $P_{j_2}$, for $1 \leq i_1 \neq i_2 \leq 3$ and $1 \leq j_1 \neq j_2 \leq 3$, then it is easy to find two internally disjoint trees connecting $\{v_1, v_2, v_3\}$ in $G - e$. See Figure 14.

If all of the 3 vertices $y_1, y_2, y_3$ are on the same path $P_3$ and $e = uv$, then either $V(v_1 P_3 u)$ or $V(v P_3 v_2)$ contains at least two of them. Without loss of generality, we may assume that $y_1$ and $y_2$ are both contained in $V(v_1 P_3 u)$ and $y_1$ is closer to $v_1$ than $y_2$. See Figure 15(a). Then there exist three internally disjoint $v_1 v_2$-paths $P_1, P_2$ and $P_3' = v_1 P_3 y_1 M_1 v_3 M_2 y_2 P_3 v_2$ in $G$ such that $e \in E(P_3')$ and $v_3 \in V(P_3')$, which was solved by Subcase 2.2.

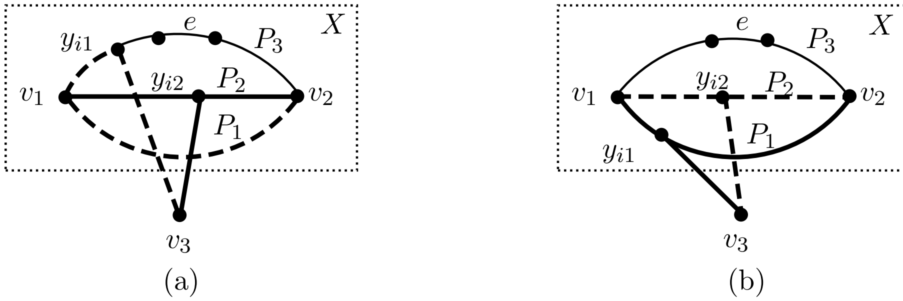


Figure 14: The graphs for Subcase 2.3 where there are two terminal vertices on two distinct $v_1v_2$-paths

If all of the three vertices $y_1, y_2, y_3$ are on the same path $P_1$ or $P_2$, without loss of generality, we may let $y_1, y_2, y_3 \in V(P_1)$ and let $y_1$ be the nearest vertex to $v_1$ among the three vertices. See Figure 15(b). Then there exist three internally disjoint $v_1v_2$-paths $P_1' = v_1P_1y_1M_1v_3M_2y_2P_1v_2$, $P_2$ and $P_3$ in $G$ such that $e \in E(P_3)$ and $v_3 \in V(P_1')$, which was solved by Subcase 2.1.

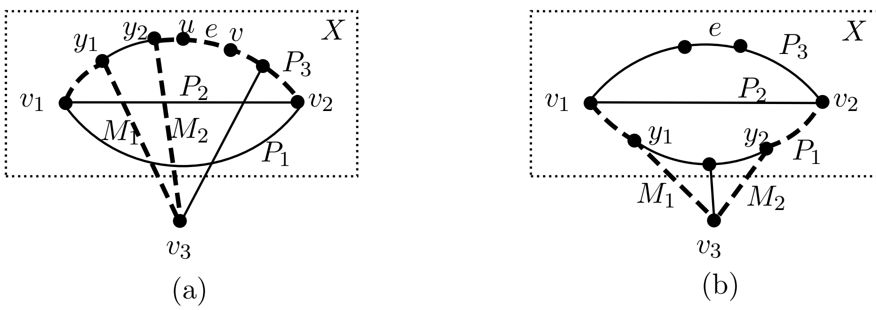


Figure 15: The graphs for Subcase 2.3 where the three terminal vertices are all on a same path

From the above, we can always find two internally disjoint trees connecting $\{v_1, v_2, v_3\}$ in $G - e$, where $v_1, v_2, v_3$ are any 3 vertices of $G - e$. It follows that $\kappa_3(G - e) \geq 2$. The proof is complete. ∎

**Lemma 3.10.** *If $G$ is a planar minimally 3-connected graph, then $\kappa_3(G) = 2$.*

*Proof.* Obviously, $\kappa_3(G) \geq 2$. So it suffices to show that $\kappa_3(G) \leq 2$.

If there are two adjacent vertices of degree 3, then by Lemma 3.1 we can get $\kappa_3(G) \leq 2$, as required.

Otherwise, any two vertices of degree 3 are not adjacent. Let $T$ be the set of vertices of degree 3, and so $G[T]$ is an empty graph. By Lemma 3.4, we can get that $G - T$ is a forest. Let $F_1$ be a component of the forest and let $\partial(F_1)$ denote the edge cut of $G$ associated with $V(F_1)$. Since the degree of any vertex in $V(F_1)$ is at least 4 in $G$, then $\sum_{v \in V(F_1)} d_G(v) \geq 4|F_1|$. Moreover, $\sum_{v \in V(F_1)} d_G(v) = \sum_{v \in V(F_1)} d_{F_1}(v) + |\partial(F_1)| =$

$2(|F_1|-1)+|\partial(F_1)|$. It follows that $|\partial(F_1)| \geq 4|F_1|-2(|F_1|-1) = 2|F_1|+2 > 3$. Let $N(F_1)=\{u|(v,u)\in E(G), v\in V(F_1), u\notin V(F_1)\}$. Then we know $N(F_1)\subseteq T$. If there are two vertices $v_1, v_2$ in $V(F_1)$ adjacent to a same vertex $u$ of $T$, namely $v_1u, v_2u \in \partial(F_1)$, there exists a cycle $C = v_1Pv_2uv_1$, where $P$ is a $v_1v_2$-path in $F_1$. There is only one vertex of degree 3 in $V(C)$. But, by Lemma 3.3 we know that each cycle of a minimally 3-connected graph contains at least two vertices of degree 3, a contradiction. Therefore, any two vertices in $V(F_1)$ can not be adjacent to a same vertex of $T$, namely $|T| \geq |\partial(F_1)| > 3$. Then, in $G$ there are three vertices of degree 3. By Lemma 3.2, we can get $\kappa_3(G)\neq 3$, namely $\kappa_3(G)\leq 2$, as required.

So for any planar minimally 3-connected graph $G$, $\kappa_3(G)=2$. ∎

**Lemma 3.11.** *Let $G$ be a 4-connected graph and let $H$ be a graph obtained from $G$ by adding a new vertex $w$ and joining it to three vertices of $G$. Then $\kappa_3(H)=\kappa(H)=3$.*

*Proof.* Since $G$ is a 4-connected graph which is also a 3-connected graph, then by Lemma 3.6 $H$ is still 3-connected. Moreover, there is a vertex $w$ of degree 3 in $H$. So we can get $\kappa(H)=3$ and it follows that $\kappa_3(H)\leq 3$ by Theorem 2.2. Hence it suffices to show that $\kappa_3(H)\geq 3$. Let $v_1, v_2$ and $v_3$ be any 3 vertices of $H$.

*Case 1:* If $v_1, v_2, v_3 \in V(G)$, that is, $v_i \neq w$ for $1\leq i\leq 3$, then restricted on $G$, we have $\kappa_G(\{v_1,v_2,v_3\})\geq 3$, since $\kappa(G)\geq 4$. So it is obvious that $\kappa_H(\{v_1,v_2,v_3\})\geq 3$.

*Case 2:* If $v_i = w$, for some $1\leq i\leq 3$, without loss of generality, we may assume $v_3 = w$. Since $\kappa(G)\geq 4$, there are four internally disjoint $v_1v_2$-paths $P_1, P_2, P_3, P_4$ of $G$. Obviously, the four paths still exist in $H$. Let $X = V(P_1\cup P_2\cup P_3\cup P_4)$. Since $\kappa(H)=3$, $w$ is not in $X$ and $|X|\geq 3$, then there exists a 3-fan $\{M_1, M_2, M_3\}$ from $w$ to $X$ by the Fan Lemma.

If the terminal vertices $y_1, y_2, y_3$ of $M_1, M_2, M_3$ are on three of the four $v_1v_2$-paths, without loss of generality, let $y_1\in V(P_1), y_2\in V(P_2)$ and $y_3\in V(P_3)$. Then there are three internally disjoint trees connecting $\{v_1, v_2, w\}$, that is, $T_1 = P_1\cup M_1, T_2 = P_2\cup M_2$ and $T_3 = P_3\cup M_3$.

Otherwise, there are two terminal vertices on a same $v_1v_2$-path. Without loss of generality, we may assume that $y_1, y_2\in V(P_1)$ and $y_1$ is closer to $v_1$ than $y_2$ on $P_1$. See Figure 16. Let $\hat{P}_1 = v_1P_1y_1M_1wM_2y_2P_1v_2$. Since $\kappa(H)=3$, it is easy to see that $\{\hat{P}_1, P_2, P_3, P_4\}$ is a $(4,1)$-original-path-bundle connecting $\{v_1, v_2, w\}$ in $H$. Then, employing the Path-Bundle Transformation, we can get a $(4,1)$-reduced-path-bundle $\{P_1', P_2', P_3', P_4'\}\cup\{M_1\}$ of $H$. Let $y$ be the terminal vertex of $M_1$ and without loss of generality, let $y\in V(P_2')$. Then $T_1 = P_2'\cup M_1, T_2 = wP_1'v_1P_3'v_2$ and $T_3 = wP_1'v_2P_4'v_1$ are three internally disjoint trees connecting $\{v_1, v_2, w\}$.

Therefore, for Case 2 we can always find three internally disjoint trees connecting $\{v_1, v_2, v_3 = w\}$ in $H$, namely $\kappa_H(\{v_1,v_2,v_3\})\geq 3$, completing the proof. ∎

**Lemma 3.12.** *If $G$ is a planar minimally 4-connected graph, then $\kappa_3(G)=3$.*

*Proof.* It is clear that $\kappa(G)=4$ and $3\leq\kappa_3(G)\leq 4$. So it suffices to show that $\kappa_3(G)\neq 4$.

Since $G$ is planar and $\kappa(G)=4$, obviously $|G| = n > 5$.

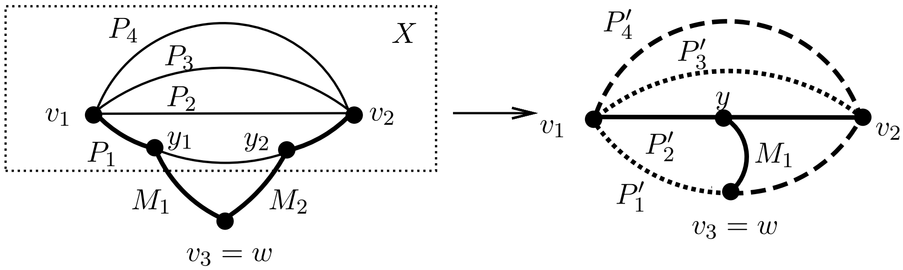


Figure 16: The graphs for Lemma 3.11

If $n = 6$, since $\kappa(G) = 4$, the degree of any vertex is at least 4. By Lemma 3.7, we know that $|E(G)| = m \leq 3n - 6$. So $6 \times 4 = 24 \leq \sum d(v) = 2m \leq 6n - 12 = 24$, which means that the degree of every vertex is 4.

If $n \geq 7$, let $U$ be the set of vertices of degree 4. Since $G$ is a minimally 4-connected graph, then by Lemma 3.5, $|U| \geq \frac{3n+2}{7} > 3$. So there exist three vertices of degree 4 in $G$.

Therefore, in either case $G$ must contain three vertices of degree 4. Then, by Lemma 3.2 we can get $\kappa_3(G) \neq 4$, as required. ∎

If $G$ is a connected planar graph, then $1 \leq \kappa(G) \leq 5$ and by Theorem 3.1, $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$. Now for each $1 \leq \kappa(G) \leq 5$, we give some classes of planar graphs attaining the bounds of $\kappa_3(G)$, respectively.

*Case 1:* $\kappa(G) = 1$**.** For any graph $G$ with $\kappa(G) = 1$, obviously $\kappa_3(G) \geq 1$ and so $\kappa_3(G) = 1 = \kappa(G)$. Therefore, all planar graphs with connectivity 1 can attain the upper bound, but can not attain the lower bound.

*Case 2:* $\kappa(G) = 2$**.** Let $G$ be a planar graph with connectivity 2 and having two adjacent vertices of degree 2. Then by Lemma 3.1, $\kappa_3(G) \leq 1$ and so $\kappa_3(G) = 1 = \kappa(G) - 1$. Therefore, this class of graphs attain the lower bound. For example, for any cycle $C$, we have $\kappa(C) = 2$ and $\kappa_3(C) = 1$.

Let $G$ be a planar minimally 3-connected graph. By the definition, for any edge $e \in E(G)$, we can get $\kappa(G - e) = 2$. Then by Theorem 2.2 and Lemma 3.9, it follows that $\kappa_3(G - e) = 2$. Therefore, the 2-connected planar graph $G - e$ attains the upper bound.

*Case 3:* $\kappa(G) = 3$**.** For any planar minimally 3-connected graph $G$, we know that $\kappa(G) = 3$ and by Lemma 3.10, $\kappa_3(G) = 2 = \kappa(G) - 1$. So this class of graphs attain the lower bound.

Let $G$ be a planar 4-connected graph and let $H$ be a graph obtained from $G$ by adding a new vertex $w$ in the interior of a face for some planar embedding of $G$ and joining it to three vertices on the boundary of the face. Then $H$ is still planar and by Lemma 3.11, we can immediately get $\kappa_3(H) = \kappa(H) = 3$, which means that $H$ attains the upper bound.

*Case 4:* $\kappa(G) = 4$**.** For any planar minimally 4-connected graph $G$, we know that $\kappa(G) = 4$ and by Lemma 3.12, $\kappa_3(G) = 3 = \kappa(G) - 1$. So this class of graphs attain the lower bound.

For every graph in Figure 17, the vertex in the center has degree 4 and it can be checked that for any two vertices there always exist 4 pairwise internally disjoint paths connecting them, which means $\kappa = 4$. It can also be checked that for any three vertices there always exist 4 pairwise internally disjoint trees connecting them. Combined with Theorem 2.2, we can get $\kappa_3 = 4$. Therefore, the graphs attain the upper bound. Moreover, we can construct a series of graphs according to the pattern of Figure 17, which attain the upper bound.

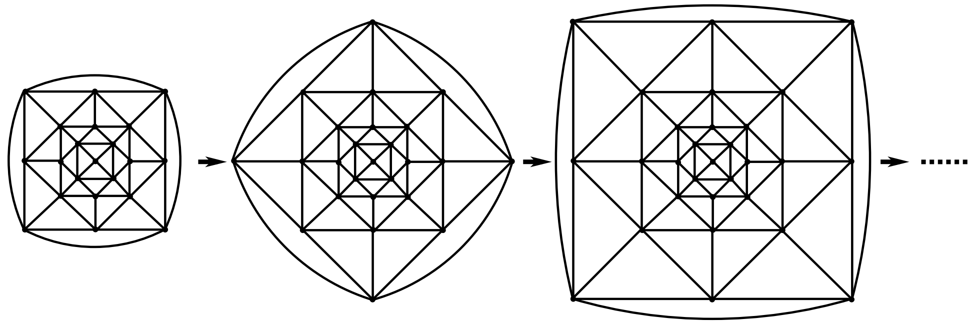

Figure 17: The graphs for the upper bound of Case 4

*Case 5:* $\kappa(G) = 5$**.** For any planar graph $G$ with $\kappa(G) = 5$, if there are at most two vertices of degree 5, then by Lemma 3.7, $2 \times 5 + (n-2) \times 6 \leq \sum d(v) = 2m \leq 6n - 12$, namely $6n - 2 \leq 6n - 12$, a contradiction. So there exist three vertices of degree 5. By Lemma 3.2, we get $\kappa_3(G) \neq 5$, namely $\kappa_3(G) = 4$. So, any planar graph $G$ with connectivity 5 can attain the lower bound, but obviously can not attain the upper bound.

We conclude that for $2 \leq \kappa \leq 4$, there exist some classes of planar graphs which can attain the lower bound and the upper bound of $\kappa_3$, respectively; for $\kappa = 1$, any graph can only attain the upper bound of $\kappa_3$; for $\kappa = 5$, any planar graph can only attain the lower bound of $\kappa_3$. ∎

# 4 Algorithmic aspects for $\kappa_3(G)$

As well-known, for any graph $G$, we have polynomial-time algorithms to get the connectivity $\kappa(G)$. A natural question is whether there is a polynomial-time algorithm to get the $\kappa_3(G)$, or more generally, $\kappa_k(G)$. At the moment, we do not know if such an algorithm exists for general graphs. But, given a fixed positive integer $k$, we have a polynomial-time algorithm to decide whether $\kappa_3(G) \geq k$.

At first, we prove the following theorem which can be used to deduce algorithms of this section.

**Theorem 4.1.** *Given a fixed positive integer $k$, for any graph $G$ the problem of deciding whether $G$ contains $k$ internally disjoint trees connecting $\{v_1, v_2, v_3\}$ can be solved by a polynomial-time algorithm, where $v_1, v_2, v_3$ are any three vertices of $V(G)$.*

*Proof.* A path $P$ is called a concise-path connecting $\{v_1, v_2, v_3\}$, if $P$ contains the three vertices $v_1, v_2, v_3$ and the ends of $P$ are also in $\{v_1, v_2, v_3\}$. See Type II of Figure 18.

First note that the tree we really want is a 3-fan from $t$ to $\{v_1, v_2, v_3\}$ or a concise-path connecting $\{v_1, v_2, v_3\}$, where $t \in V(G - \{v_1, v_2, v_3\})$, as shown in Figure 18.

Otherwise, let $T$ be an arbitrary tree connecting $\{v_1, v_2, v_3\}$. Then $T$ must contain a $v_1v_2$-path $P$.

If $v_3 \in V(P)$, then $P \subseteq T$ is exactly a tree connecting $\{v_1, v_2, v_3\}$ and it is also a concise-path connecting $\{v_1, v_2, v_3\}$.

If $v_3 \notin V(P)$, then $T$ must contain a $(v_3, V(P))$-path $M$. Then $(P \cup M) \subseteq T$ is a tree connecting $\{v_1, v_2, v_3\}$. Let $t$ be the terminal vertex of $M$. If $t \in \{v_1, v_2\}$, then $P \cup M$ is a concise-path connecting $\{v_1, v_2, v_3\}$. If $t \in V(P) - \{v_1, v_2\}$, then $P \cup M$ is a 3-fan from $t$ to $\{v_1, v_2, v_3\}$.

Therefore, we only need trees $T$ belonging to one of the two types in Figure 18.

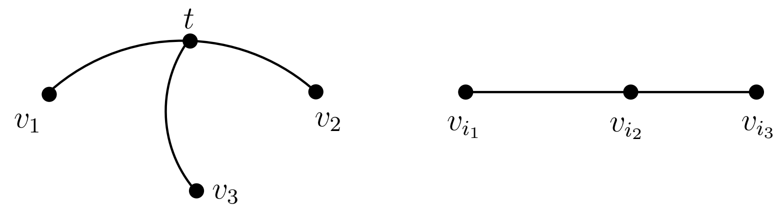


Figure 18: Two types of trees we really want, where $\{v_{i_1}, v_{i_2}, v_{i_3}\} = \{v_1, v_2, v_3\}$

If $k = 1$, as long as $v_1, v_2$ and $v_3$ are contained in a same component of $G$, there exists one tree connecting $\{v_1, v_2, v_3\}$. This case is trivial. So we first look at the case $k = 2$.

For $k = 2$, if $G$ contains two internally disjoint trees $T_1, T_2$ connecting $\{v_1, v_2, v_3\}$, since both $T_1$ and $T_2$ belong to one of the two types in Figure 18, then $T_1 \cup T_2$ has three main types: $T_1 \cup T_2$ consists of two 3-fans from two distinct vertices of $V(G - \{v_1, v_2, v_3\})$ to $\{v_1, v_2, v_3\}$, respectively; $T_1 \cup T_2$ consists of one 3-fan from one vertex of $V(G - \{v_1, v_2, v_3\})$ to $\{v_1, v_2, v_3\}$ and one concise-path connecting $\{v_1, v_2, v_3\}$; $T_1 \cup T_2$ consists of two concise-paths connecting $\{v_1, v_2, v_3\}$. We call them Type I, Type II and Type III, respectively, as shown in Figure 19. Moreover, Type III can be divided into two main subtypes: the two concise-paths have same ends; the two concise-paths have different ends.

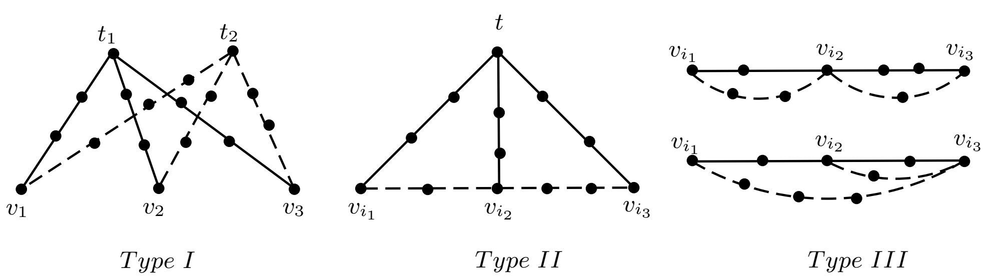


Figure 19: Three main types of $T_1 \cup T_2$, where $\{v_{i_1}, v_{i_2}, v_{i_3}\} = \{v_1, v_2, v_3\}$

Our algorithm is to check all possible types until two internally disjoint trees are found. Otherwise, we get $\kappa(\{v_1, v_2, v_3\}) < 2$.

**stage 1-Type I** Let $\{t_1, t_2\}$ be a vertex pair of $V(G - \{v_1, v_2, v_3\})$ and $S = \{v_1, v_2, v_3\}$. If $G$ contains two internally disjoint 3-fans from $t_1$ to $S$ and from $t_2$ to $S$, respectively, then the procedure terminates and we get $\kappa(\{v_1, v_2, v_3\}) \geq 2$. Otherwise, we continue to check another vertex pair. The procedure will terminate once we get two internally disjoint 3-fans. After all vertex pairs of $V(G - \{v_1, v_2, v_3\})$ are checked, we turn to stage 2-Type II.

Now the problem becomes that given a vertex pair $\{t_1, t_2\}$ of $V(G-\{v_1, v_2, v_3\})$, decide whether there are two internally disjoint 3-fans from $t_1$ to $S$ and from $t_2$ to $S$, respectively. we show that this problem can be reduced in polynomial time to the $k$-linkage problem, namely, the problem whether there exists an $XY$-linkage for given sets $X$, $Y$ and any fixed value of $|X| = |Y| = k$.

At first, for each $i \in \{1, 2, 3\}$, we replace the vertex $v_i$ by two new vertices $v_{i_1}, v_{i_2}$ and let them be adjacent to all the neighbors of $v_i$, namely, duplicating the vertex $v_i$. For each $i \in \{1, 2\}$, we replace the vertex $t_i$ by three new vertices $t_{i_1}, t_{i_2}, t_{i_3}$ and let them be adjacent to all the neighbors of $t_i$, namely, duplicating the vertex $t_i$ twice. Denote the new graph by $G'$. See Figure 20. Let $X = \{t_{1_1}, t_{1_2}, t_{1_3}, t_{2_1}, t_{2_2}, t_{2_3}\}$ and $Y = \{v_{1_1}, v_{2_1}, v_{3_1}, v_{1_2}, v_{2_2}, v_{3_2}\}$. If there exists an $XY$-linkage in $G'$, it is easy to see that $t_{1_1}P_1v_{1_1} \cup t_{1_2}P_2v_{2_1} \cup t_{1_3}P_3v_{3_1}$ and $t_{2_1}P_4v_{1_2} \cup t_{2_2}P_5v_{2_2} \cup t_{2_3}P_6v_{3_2}$ can be converted into two internally disjoint 3-fans from $t_1$ to $S$ and from $t_2$ to $S$ in $G$. Conversely, in $G$ any two internally disjoint 3-fans from $t_1$ to $S$ and from $t_2$ to $S$ can be converted into an $XY$-linkage in $G'$.

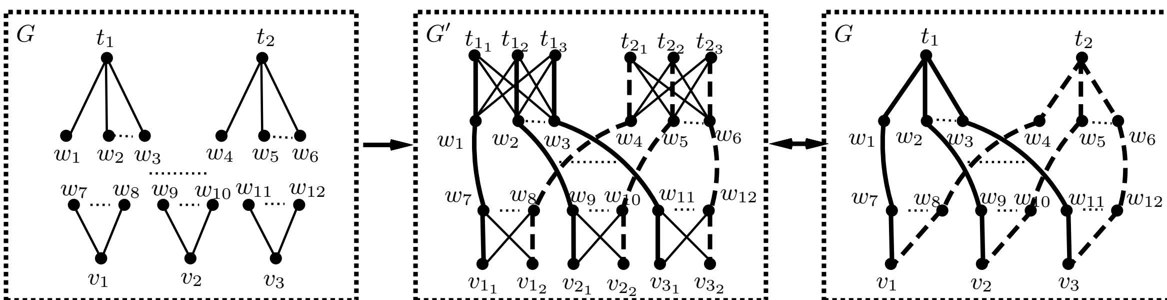


Figure 20: An $XY$-linkage of $G'$ is equivalent to two internally disjoint 3-fans of $G$

Note that if there is an edge $e$ incident with two vertices of $\{t_1, t_2, v_1, v_2, v_3\}$, subdivide $e$ by a new vertex $w$ and then implement the vertex duplications. Since at most one path can contain $w$, the operation can ensure that the edge $e$ of $G$ is used only once.

Since the $k$-linkage problem has a polynomial-time algorithm, which has a running time $O(n^3)$, see [5], then the problem whether there are two internally disjoint 3-fans from $t_1$ to $S$ and from $t_2$ to $S$, respectively, has a polynomial-time algorithm, which also has a running time $O(n^3)$.

There are $O(n^2)$ vertex pairs of $V(G-\{v_1, v_2, v_3\})$. So stage 1-Type I needs a running time at most $O(n^5)$.

**stage 2-Type II** Let $\{t, v_{i_2}\}$ be a vertex pair such that $t \in V(G - \{v_1, v_2, v_3\})$ and

$v_{i_2} \in \{v_1, v_2, v_3\}$. Let $S = \{v_1, v_2, v_3\}$. We check whether there is a 3-fan from $t$ to $S$ and a $v_{i_1}v_{i_3}$-path containing $v_{i_2}$ such that the fan and the path have no vertices in common except $v_1, v_2, v_3$, where $\{v_{i_1}, v_{i_2}, v_{i_3}\} = \{v_1, v_2, v_3\}$. If so, the procedure terminates and we get $\kappa(\{v_1, v_2, v_3\}) \geq 2$. Otherwise, we continue to check another vertex pair such that one is in $V(G - \{v_1, v_2, v_3\})$ and the other is in $\{v_1, v_2, v_3\}$. The procedure will terminate once we get such a 3-fan and a concise-path. After all such vertex pairs are checked, we turn to stage 3-Type III.

Now the problem becomes that given a vertex pair $\{t, v_{i_2}\}$ such that $t \in V(G - \{v_1, v_2, v_3\})$ and $v_{i_2} \in \{v_1, v_2, v_3\}$, decide whether there is a 3-fan from $t$ to $S$ and a $v_{i_1}v_{i_3}$-path containing $v_{i_2}$ such that the fan and the path have no vertices in common except $v_1, v_2, v_3$, where $\{v_{i_1}, v_{i_2}, v_{i_3}\} = \{v_1, v_2, v_3\}$. This problem can also be reduced to the $k$-linkage problem. The method used here is the same as that for stage 1-Type I. Without loss of generality, let $v_{i_2} = v_2$.

For $j = 1$ and 3, replace the vertex $v_j$ by two new vertices $v_{j_1}, v_{j_2}$ and let them be adjacent to all the neighbors of $v_j$. Replace the vertex $t$ by three new vertices $t_1, t_2, t_3$ and let them be adjacent to all the neighbors of $t$. Replace the vertex $v_2$ by three new vertices $v_{2_1}, v_{2_2}, v_{2_3}$ and let them be adjacent to all the neighbors of $v_2$. Denote the new graph by $G'$. Then let $X = \{t_1, t_2, t_3, v_{2_2}, v_{2_3}\}$ and $Y = \{v_{1_1}, v_{2_1}, v_{3_1}, v_{1_2}, v_{3_2}\}$. If there exists an $XY$-linkage in $G'$, it is easy to see that $t_1P_1v_{1_1} \cup t_2P_2v_{2_1} \cup t_3P_3v_{3_1}$ and $v_{2_2}P_4v_{1_2} \cup v_{2_3}P_5v_{3_2}$ can be converted into a 3-fan from $t$ to $S$ and a $v_1v_3$-path containing $v_2$ in $G$ such that the fan and the concise-path have no vertices in common except $v_1, v_2, v_3$. Conversely, in $G$ a 3-fan from $t$ to $S$ and a $v_1v_3$-path containing $v_2$ can be converted into an $XY$-linkage in $G'$.

There are $O(n)$ vertex pairs such that one vertex is in $V(G - \{v_1, v_2, v_3\})$ and the other is in $\{v_1, v_2, v_3\}$. So stage 2-Type II needs a running time at most $O(n^4)$.

**stage 3-Type III** we check whether there are two $v_{i_1}v_{i_3}$-paths both containing $v_{i_2}$, or there is a $v_{i_1}v_{i_3}$-path containing $v_{i_2}$ and a $v_{i_1}v_{i_2}$-path containing $v_{i_3}$, where $\{v_{i_1}, v_{i_2}, v_{i_3}\} = \{v_1, v_2, v_3\}$. No matter which case happens, the two paths have no vertices in common except $v_1, v_2, v_3$. Then the operation is similar. Duplicate the vertices and convert the problem to the $k$-linkage problem. It can be checked that there are six kinds of concise-path pairs. So stage 3-Type III needs a running time at most $O(n^3)$.

The procedure terminates when either we find two internally disjoint trees connecting $\{v_1, v_2, v_3\}$ in some type, or there are no such two trees when all the possibilities are checked. For the former case, we get $\kappa(\{v_1, v_2, v_3\}) \geq 2$, and for the latter case, we get $\kappa(\{v_1, v_2, v_3\}) < 2$. Moreover, the algorithm needs a running time at most $O(n^5)$.

Now we turn to the case $k = 3$.

For $k = 3$, if $G$ contains three internally disjoint trees $T_1, T_2, T_3$ connecting $\{v_1, v_2, v_3\}$, since all of $T_1, T_2$ and $T_3$ belong to one of the two types in Figure 18, then $T_1 \cup T_2 \cup T_3$ has four main types: $T_1 \cup T_2 \cup T_3$ consists of three 3-fans; $T_1 \cup T_2 \cup T_3$ consists of two 3-fans and a concise-path connecting $\{v_1, v_2, v_3\}$; $T_1 \cup T_2 \cup T_3$ consists of one 3-fan and two concise-paths connecting $\{v_1, v_2, v_3\}$; $T_1 \cup T_2 \cup T_3$ consists of three concise-paths connecting $\{v_1, v_2, v_3\}$. We call them Type 1, Type 2, Type 3 and Type 4, respectively, as shown in Figure 21.

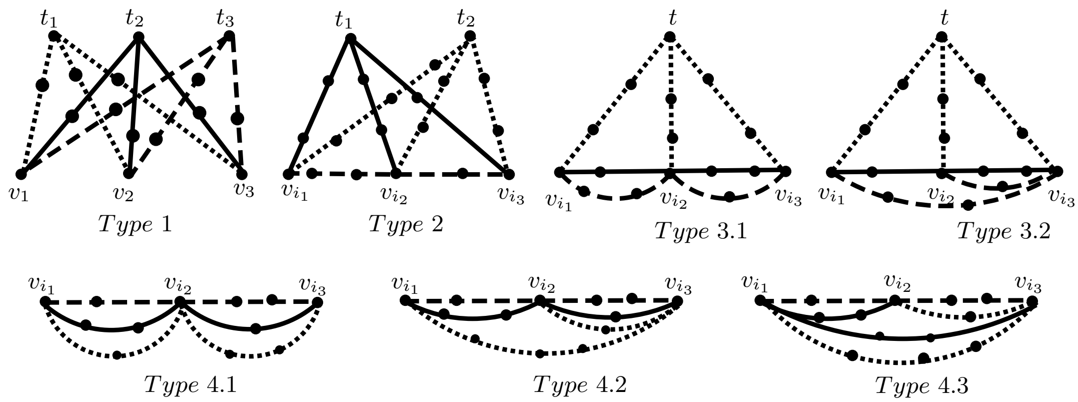


Figure 21: Four main types of $T_1 \cup T_2 \cup T_3$, where $\{v_{i_1}, v_{i_2}, v_{i_3}\} = \{v_1, v_2, v_3\}$

The method to deal with this case is similar to that for the case $k = 2$. We still implement the vertex duplications and convert the problem into the $k$-linkage problem.

For $k \geq 4$, the operation is also similar.

If $G$ contains $k$ internally disjoint trees $T_1, T_2, \ldots, T_k$ connecting $\{v_1, v_2, v_3\}$, then $T_1 \cup T_2 \ldots \cup T_k$ has $(k+1)$ main types, where if $T_1 \cup T_2 \ldots \cup T_k$ is of Type $i$, it consists of $(k+1-i)$ 3-fans and $(i-1)$ concise-paths connecting $\{v_1, v_2, v_3\}$.

We say that two concise-paths connecting $\{v_1, v_2, v_3\}$ have the same structure, if they have the same ends. For example, $v_1P_1v_2P_2v_3$ and $v_1P_1'v_2P_2'v_3$ have the same structure, even if $P_1 \neq P_1'$ and $P_2 \neq P_2'$. So, any concise-path connecting $\{v_1, v_2, v_3\}$ has only three structures having ends $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$, respectively. Therefore, the union of $(i-1)$ concise-paths has $\binom{i-1+3-1}{3-1}$ different subtypes. ($m_1$ concise-paths belong to Structure 1, $m_2$ concise-paths belong to Structure 2 and $m_3$ concise-paths belong to Structure 3, where $0 \leq m_1, m_2, m_3 \leq i-1$ and $m_1 + m_2 + m_3 = i - 1$.)

Moreover, for Type $i$ we need to check at most $\binom{n-3}{k+1-i}$ $(k+1-i)$-subsets of $V(G - \{v_1, v_2, v_3\})$ for $(k+1-i)$ 3-fans.

Therefore, the time used for checking Type $i$ is at most $O\big(\binom{n-3}{k+1-i}\binom{i-1+3-1}{3-1}n^3\big)$, where $1 \leq i \leq k+1$.

Since $k$ is a fixed positive integer, we conclude that the algorithm can be implemented to run in polynomial-time ($O(n^{k+3})$). The proof is now complete. ∎

Now we can easily obtain our following main result.

**Theorem 4.2.** *Given a fixed positive integer $k$, for any graph $G$ the problem of deciding whether $\kappa_3(G) \geq k$ can be solved by a polynomial-time algorithm.*

*Proof.* We know that $\kappa_3(G) = min\{\kappa(S)\}$, where the minimum is taken over all 3-subsets $S$ of $V(G)$. Therefore, for every 3-subset $S$ of $V(G)$, if there always exist $k$ internally disjoint trees connecting $S$, namely, $\kappa(S) \geq k$, we can conclude that $\kappa_3(G) \geq k$. Otherwise, once there exists a 3-subset $S$ such that $\kappa(S) < k$, we can get $\kappa_3(G) < k$. So by Theorem 4.1, a polynomial-time algorithm to decide whether $\kappa_3(G) \geq k$ follows immediately and it can be implemented to run in polynomial-time ($O(n^{k+6})$). ∎

The following two corollaries are immediate.

**Corollary 4.1.** *Given a fixed positive integer $\kappa$, for any graph $G$ with connectivity $\kappa$, the problem of determining $\kappa_3(G)$ can be solved in polynomial time.*

*Proof.* From Theorem 2.2, we see that $\kappa_3(G) \leq \kappa$. Then, from Theorem 4.2, we need at most to try every $k$ with $k \leq \kappa$ to determine the value of $\kappa_3(G)$. The complexity is bounded by $O(\kappa n^{\kappa+6})$. ∎

**Corollary 4.2.** *Given a fixed positive integer $\delta$, for any graph $G$ with minimum degree $\delta$, the problem of determining $\kappa_3(G)$ can be solved in polynomial time.*

*Proof.* It follows from the fact that the value of the connectivity of a graph is at most its minimum degree. ∎

Our next result also follows easily.

**Theorem 4.3.** *For a planar graph $G$ with connectivity $\kappa(G)$, the problem of determining $\kappa_3(G)$ has a polynomial-time algorithm and its complexity is bounded by $O(n^8)$.*

*Proof.* Since for a planar graph $G$, from Theorem 3.1 we have $\kappa_3(G) = \kappa(G)$ or $\kappa(G) - 1$, we only need to check whether $\kappa_3(G) \geq \kappa(G)$.

Actually, if $\kappa(G) = 1$, we can immediately get $\kappa_3(G) = \kappa(G)$. If $\kappa(G) = 5$, we can immediately get $\kappa_3(G) = \kappa(G) - 1$. Then if $2 \leq \kappa(G) \leq 4$, for each 3-subset $S$ of $V(G)$, we want to decide whether $G$ contains $\kappa(G)$ internally disjoint trees connecting $S$, namely to decide whether $\kappa(S) \geq \kappa(G)$. The algorithm for Theorem 4.1 can be used here.

But we need to note that for a planar graph, if there are three internally disjoint 3-fans from three distinct vertices of $V(G - S)$ to $S$, respectively, then $G$ contains a subdivision of $K_{3,3}$, which is impossible. So, for $3 \leq \kappa(G) \leq 4$, when check whether there are $\kappa(G)$ internally disjoint trees connecting $S$, we only need to consider the main types which contain at most two 3-fans, namely Type $i$, where $k - 1 \leq i \leq k + 1$. Therefore, for each 3-subset $S$ of $V(G)$, the algorithm to decide whether $\kappa(S) \geq \kappa(G)$ needs a running time at most $O(n^5)$.

We conclude that for a planar graph $G$, $\kappa_3(G)$ can be obtained in a running time $O(n^8)$. The proof is complete. ∎

The above complexity is not very good so the problem of finding a more efficient algorithm is an interesting one.

As we mentioned at the beginning of this section, the complexity of the problem of determining $\kappa_3(G)$ for a general graph $G$ is not known: Can it be solved in polynomial time or NP-hard? Nevertheless, from Theorems 2.2 and 2.3 we have a polynomial-time algorithm to determine it approximately with a constant ratio, which we formulate into the following result.

**Theorem 4.4.** *The problem of determining $\kappa_3(G)$ for any graph $G$ can be solved by a polynomial-time approximation algorithm with a constant ratio about $\frac{3}{4}$.*

*Proof.* From Theorems 2.2 and 2.3 we can get

$$\frac{3}{4}\kappa(G) - \frac{3}{4}r + \lceil r/2 \rceil \leq \kappa_3(G) \leq \kappa(G),$$

where $r \in \{0, 1, 2, 3\}$. Therefore, if we take $\frac{3}{4}\kappa(G) - \frac{1}{2}$ as an approximate solution for $\kappa_3(G)$, then from the above we get the following inequalities

$$\frac{3}{4}\kappa_3(G) - \frac{1}{2} \leq \frac{3}{4}\kappa(G) - \frac{1}{2} \leq \frac{3}{4}\kappa(G) - \frac{3}{4}r + \lceil r/2 \rceil \leq \kappa_3(G),$$

which means that the approximate solution has a ratio about $\frac{3}{4}$. Since the value of the connectivity $\kappa(G)$ of $G$ can be determined in polynomial time, the proof is complete. ∎

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# Appendix

## An example of the Path-Bundle Transformation

Let $G$ be a graph with $\kappa(G) = 5$. Figure 22 illustrates a subgraph of $G$, in which $\{P_1, P_2, P_3, P_4, P_5\}$ is a $(5, 2)$-original-path-bundle connecting $\{v_1, v_2, v_3\}$ of $G$.

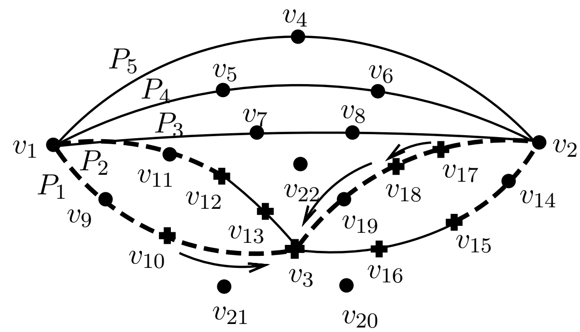


Figure 22: A subgraph of $G$

Now employing the Path-Bundle Transformation, we will get a $(5, 2)$-reduced-path-bundle connecting $\{v_1, v_2, v_3\}$.

Let $X = V(P_3 \cup P_4 \cup P_5)$. Since $G$ is 5-connected and $|X| \geq 5$, there is a 5-fan $\{M_1, M_2, M_3, M_4, M_5\}$ from $v_3$ to $X$ by the Fan Lemma, where

$M_1 = v_3v_{20}v_{15}v_{17}v_4$;

$M_2 = v_3v_{21}v_5$;

$M_3 = v_3v_{16}v_{12}v_{18}v_{10}v_6$;

$M_4 = v_3v_7$;

$M_5 = v_3v_{13}v_{22}v_2$;

Then $N = V(M_1 \cup \ldots \cup M_5) \cap V(P_1 \cup P_2 - \{v_1, v_2\}) = \{v_3, v_{10}, v_{12}, v_{13}, v_{15}, v_{16}, v_{17}, v_{18}\}$.

$T_1 = (v_{10}, v_3)$;

$T_2 = (v_{12}, v_{13}, v_3)$;

$T_3 = (v_{15}, v_{16}, v_3)$;

$T_4 = (v_{17}, v_{18}, v_3)$;

**Finding stage:** For each $1 \leq i \leq 4$, mark the first vertex in $T_i$, namely, $v_{10}, v_{12}, v_{15}, v_{17}$ and mark the corresponding $(v_3, X)$-paths $M_{v_{10}}, M_{v_{12}}, M_{v_{15}}, M_{v_{17}}$, namely, $M_3, M_3, M_1, M_1$. (here $M_{v_i}$ denote the path containing the vertex $v_i$.)

We can see that the marked $(v_3, X)$-path $M_3$ contains two marked vertices $v_{10}, v_{12}$. Since $v_{12}$ is nearer to $v_3$ than $v_{10}$ on $M_3$, keep the mark of $v_{12}$ and unmark $v_{10}$. Then, since $v_{10} \in T_1$, we mark the next vertex $v_3$ in $T_1$ and mark the corresponding $(v_3, X)$-path $M_{v_3(T_1)}$. We know that $v_3$ is a vertex of any path of $M_1, M_2, \ldots, M_5$. So we can choose $M_2$ which has not been marked and let $M_{v_3(T_1)} = M_2$.

The marked path $M_1$ also contains two marked vertices $v_{15}, v_{17}$. Since $v_{15}$ is nearer to

$v_3$ than $v_{17}$ on $M_1$, keep the mark of $v_{15}$ and unmark $v_{17}$. Then, since $v_{17} \in T_4$, we mark the next vertex $v_{18}$ in $T_4$ and mark the corresponding $(v_3, X)$-path $M_{v_{18}}$, namely $M_3$.

Now the updated four marked vertices are $v_3, v_{12}, v_{15}, v_{18}$ contained in $T_1, T_2, T_3, T_4$, respectively, and the corresponding marked paths are $M_2, M_3, M_1, M_3$.

Again, the marked path $M_3$ contains two marked vertices $v_{12}, v_{18}$. Since $v_{12}$ is nearer to $v_3$ than $v_{18}$ on $M_3$, keep the mark of $v_{12}$ and unmark $v_{18}$. Then, since $v_{18} \in T_4$, we mark the next vertex $v_3$ in $T_4$ and mark the corresponding $(v_3, X)$-path $M_{v_3(T_4)}$. We can choose $M_4$ which has not been marked and let $M_{v_3(T_4)} = M_4$.

Now the updated four marked vertices are $v_3, v_{12}, v_{15}, v_3$ and the corresponding marked paths are $M_2, M_3, M_1, M_4$, which are four pairwise distinct $(v_3, X)$-paths. The Finding stage is completed and we proceed to the Adjusting stage.

Note that in the end there are only four marked vertices $q_1 = v_3, q_2 = v_{12}, q_3 = v_{15}, q_4 = v_3$ and they belong to $T_1, T_2, T_3, T_4$, respectively (Remark 2.4).

We can also see that both $v_{15}$ and $v_{17}$ are vertices on $M_1$, $v_{15}$ is one of the final marked vertices and $v_{17}$ was ever marked and unmarked at some step. Obviously $v_{15}$ is closer to $v_3$ than $v_{17}$ on $M_1$ (Remark 2.6).

**Adjusting stage:** The dashed lines in Figure 22 denote paths $v_1P_1q_1, v_1P_2q_2, v_2P_1q_3$ and $v_2P_2q_4$, respectively. We can get

$P'_1 = v_1P_1v_3M_2v_3M_1v_{15}P_1v_2 = v_1P_1v_3M_1v_{15}P_1v_2 = v_1v_9v_{10}v_3v_{20}v_{15}v_{14}v_2$;

$P'_2 = v_1P_2v_{12}M_3v_3M_4v_3P_2v_2 = v_1P_2v_{12}M_3v_3P_2v_2 = v_1v_{11}v_{12}v_{16}v_3v_{19}v_{18}v_{17}v_2$;

$P'_3 = P_3 = v_1v_7v_8v_2$;

$P'_4 = P_4 = v_1v_5v_6v_2$;

$P'_5 = v_1v_4v_2$;

Moreover, there is a $(v_3, X)$-path $M_5 = v_3v_{13}v_{22}v_2$ left that has not been marked so far.

It can be checked that $\{P'_1, P'_2, P'_3, P'_4, P'_5\} \cup M_5$ is exactly a $(5, 2)$-reduced-path-bundle connecting $\{v_1, v_2, v_3\}$ of $G$.