

An Operator Approach to the Al-Salam-Carlitz Polynomials

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Abstract. We present an operator approach to Rogers-type formulas and Mehler's formulas for the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$. By using the q -exponential operator, we obtain a Rogers-type formula which leads to a linearization formula. With the aid of a bivariate augmentation operator, we get a simple derivation of Mehler's formula due to Al-Salam and Carlitz. By means of the Cauchy companion augmentation operator, we obtain an equivalent form of Mehler's formula. We also give several identities on the generating functions for products of the Al-Salam-Carlitz polynomials which are extensions of formulas for Rogers-Szegö polynomials.

Keywords: Al-Salam-Carlitz polynomial, q -exponential operator, homogeneous q -shift operator, Cauchy companion operator, Rogers-type formula, Mehler's formula

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1. Introduction

The Al-Salam-Carlitz polynomials are q -orthogonal polynomials whose applications and generalizations arise in many applications such as the q -harmonic oscillator, theta functions, quantum groups and coding theory; see for example [4, 5, 7, 16, 17]. This paper presents an operator approach to the Rogers-type formulas and Mehler's formula for the Al-Salam-Carlitz polynomials. These polynomials are a generalization of the classical Rogers-Szegö polynomials which have been extensively studied, see for example [6, 8, 9, 11, 14]. There are two classical formulas concerning the Rogers-Szegö polynomials, namely, Mehler's formula and the Rogers formula, in connection with the Poisson kernel formula and the linearization formula.

It is natural to study the Rogers-type formulas and Mehler's formula beyond the Rogers-Szegö polynomials. In fact, Mehler's formula for the Al-Salam-Carlitz polynomials has been

derived by Al-Salam and Carlitz [2], which requires a terminating condition on a ${}_3\phi_2$ series as mentioned by Askey and Suslov [4]. Using our operator approach, we deduce a new formula in a similar form. We also derive some Rogers-type formulas, one of which leads to a linearization formula. In addition, we obtain several identities on the generating functions of products of the Al-Salam-Carlitz polynomials as extensions of the formulas for the Rogers-Szegő polynomials.

We adopt the common notation on q -series in Gasper and Rahman [16]. The set of integers is denoted by \mathbb{Z} . Throughout this paper, q is a fixed nonzero complex number with $|q| < 1$. The q -shifted factorial is defined for any complex parameter a by

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k) \quad \text{and} \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad n \in \mathbb{Z}.$$

We shall use the compact notation

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \quad \text{for } n \in \mathbb{Z} \text{ or } n = \infty.$$

The q -binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_k}.$$

The basic hypergeometric series ${}_r\phi_s$ is defined as follows,

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} x^n. \quad (1.1)$$

Note that the ${}_r\phi_s$ series converges absolutely for all x if $r \leq s$ and for $|x| < 1$ if $r = s + 1$. It diverges for $x \neq 0$ if $0 < |q| < 1$ and $r > s + 1$ unless it terminates.

This paper is primarily concerned with the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ which can be defined in terms of a ${}_2\phi_1$ series

$$U_n(x, y, a; q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, y/x \\ 0 \end{matrix}; q, \frac{qx}{a} \right). \quad (1.2)$$

The following generating function for the the Al-Salam-Carlitz polynomials has been given by Al-Salam and Carlitz [2],

$$\sum_{n=0}^{\infty} U_n(x, y, a; q) \frac{t^n}{(q; q)_n} = \frac{(at, yt; q)_\infty}{(xt; q)_\infty}, \quad (1.3)$$

where $|xt| < 1$. Since that the right-hand side of (1.3) is symmetric in a and y , the polynomials $U_n(x, y, a; q)$ are symmetric in a and y , that is,

$$U_n(x, y, a; q) = U_n(x, a, y; q). \quad (1.4)$$

In terms of of the Cauchy polynomials

$$P_n(x, y) = (x - y)(x - qy) \cdots (x - q^{n-1}y),$$

with the generating function

$$\sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}}, \quad |xt| < 1, \quad (1.5)$$

the Al-Salam-Carlitz polynomials can be expressed as

$$U_n(x, y, a; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} a^k P_{n-k}(x, y). \quad (1.6)$$

The above definition is essentially the same as the original definition of the Al-Salam-Carlitz polynomials $u_n^{(a)}(x; q)$,

$$u_n^{(a)}(x; q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}; q, \frac{qx}{a} \right). \quad (1.7)$$

Clearly, we have the following relation

$$U_n(x, y, a; q) = y^n u_n^{(a/y)}(x/y; q). \quad (1.8)$$

The Al-Salam-Carlitz polynomials are related to several q -orthogonal polynomials, such as the q -Bessel polynomials $B_n(x, b; q)$ due to Abdi [1], and the Stieltjes-Wigert polynomials $S_n(x; q)$ [18, p.116]. In particular, the Al-Salam-Carlitz polynomials are connected to the bivariate Rogers-Szegö polynomials [12]

$$h_n(x, y|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, y),$$

which have the generating function

$$\sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(t, xt; q)_{\infty}}, \quad |t| < 1, |xt| < 1. \quad (1.9)$$

On the other hand, although the Al-Salam-Carlitz polynomials can be expressed in terms of the bivariate Rogers-Szegö polynomials

$$U_n(x, y, a; q) = (-1)^n q^{\binom{n}{2}} a^n h_n \left(\frac{y}{a}, \frac{x}{a} \middle| q^{-1} \right), \quad (1.10)$$

as noted by Carlitz [9], it often happens that an infinite q -series identity no longer holds when q is replaced by q^{-1} . In fact, it turns out to be the case for the Rogers formula and Mehler's formula for the polynomials $h_n(x, y|q)$. This suggests that there is a need for a direct approach to deal with the Al-Salam-Carlitz polynomials.

This paper is organized as follows. In Section 2, we give an overview of the q -exponential operator $T(bD_q)$, and derive a Rogers-type formula for $U_n(x, y, a; q)$ which leads to a linearization formula. In Section 3, we construct a homogeneous q -shift operator $\mathbb{F}(aD_{xy})$ and apply it to derive Mehler's formula. In Section 4, we make use of the Cauchy companion operator to obtain two Rogers-type formulas and an equivalent form of Mehler's formula. In the last section, we provide three generating function identities for products of $U_n(x, y, a; q)$.

2. A Rogers-Type Formula

In this section, we give a Rogers-type formula for the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ by using the q -exponential operator $T(bD_q)$. As a consequence, we obtain a linearization formula for $U_n(x, y, a; q)$.

The q -differential operator, or the q -derivative, acting on the variable a , is defined by

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}.$$

The q -shift operator, denoted by η , is defined by

$$\eta\{f(a)\} = f(aq) \quad \text{and} \quad \eta^{-1}\{f(a)\} = f(aq^{-1}),$$

see, for example, [3, 19]. The operator θ is defined by

$$\theta = \eta^{-1}D_q, \tag{2.1}$$

see Roman [19]. Recall the q -Leibniz rule for D_q , see [19],

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(a)\} D_q^{n-k}\{g(q^k a)\}.$$

By convention, D_q^0 is understood as the identity, that is, $D_q^0\{f(a)\} = f(a)$. Chen and Liu [13] introduced the following q -exponential operator

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q; q)_n}$$

for proving basic hypergeometric identities from their special cases. This method is called parameter augmentation. The following lemma for the q -exponential operator $T(bD_q)$ is easy to verify.

Lemma 2.1. *We have*

$$T(bD_q)\{a^n\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} b^k a^{n-k}. \tag{2.2}$$

From the q -Leibniz rule for D_q , Zhang and Wang [20] derived the following formula.

Lemma 2.2. *Let n be a nonnegative integer. Then*

$$D_q^n \left\{ \frac{(at; q)_\infty}{(av; q)_\infty} \right\} = v^n (t/v; q)_n \frac{(atq^n; q)_\infty}{(av; q)_\infty}.$$

Based on the above relation, we obtain the following identity.

Lemma 2.3. *We have*

$$T(bD_q) \left\{ \frac{(as, at; q)_\infty}{(av; q)_\infty} \right\} = \frac{(as, at; q)_\infty}{(av; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (av; q)_k (bs)^k}{(q; q)_k (as, at; q)_k} {}_2\phi_1 \left(\begin{matrix} t/v, 0 \\ atq^k \end{matrix}; q, bv \right). \tag{2.3}$$

Proof. By the definition of $T(bD_q)$ and the q -Leibniz rule for D_q , we have

$$\begin{aligned}
T(bD_q) \left\{ \frac{(as, at; q)_\infty}{(av; q)_\infty} \right\} &= \sum_{n=0}^{\infty} \frac{b^n}{(q; q)_n} D_q^n \left\{ \frac{(as, at; q)_\infty}{(av; q)_\infty} \right\} \\
&= \sum_{n=0}^{\infty} \frac{b^n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \{(as; q)_\infty\} D_q^{n-k} \left\{ \frac{(atq^k; q)_\infty}{(avq^k; q)_\infty} \right\} \\
&= \sum_{n=0}^{\infty} \frac{b^n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} (-1)^k q^{\binom{k}{2}} s^k (asq^k; q)_\infty (vq^k)^{n-k} (t/v; q)_{n-k} \frac{(atq^n; q)_\infty}{(avq^k; q)_\infty} \\
&= \frac{(as, at; q)_\infty}{(av; q)_\infty} \sum_{n=0}^{\infty} \frac{b^n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} s^k \frac{(av; q)_k (t/v; q)_{n-k} v^{n-k}}{(as; q)_k (at; q)_n} \\
&= \frac{(as, at; q)_\infty}{(av; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (av; q)_k (bs)^k}{(q, as, at; q)_k} \sum_{n=0}^{\infty} \frac{(t/v; q)_n (bv)^n}{(q; q)_n (atq^k; q)_n} \\
&= \frac{(as, at; q)_\infty}{(av; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (av; q)_k (bs)^k}{(q; q)_k (as, at; q)_k} {}_2\phi_1 \left(\begin{matrix} t/v, 0 \\ atq^k \end{matrix}; q, bv \right).
\end{aligned}$$

This completes the proof. ■

Now we are ready to give a Rogers-type formula for the polynomials $U_n(x, y, a; q)$.

Theorem 2.4. *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} U_{n+m}(x, y, a; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\
&= \frac{(as, ys; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (xs; q)_k (at)^k}{(q; q)_k (as, ys; q)_k} {}_2\phi_1 \left(\begin{matrix} y/x, 0 \\ ysq^k \end{matrix}; q, xt \right), \tag{2.4}
\end{aligned}$$

provided that $\max\{|xs|, |xt|\} < 1$.

Proof. Setting $m \rightarrow m - n$, exchanging the order of the sum on the left hand side of (2.4), and applying the operator identity (2.2), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} U_{n+m}(x, y, a; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\
&= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} U_m(x, y, a; q) \frac{t^n}{(q; q)_n} \frac{s^{m-n}}{(q; q)_{m-n}} = \sum_{m=0}^{\infty} \frac{U_m(x, y, a; q)}{(q; q)_m} \sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix} t^n s^{m-n} \\
&= \sum_{m=0}^{\infty} \frac{U_m(x, y, a; q)}{(q; q)_m} T(tD_q) \{s^m\} = T(tD_q) \left\{ \sum_{m=0}^{\infty} \frac{U_m(x, y, a; q)}{(q; q)_m} s^m \right\} \quad (|xs| < 1) \\
&= T(tD_q) \left\{ \frac{(as, ys; q)_\infty}{(xs; q)_\infty} \right\},
\end{aligned}$$

where D_q acts on the parameter s . Applying Lemma 2.3, we complete the proof. ■

From the above Rogers-type formula, we obtain the following linearization formula for $U_n(x, y, a; q)$.

Theorem 2.5. *For $n, m \geq 0$, we have*

$$U_{n+m}(x, y, a; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} (aq^m)^k P_{n-k}(x, y) U_m(x, yq^{n-k}, a; q).$$

Proof. Rewrite the Rogers-type formula (2.4) as follows

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} U_{n+m}(x, y, a; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (at)^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{(y/x; q)_n}{(q; q)_n} (xt)^n \frac{(asq^k, ysq^{n+k}; q)_{\infty}}{(xsq^k; q)_{\infty}} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (at)^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{(y/x; q)_n}{(q; q)_n} (xt)^n \sum_{l=0}^{\infty} U_l(x, yq^n, a; q) \frac{(sq^k)^l}{(q; q)_l}. \end{aligned}$$

Equating the coefficients of $t^n s^m$ in the above equation, we obtain the desired identity. \blacksquare

3. Mehler's Formula

In this section, we aim to introduce the homogeneous q -shift operator which can be used to give a simple derivation of Mehler's formula for $U_n(x, y, a; q)$ due to Al-Salam and Carlitz.

Recall that the homogeneous q -difference operator D_{xy} introduced by Chen, Fu and Zhang [12] is given by

$$D_{xy}\{f(x, y)\} = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}. \quad (3.1)$$

Based on the operator D_{xy} , we construct the following homogeneous q -shift operator

$$\mathbb{F}(aD_{xy}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (aD_{xy})^n}{(q; q)_n}. \quad (3.2)$$

The q -difference operator D_{xy} has the following basic properties.

Proposition 3.1. *We have*

$$D_{xy}^k \{P_n(x, y)\} = \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(x, y), \quad (3.3)$$

$$D_{xy}^k \left\{ \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \right\} = t^k \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}}. \quad (3.4)$$

Invoking (3.3), the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ can be expressed in terms of the homogeneous q -shift operator $\mathbb{F}(aD_{xy})$.

Theorem 3.2. *We have*

$$U_n(x, y, a; q) = \mathbb{F}(aD_{xy}) \{P_n(x, y)\}. \quad (3.5)$$

Using (3.4), it is easy to derive the following relation.

Proposition 3.3. *We have*

$$\mathbb{F}(aD_{xy}) \left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} \right\} = \frac{(at, yt; q)_\infty}{(xt; q)_\infty}. \quad (3.6)$$

Combining (3.5) and (3.6), the generating function of $U_n(x, y, a; q)$ can be derived as follows,

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, a; q) \frac{t^n}{(q; q)_n} &= \mathbb{F}(aD_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \right\} \quad (|xt| < 1) \\ &= \mathbb{F}(aD_{xy}) \left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} \right\} = \frac{(at, yt; q)_\infty}{(xt; q)_\infty}. \end{aligned}$$

The following identity will be used to derive Mehler's formula.

Theorem 3.4. *Assume $|xs| < 1$. We have*

$$\mathbb{F}(aD_{xy}) \left\{ \frac{P_n(x, y)(ys; q)_\infty}{(ys; q)_n(xs; q)_\infty} \right\} = \frac{(ysq^n, as; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \frac{(xs; q)_k (y/x; q)_{n-k}}{(as; q)_k} x^{n-k} a^k. \quad (3.7)$$

Proof. Applying (3.5), the left hand side of the Rogers-type formula (2.4) equals

$$\begin{aligned} &\mathbb{F}(aD_{xy}) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n+m}(x, y) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \right\} \\ &= \mathbb{F}(aD_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \sum_{m=0}^{\infty} P_m(x, q^n y) \frac{s^m}{(q; q)_m} \right\} \quad (|xs| < 1) \\ &= \mathbb{F}(aD_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \frac{(yq^n s; q)_\infty}{(xs; q)_\infty} \right\} \\ &= \sum_{n=0}^{\infty} \mathbb{F}(aD_{xy}) \left\{ \frac{P_n(x, y)(ys; q)_\infty}{(ys; q)_n(xs; q)_\infty} \right\} \frac{t^n}{(q; q)_n}. \end{aligned} \quad (3.8)$$

On the other hand, the right hand side of (2.4) can be restated as

$$\frac{(as, ys; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (xs; q)_k (at)^k}{(q; q)_k (as, ys; q)_k} \sum_{l=0}^{\infty} \frac{(y/x; q)_l}{(q, ysq^k; q)_l} (xt)^l. \quad (3.9)$$

Equating the coefficients of t^n in (3.8) and (3.9), we complete the proof. ■

Applying the above operator identity, we obtain Mehler's formula involving a terminating ${}_3\phi_2$ series.

Theorem 3.5. *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} U_n(x, y, a; q) U_n(u, v, b; q) \frac{t^n}{(q; q)_n} \\ &= \frac{(abt, ybt, avt; q)_{\infty}}{(xbt, aut; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} y/x, v/u, q/abt \\ q/xbt, q/aut \end{matrix}; q, q \right), \end{aligned} \quad (3.10)$$

where $y/x = q^{-r}$ or $v/u = q^{-r}$ for a nonnegative integer r , and $\max\{|xbtq^{-r}|, |autq^{-r}|\} < 1$.

Proof. Using (3.5), we find

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} U_n(x, y, a; q) U_n(u, v, b; q) \frac{t^n}{(q; q)_n} \\ &= \mathbb{F}(aD_{xy}) \left\{ \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} P_n(x, y) U_n(u, v, b; q) \frac{t^n}{(q; q)_n} \right\} \\ &= \mathbb{F}(aD_{xy}) \left\{ \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} P_n(x, y) \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} b^k P_{n-k}(u, v) \right) \frac{t^n}{(q; q)_n} \right\} \\ &= \mathbb{F}(aD_{xy}) \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(u, v) P_n(x, y) t^n}{(q; q)_n} \sum_{k=0}^{\infty} P_k(x, q^n y) \frac{(btq^{-n})^k}{(q; q)_k} \right\}. \end{aligned} \quad (3.11)$$

The terminating condition $v/u = q^{-r}$ or $y/x = q^{-r}$ implies that the first sum in (3.11) is finite. Utilizing (1.5), we see that (3.11) equals

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(u, v) t^n}{(q; q)_n} \mathbb{F}(aD_{xy}) \left\{ \frac{P_n(x, y)}{(ybtq^{-n}; q)_n} \frac{(ybtq^{-n}; q)_{\infty}}{(xbtq^{-n}; q)_{\infty}} \right\}.$$

Applying (3.7) with $s \rightarrow btq^{-n}$, the above sum equals

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(u, v) t^n}{(q; q)_n} \frac{(ybt, abtq^{-n}; q)_{\infty}}{(xbtq^{-n}; q)_{\infty}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \frac{(xbtq^{-n}; q)_k (y/x; q)_{n-k}}{(abtq^{-n}; q)_k} x^{n-k} a^k \\ &= \frac{(abt, ybt; q)_{\infty}}{(xbt; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (abtq^{-n}, y/x; q)_n P_n(u, v) (xt)^n}{(q; q)_n (xbtq^{-n}; q)_n} \sum_{k=0}^{\infty} P_k(u, vq^n) \frac{(atq^{-n})^k}{(q; q)_k}, \end{aligned} \quad (3.12)$$

Under the terminating condition, the above sum further simplifies to the right hand side of (3.10). This completes the proof. \blacksquare

We remark that the second sums in (3.11) and (3.12) do not converge when n tends to infinity. To avoid this problem, we may restrict our attention to the case that $v/u = q^{-r}$ or $y/x = q^{-r}$, where r is a nonnegative integer, as noticed by Askey and Suslov [4] and Fang [15].

Owing to the symmetry property and the relation (1.8), Mehler's formula for the Al-Salam-Carlitz polynomials $u_n^{(a)}(x; q)$ given by Al-Salam and Carlitz [2] can be recovered from the above theorem by setting $y/a \rightarrow a, x/a \rightarrow x, v/b \rightarrow b, u/b \rightarrow y, -abt \rightarrow t$.

Corollary 3.5.1 (Mehler's formula for $u_n^{(a)}(x|q)$).

$$\sum_{n=0}^{\infty} q^{-\binom{n}{2}} u_n^{(a)}(x; q) u_n^{(b)}(y; q) \frac{t^n}{(q; q)_n} = \frac{(-t, -at, -bt; q)_{\infty}}{(-xt, -yt; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} a/x, b/y, -q/t \\ -q/xt, -q/yt \end{matrix}; q, q \right), \quad (3.13)$$

where $a/x = q^{-r}$ or $b/y = q^{-r}$ for a nonnegative integer r , and $\max\{|xtq^{-r}|, |ytq^{-r}|\} < 1$.

4. The Cauchy Companion Operator

In this section, we apply the Cauchy companion operator defined by Chen in his thesis [10] to derive the Rogers-type formulas and an alternative form of Mehler's formula. Recall that the Cauchy augmentation operator is defined by Chen and Gu [11],

$$T(a, b; D_q) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (bD_q)^n.$$

Moreover, Chen [10] introduced the Cauchy companion operator

$$E(a, b; \theta) = \sum_{n=0}^{\infty} \frac{(a; q)_n (-b\theta)^n}{(q; q)_n}. \quad (4.1)$$

As observed by Chen [10], when applying $E(a, b; \theta)$ to the product $(cs, ct; q)_{\infty}/(cv; q)_{\infty}$, one does not get a valid identity by directly using q -Leibniz rule because of the convergence consideration. Instead, we may use the following expansion for D_q^n

$$D_q^n \{f(c)\} = c^{-n} q^{-\binom{n}{2}} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} f(cq^k).$$

In this way, we can deduce an alternative expansion for $E(a, b; \theta)$ which converges absolutely

$$E(a, b; \theta) \{f(c)\} = \frac{(abq/c; q)_{\infty}}{(bq/c; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a; q)_k f(cq^{-k}) q^{\binom{k}{2}}}{(q, abq/c; q)_k} \left(-\frac{bq}{c}\right)^k, \quad (4.2)$$

where $|bq/c| < 1$, see [10].

Proposition 4.1. *Assume that the operator $E(a, b; \theta)$ acts on the parameter c . Then we have*

$$E(a, b; \theta) \{c^n\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k (-bq)^k c^{n-k} q^{\binom{k}{2}} q^{-nk}, \quad (n \geq 0), \quad (4.3)$$

$$E(a, b; \theta) \left\{ \frac{(ct; q)_{\infty}}{(cv; q)_{\infty}} \right\} = \frac{(ct; q)_{\infty}}{(cv; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} a, t/v \\ q/cv \end{matrix}; q, \frac{bq}{c} \right), \quad (|bq/c| < 1), \quad (4.4)$$

$$E(a, b; \theta) \left\{ \frac{(cs, ct; q)_\infty}{(cv; q)_\infty} \right\} = \frac{(abq/c, cs, ct; q)_\infty}{(bq/c, cv; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, q/cs, q/ct \\ abq/c, q/cv \end{matrix}; q, \frac{bst}{v} \right),$$

$$(\max\{|bq/c|, |bst/v|\} < 1). \quad (4.5)$$

In light of the property (4.3), we obtain the following operator representation of $U_n(x, y, a; q)$.

Theorem 4.2. *Assume that the operator $E(y/x, x; \theta)$ acts on the parameter a . Then we have*

$$E(y/x, x; \theta) \{(-1)^n q^{\binom{n}{2}} a^n\} = U_n(x, y, a; q). \quad (4.6)$$

The first step in the derivation of the following theorems is to apply the above operator identity. We have to stress that special attention must be paid to the absolute convergence of the multi-sums. Otherwise, one may be led to formal identities that do not converge. By the definition (4.2), we have

$$E(y/x, x; \theta) \{(-1)^n q^{\binom{n}{2}} a^n\} = (-1)^n q^{\binom{n}{2}} \frac{(yq/a; q)_\infty}{(xq/a; q)_\infty} \sum_{k=0}^{\infty} \frac{(y/x; q)_k q^{\binom{k}{2}}}{(q, yq/a; q)_k} \left(-\frac{xq}{a}\right)^k a^n q^{-nk}.$$

The absolute convergence of the involved multi-sum is clearly affected by the factor q^{-nk} . As will be seen in the proof of the following theorem, the convergence issue should be taken into account.

Theorem 4.3. *We have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n q^{-\binom{n}{2}-nm} U_{n+m}(x, y, a; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(as; q)_\infty}{(at; q)_\infty} {}_2\phi_1 \left(\begin{matrix} y/x, s/t \\ q/at \end{matrix}; q, \frac{xq}{a} \right), \quad (4.7)$$

provided that $y/x = q^{-r}$ for a nonnegative integer r , and $\max\{|atq^{-r}|, |xq/a|\} < 1$.

Proof. By (4.6), the left hand side of (4.7) can be written as

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n q^{-\binom{n}{2}-nm} U_{n+m}(x, y, a; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m q^{\binom{m}{2}} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} E(y/x, x; \theta) \{a^{n+m}\}. \end{aligned} \quad (4.8)$$

By using the definition (4.2), the above equation comes to a triple summation,

$$\frac{(yq/a; q)_\infty}{(xq/a; q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m q^{\binom{m}{2}} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \sum_{k=0}^{\infty} \frac{(y/x; q)_k q^{\binom{k}{2}}}{(q, yq/a; q)_k} \left(-\frac{xq}{a}\right)^k a^{n+m} q^{-(n+m)k}.$$

Under the condition that this triple summation is absolutely convergent, we can move the operator $E(y/x, x; \theta)$ in (4.8) out of the double sum. That is to say, the operator $E(y/x, x; \theta)$ can act as a linear operator. As for the sums over m and n , the convergence depends on the following factors

$$q^{\binom{m}{2}-mk}, \quad q^{-nk}.$$

If we restrict our attention to the range $k \leq r$, for a nonnegative integer r , then the sum over m is convergent. If we further assume that $|atq^{-r}| < 1$, then the sum over n is also convergent. Thus, under the conditions that $y/x = q^{-r}$ and $|atq^{-r}| < 1$, we find that (4.8) equals

$$\begin{aligned} E(y/x, x; \theta) & \left\{ \sum_{n=0}^{\infty} \frac{a^n t^n}{(q; q)_n} \sum_{m=0}^{\infty} (-1)^m q^{\binom{m}{2}} \frac{(as)^m}{(q; q)_m} \right\} \\ & = E(y/x, x; \theta) \left\{ \frac{(as; q)_{\infty}}{(at; q)_{\infty}} \right\}. \end{aligned}$$

Using (4.4), we complete the proof. ■

Applying the operator $E(a, b; \theta)$ one more time, we obtain the following triple sum identity.

Theorem 4.4. *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k}{2} - (m+n)k - mn} U_{n+m+k}(x, y, a; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{v^k}{(q; q)_k} \\ & = \frac{(yq/a, as, at; q)_{\infty}}{(xq/a, av; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} y/x, q/as, q/at \\ yq/a, q/av \end{matrix}; q, \frac{xst}{v} \right), \end{aligned} \quad (4.9)$$

provided that $y/x = q^{-r}$ and $\max\{|avq^{-r}|, |xq/a|, |xst/v|\} < 1$.

Proof. By the operator identity (4.6), the left hand side of (4.9) equals

$$\begin{aligned} E(y/x, x; \theta) & \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{m+n} q^{\binom{n}{2} + \binom{m}{2}} \frac{(at)^n}{(q; q)_n} \frac{(as)^m}{(q; q)_m} \frac{(av)^k}{(q; q)_k} \right\} \quad (|av| < 1) \\ & = E(y/x, x; \theta) \left\{ \frac{(as, at; q)_{\infty}}{(av; q)_{\infty}} \right\}. \end{aligned}$$

Applying the operator identity (4.5), we complete the proof. ■

Setting $s \rightarrow 0$ and $t \rightarrow s, v \rightarrow t$ and applying Jackson's transformation formula [16, III.4], the triple sum (4.9) reduces to the Rogers-type formula (4.7). The Cauchy companion operator also applies to other Rogers-type formulas, including the one given in the previous section.

We now present the equivalent form of Mehler's formula.

Theorem 4.5. *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} U_n(x, y, a; q) U_n(u, v, b; q) \frac{t^n}{(q; q)_n} \\ & = \frac{(yq/a, abt, avt; q)_{\infty}}{(xq/a, aut; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} y/x, q/abt, q/avt \\ yq/a, q/aut \end{matrix}; q, \frac{xbvt}{u} \right), \end{aligned} \quad (4.10)$$

provided that $y/x = q^{-r}$ or $v/u = q^{-r}$ and $\max\{|autq^{-r}|, |xq/a|, |xbvt/u|\} < 1$.

Proof. Using (4.6) and (1.3), we find

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} U_n(x, y, a; q) U_n(u, v, b; q) \frac{t^n}{(q; q)_n} \\ &= E(y/x, x; \theta) \left\{ \sum_{n=0}^{\infty} U_n(u, v, b; q) \frac{(at)^n}{(q; q)_n} \right\} \\ &= E(y/x, x; \theta) \left\{ \frac{(avt; abt; q)_{\infty}}{(aut; q)_{\infty}} \right\}. \end{aligned}$$

Using the operator identity (4.5), we complete the proof. ■

Note that the above form of Mehler's formula is equivalent to (3.10) in view of the transformation formula derived by substituting $a \rightarrow q/avt$, $b \rightarrow y/x$, $c \rightarrow q/abt$, $d \rightarrow q/aut$ and $e \rightarrow yq/a$ into the transformation formula [16, Appendix (III.34)]:

$$\begin{aligned} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right) &= \frac{(e/b, e/c; q)_{\infty}}{(e, e/bc; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} d/a, b, c \\ d, bcq/e \end{matrix}; q, q \right) \\ &+ \frac{(d/a, b, c, de/bc; q)_{\infty}}{(d, e, bc/e, de/abc; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} e/b, e/c, de/abc \\ de/bc, eq/bc \end{matrix}; q, q \right). \end{aligned}$$

The second term on the right hand side vanishes because under the above substitution $(d/a; q)_{\infty} = (v/u; q)_{\infty} = (q^{-r}; q)_{\infty} = 0$.

5. Generating Functions for Products of $U_n(x, y, a; q)$

The objective of this section is to give several generating function formulas for products of the Al-Salam-Carlitz polynomials by using the Cauchy companion operator. Keep in mind that the Al-Salam-Carlitz polynomials are extensions of the Rogers-Szegő polynomials defined by

$$g_n(a|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} a^k.$$

It is easily seen that

$$U_n(0, 1, a; q) = (-1)^n q^{\binom{n}{2}} g_n(a|q).$$

Theorem 5.1. *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^{n+m} q^{-\binom{n+m}{2}} U_{n+m}(x, y, a; q) U_n(u, v, b; q) \frac{t^n}{(q; q)_n} \\ &= \frac{(yq/a, abt, avt; q)_{\infty}}{(xq/a, aut; q)_{\infty}} a^m {}_3\phi_2 \left(\begin{matrix} y/x, q/abt, q/avt \\ yq/a, q/aut \end{matrix}; q, \frac{xbvt}{uq^m} \right), \end{aligned}$$

where $y/x = q^{-r}$ for a nonnegative integer r and $\max\{|xq/a|, |autq^{-r}|, |\frac{xbvt}{uq^m}|\} < 1$.

Proof. By the operator identity (4.6) acting on the parameter a , we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^{n+m} q^{-\binom{n+m}{2}} U_{n+m}(x, y, a; q) U_n(u, v, b; q) \frac{t^n}{(q; q)_n} \\ &= E(y/x, x; \theta) \left\{ a^m \sum_{n=0}^{\infty} U_n(u, v, b; q) \frac{(at)^n}{(q; q)_n} \right\} \end{aligned} \quad (5.1)$$

Employing the generating function (1.3) with $|aut| < |q^r| < 1$ and the operator identity (4.2) with $|xq/a| < 1$, we see that (5.1) equals

$$\begin{aligned} & E(y/x, x; \theta) \left\{ a^m \frac{(avt, abt; q)_{\infty}}{(aut; q)_{\infty}} \right\} \\ &= \frac{(yq/a; q)_{\infty}}{(xq/a; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(y/x; q)_k q^{\binom{k}{2}}}{(q, yq/a; q)_k} \left(-\frac{xq}{a} \right)^k a^m q^{-mk} \frac{(avtq^{-k}, abtq^{-k}; q)_{\infty}}{(autq^{-k}; q)_{\infty}}, \end{aligned}$$

then the proof is completed by simplification. \blacksquare

Letting $y, v \rightarrow 1, x, u \rightarrow 0$, and applying the transformation formula for ${}_2\phi_1$ series [16, Appendix III.2], we are led to the following formula due to Cao [8, Theorem 4.1],

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} g_{n+m}(a|q) g_n(b|q) \frac{t^n}{(q; q)_n} \\ &= \frac{(abt, at, bt, t; q)_{\infty}}{(abt^2/q; q)_{\infty}} \frac{(q/t; q)_m}{(bt/q)^m (q^2/abt^2; q)_m} {}_2\phi_1 \left(\begin{matrix} q^{-m}, q/abt \\ tq^{-m} \end{matrix}; q, bt \right). \end{aligned}$$

Theorem 5.2. Assume $\max\{|xq/a|, |uq/b|\} < 1$. We have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k}{2} - (n+m)k} U_{n+k}(x, y, a; q) U_{m+k}(u, v, b; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{z^k}{(q; q)_k} \\ &= \frac{(yq/a, vq/b, bs, at, abz; q)_{\infty}}{(xq/a, uq/b; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} (y/x, q/at, q/abz; q)_k}{(q, yq/a; q)_k} \left(\frac{xabtz}{q} \right)^k \\ & \quad \times {}_3\phi_1 \left(\begin{matrix} v/u, q/bs, q^{k+1}/abz \\ vq/b \end{matrix}; q, \frac{abusz}{q^{k+1}} \right), \end{aligned}$$

where $y/x = q^{-\alpha}$ for a nonnegative integer α and $\max\{|usq^{-\alpha}|, |auzq^{-\alpha}|, |abusz/q^{\alpha+1}|\} < 1$.

Setting $y, v \rightarrow 1, x, u \rightarrow 0$, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n+m+k} q^{\binom{n}{2} + \binom{m}{2} + \binom{k}{2}} g_{n+k}(a|q) g_{m+k}(b|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{z^k}{(q; q)_k} \\ &= (q/a, q/b, bs, at, abz; q)_{\infty} \sum_{k=0}^{\infty} \frac{(q/at, q/abz; q)_k}{(q, q/a; q)_k} \left(\frac{abtz}{q} \right)^k {}_2\phi_1 \left(\begin{matrix} q/bs, q^{k+1}/abz \\ q/b \end{matrix}; q, \frac{absz}{q^{k+1}} \right). \end{aligned}$$

Under the conditions $q/at = q^{-\beta}$ and $|absz/q^{\beta+1}| < 1$ for a nonnegative integer β , the ${}_2\phi_1$ series in the above expression can be summed by the q -Gauss formula [16, Appendix (II.8)].

It follows that

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n+m+k} q^{\binom{n}{2} + \binom{m}{2} + \binom{k}{2}} g_{n+k}(a|q) g_{m+k}(b|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{z^k}{(q; q)_k} \\ &= \frac{(q/a, s, at, az, bs, abz; q)_{\infty}}{(absz/q; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} q^{-n}, q/az, q/abz \\ q/a, q^2/absz \end{matrix}; q, \frac{atz}{s} \right). \end{aligned}$$

It should be mentioned that the terminating condition $t/v = q^{-r}$ is missing in the operator identity of Zhang and Wang [20, Theorem 2.5], and the same convergence conditions are required for the identity of Cao [8, Eq. (2.9)].

Theorem 5.3. *Assume $\max\{|xq/a|, |vq/b|\} < 1$. We have*

$$\begin{aligned} & \sum_{k=0}^{\infty} (-1)^{m+n+k} q^{-\binom{n+k}{2} - \binom{m}{2} - mk} U_{n+k}(x, y, a; q) U_{m+k}(u, v, b; q) \frac{z^k}{(q; q)_k} \\ &= \frac{(yq/a, vq/b, abz; q)_{\infty}}{(xq/a, uq/b; q)_{\infty}} a^n b^m \sum_{k=0}^{\infty} \frac{(y/x, q/abz; q)_k}{(q, yq/a; q)_k} \left(\frac{xbz}{q^n}\right)^k {}_2\phi_1 \left(\begin{matrix} v/u, q^{k+1}/abz \\ vq/b \end{matrix}; q, \frac{auz}{q^{m+k}} \right), \end{aligned}$$

where $y/x = q^{-r}$ for a nonnegative integer r and $|auz/q^{m+r}| < 1$.

Setting $y, v \rightarrow 1, x, u \rightarrow 0$, we deduce that

$$\begin{aligned} & \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} g_{n+k}(a|q) g_{m+k}(b|q) \frac{z^k}{(q; q)_k} \\ &= (q/a, q/b, abz; q)_{\infty} a^n b^m \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (q/abz; q)_k}{(q, q/a; q)_k} \left(\frac{bz}{q^n}\right)^k {}_1\phi_1 \left(\begin{matrix} q^{k+1}/abz \\ q/b \end{matrix}; q, \frac{az}{q^{m+k}} \right), \end{aligned}$$

which can be deduced from the formula of Cao [8, Theorem 4.3] by three transformations, namely, the limiting case of [16, Appendix (III.2)] when $c \rightarrow 0$, the two transformations [16, Appendix (III.2)] and [16, Appendix (III.7)].

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