Two-tough graphs and f-factors with given properties *

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Abstract

Let G be a 2-tough graph on at least five vertices and let e_1, e_2 be any two edges of G. Katerinis and Wang [6] showed that there exists a 2-factor in G including/excluding e_1 and e_2. In this paper, we generalize their result by considering the existence of an f-factor including/excluding e_1 and e_2, where $f : V(G) \to \{1, 2\}$.

Keywords: f-factor, inclusion/exclusion property, 2-toughness.

1 Introduction

All graphs considered are simple and finite. We refer the reader to [1] for terminology and notations not defined here.

Let G be a graph. The degree of a vertex v in G is denoted by $\deg_G(v)$. For any disjoint subsets $X, Y \subseteq V(G)$, $E_G(X, Y)$ denotes the set of edges with one end in X and the other in Y. Set $e_G(X, Y) = |E_G(X, Y)|$.

For $X \subseteq V(G)$, the *neighbor set* of X in G is defined to be the set of all vertices adjacent to vertices in X; this set is denoted by $N_G(X)$. The *subgraph induced* by X, denoted by $G[X]$, has vertex set X and edge set $\{uv \in E(G) : u, v \in X\}$.

A *cut* or *vertex cut* of a connected graph G is a set of vertices whose removal renders G disconnected. A *k-vertex cut* is a vertex cut with k elements. The *connectivity* of G, $\kappa(G)$, is the minimum k for which G has a k-vertex cut. Similarly, an edge cut and edge-connectivity of G $\big($i.e., $\kappa'(G)\big)$ are defined.

For an integer-valued function f defined on a finite set X, we put

$$f(X) = \sum_{x \in X} f(x), \quad f(\emptyset) = 0.$$

Let f be an integer-valued function defined on the vertex set of a graph G. Then G has an *f-factor* if there exists a spanning subgraph F of G such that $\deg_F(v) = f(v)$ for every vertex $v \in V(G)$. In particular, if $f(v) = k$ for all $v \in V(G)$, the spanning subgraph F is called a *k-factor*.

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If G is not complete, the *toughness* $t(G)$ is defined as

$$t(G) = \min_S \left\{ \frac{|S|}{c(G-S)} \right\},$$

where the minimum is taken over all vertex cut S of G, and $c(G-S)$ denotes the number of components in $G-S$. For complete graph K_n, we set $t(K_n) = \infty$. A graph G is *k-tough* if $t(G) \geq k$.

Chvátal introduced the concept of toughness in [3], and mainly studied the relations between toughness and the existence of Hamiltonian cycles and k-factors. He conjectured that every k-tough graph G ($k \in \mathbb{Z}^+$) has a k-factor if $k|V(G)|$ is even. Enomoto, Jackson, Katerinis and Saito [4] confirmed Chvátal's conjecture and also proved that their result is sharp. Chen [2] showed a stronger result: for any k-tough graph G and for every edge e of G, the graph G contains a k-factor F_1 containing e and another k-factor F_2 excluding e. Katerinis and Wang [6] obtained the following result.

Theorem 1.1 (Katerinis and Wang, [6]). Let G be a 2-tough graph with at least five vertices, and let e_1, e_2 be a pair of arbitrarily given edges of G. Then

(a) there exists a 2-factor in G including e_1, e_2;

(b) there exists a 2-factor in G excluding e_1, e_2;

(c) there exists a 2-factor in G including e_1 and excluding e_2.

Katerinis (1990) also proved a result related to the existence of f-factor in 2-tough graphs.

Theorem 1.2 (Katerinis, [5]). Let G be a 2-tough graph and $f : V(G) \to \{1,2\}$ be a function such that $f\big(V(G)\big)$ is even. Then G has an f-factor.

Motivated by above theorems, we consider 2-tough graphs and f-factors with inclusion and/or exclusion properties.

2 Preliminary Results

The following result shows the relation between toughness, connectivity and minimum degree.

Proposition 2.1 (Chvátal, [3]). For any non-complete graph G,

$$t(G) \leq \frac{\kappa(G)}{2} \leq \frac{\delta(G)}{2}.$$

A necessary and sufficient condition for a graph G to have an f-factor was obtained by Tutte [7] in 1952.

Theorem 2.1 (Tutte's f-factor Theorem). Let G be a graph and $f : V(G) \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers. Then G has an f-factor if and only if

$$q_G(S,T;f) + \sum_{x\in T} \big(f(x) - \deg_{G-S}(x)\big) \leq f(S) \tag{2.1}$$

for all disjoint subsets $S, T \subseteq V(G)$, where $q_G(S,T;f)$ denotes the number of components C of $G-(S\cup T)$ such that $e_G\big(V(C),T\big) + f\big(V(C)\big)$ is odd. (Hereafter, we refer to these components as *odd* components.) Moreover,

$$q_G(S,T;f) + \sum_{x\in T} \big(f(x) - \deg_{G-S}(x)\big) - f(S) \equiv f\big(V(G)\big) \pmod 2. \tag{2.2}$$

The following lemmas play important roles in the proofs of the main theorems.

Lemma 2.1. Let G be a graph, $e = ab$ be an edge of G and let G' be the graph obtained from G by inserting a new vertex u on the edge e.

For a given function $f : V(G) \to \{1,2\}$, define a function $f' : V(G') \to \{1,2\}$ as follows:

$$f'(v) = \begin{cases} 2, & \text{if } v = u; \\ f(v), & \text{otherwise.} \end{cases}$$

Then, for any pair of disjoint subsets $S', T' \subseteq V(G')$,

$$q_{G'}(S',T';f') + \sum_{x\in T'} \big(f'(x) - \deg_{G'-S'}(x)\big) = q_G(S,T;f) + \sum_{x\in T} \big(f(x) - \deg_{G-S}(x)\big) + 2\varepsilon \qquad (2.3)$$

where $S = S' - \{u\}$, $T = T' - \{u\}$ and $\varepsilon = 0,1$. Moreover, if $u \notin S'$, then $\varepsilon = 1$ if and only if

(I) $e \in E(G[S])$ and $u \in T'$; or

(II) $e \in E_G\big(V(C'), S\big)$ and $u \in T'$, where C' is an odd component of $G' - (S' \cup T')$ and $V(C')$ induces an even component of $G - (S \cup T)$; or

(III) $a \in C'_1, b \in C'_2$ and $u \in T'$, where C'_1, C'_2 are two odd components of $G' - (S' \cup T')$ and $V(C'_1 \cup C'_2)$ induces an even component of $G - (S \cup T)$.

Proof. Since $\deg_{G'}(u) = 2 = f'(u)$ and $\deg_{G'}(x) = \deg_G(x)$ for any vertex $x \in V(G') - \{u\}$, $f\big(V(G')\big) \equiv f\big(V(G)\big) \pmod 2$. It follows from (2.2) that to prove (2.3), it is enough to prove $\varepsilon = 0,1$. Note that

$$\begin{aligned} & q_{G'}(S',T';f') + \sum_{x\in T'} \big(f'(x) - \deg_{G'-S'}(x)\big) \\ = \;& q_{G'}(S',T';f') + \sum_{x\in T'} \big(f'(x) - \deg_{G'}(x)\big) + e_{G'}(S',T') \\ = \;& q_{G'}(S',T';f') + \sum_{x\in T} \big(f(x) - \deg_{G}(x)\big) + e_{G'}(S',T'). \end{aligned}$$

In fact, the graph G' can be viewed as a graph obtained from G by deleting the edge ab and adding two adjacent edges ua, ub. Then

$$-1 \le e_{G'}(S',T') - e_G(S,T) \le 2.$$

There are four cases to consider.

Case 1. $e_{G'}(S',T') = e_G(S,T) + 2$. Then $\{ua, ub\} \subseteq E_{G'}(S',T')$. So $q_{G'}(S',T';f') = q_G(S,T;f)$ and $\varepsilon = 1$. If $u \notin S'$, then $e \in E(G[S])$ (see Fig. 1).

Case 2. $e_{G'}(S',T') = e_G(S,T) + 1$. Then exactly one of $\{ua, ub\}$ is in $E_{G'}(S',T')$, and hence one vertex of $\{a,b\}$, say b, is in a component of $G' - (S' \cup T')$. So $q_{G'}(S',T';f') = q_G(S,T;f) - 1$ or $q_{G'}(S',T';f') = q_G(S,T;f) + 1$, and then $\varepsilon = 0$ or 1.

If $u \notin S'$, $\varepsilon = 1$ if and only if the component C' containing b is an odd component of $G' - (S' \cup T')$ (see Fig. 2). Note that in this case, $V(C')$ induces an even component of $G - (S \cup T)$.

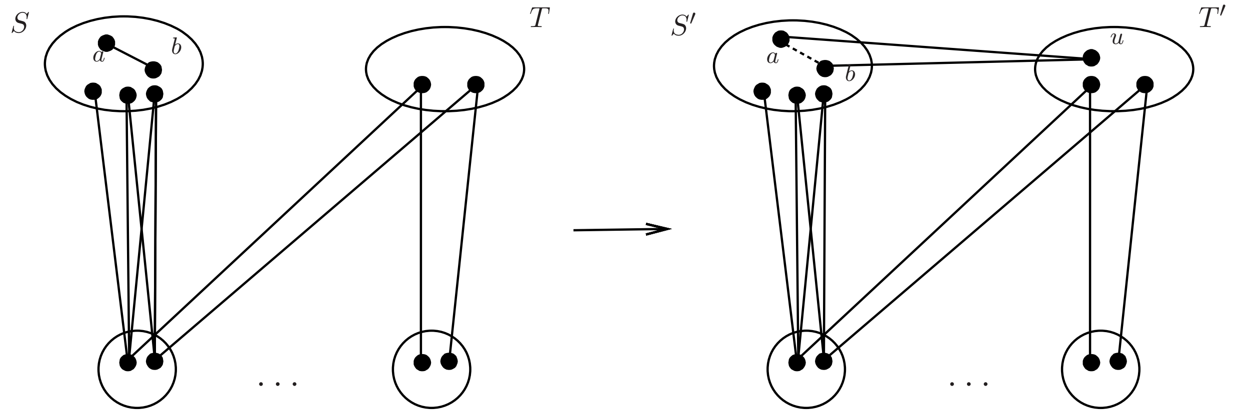
Case 3. $e_{G'}(S',T') = e_G(S,T)$. Then $\varepsilon = 0,1$, as $0 \le q_{G'}(S',T';f') - q_G(S,T;f) \le 2$. If $u \notin S'$, $\varepsilon = 1$ if and only if a and b are in two distinct odd components C'_1, C'_2 of $G' - (S' \cup T')$ (see Fig. 3). Now $V(C'_1 \cup C'_2)$ induces an even component of $G - (S \cup T)$.

Case 4. $e_{G'}(S',T') = e_G(S,T) - 1$. Then the edge $ab \in E_G(S,T)$ and vertex $u \notin S' \cup T'$. Since u has only two neighbors a and b, vertex u induced a component of $G' - (S' \cup T')$. Moreover, vertex u induced an odd component of $G' - (S' \cup T')$. Then $q_{G'}(S',T';f') = q_G(S,T;f) + 1$, and hence $\varepsilon = 0$. □

Lemma 2.2. Let G be a graph, $e = ab$ be an edge of G and $G' = G - \{e\}$. Given a function $f : V(G) \to \{1,2\}$, then, for any pair of disjoint subsets $S, T \subseteq V(G)$,

$$q_{G'}(S,T;f) + \sum_{x\in T} \big(f(x) - \deg_{G'-S}(x)\big) = q_G(S,T;f) + \sum_{x\in T} \big(f(x) - \deg_{G-S}(x)\big) + 2\varepsilon \qquad (2.4)$$

where $\varepsilon = 0,1$. Moreover, $\varepsilon = 1$ if and only if



the graph with triple (G,S,T) the graph with triple (G',S',T')

Fig. 1: Location of e in G and that of a,b,u in (G',S',T') as in type (I)

(IV) $e\in E(G[T])$; or

(V) $e\in E_G\big(V(C'),T\big)$, where C' is an odd component of $G'-(S\cup T)$ and $V(C')$ induces an even component of $G-(S\cup T)$; or

(VI) $e\in E_G\big(V(C_1'),V(C_2')\big)$, where C_1',C_2' are two odd components of $G'-(S\cup T)$ and $V(C_1'\cup C_2')$ induces an even component of $G-(S\cup T)$.

Proof. Since $G'=G-\{e\}$, $\deg_{G'}(a)=\deg_G(a)-1$ and $\deg_{G'}(b)=\deg_G(b)-1$. Hence we have

$$q_{G'}(S,T;f)+\sum_{x\in T}\big(f(x)-\deg_{G'-S}(x)\big)=q_{G'}(S,T;f)-\sum_{x\in T}\deg_{G'-S}(x)+f(T).$$

According to the locations of a,b and T, there are three cases to consider.

Case 1. $a,b\in T$. Then $q_{G'}(S,T;f)=q_G(S,T;f)$ and $\sum\limits_{x\in T}\deg_{G'-S}(x)=\sum\limits_{x\in T}\deg_{G-S}(x)-2$. Thus, $\varepsilon=1$.

Case 2. Exactly one of $\{a,b\}$, say a, is in T. If $b\in S$, then $\sum\limits_{x\in T}\deg_{G'-S}(x)=\sum\limits_{x\in T}\deg_{G-S}(x)$ and $q_G(S,T;2)=q_{G'}(S,T;2)$; if $b\notin S\cup T$, then $\sum\limits_{x\in T}\deg_{G'-S}(x)=\sum\limits_{x\in T}\deg_{G-S}(x)-1$ and $q_{G'}(S,T;f)-q_G(S,T;f)=-1$ or 1. Then $\varepsilon=0,1$. Moreover, $\varepsilon=1$ if and only if $q_{G'}(S,T;f)=q_G(S,T;f)+1$, i.e., $b\in V(C')$, where C' is an odd component of $G'-(S\cup T)$. Clearly, $V(C')$ induces an even component of $G-(S\cup T)$.

Case 3. $\{a,b\}\cap T=\emptyset$. Then $\sum\limits_{x\in T}\deg_{G'-S}(x)=\sum\limits_{x\in T}\deg_{G-S}(x)$ and $q_{G'}(S,T;f)-q_G(S,T;f)=0$ or 2; thus $\varepsilon=0,1$. Moreover, $\varepsilon=1$ if and only if $e\in E_G\big(V(C_1'),V(C_2')\big)$, where C_1',C_2' are two odd components of $G'-(S\cup T)$, and $V(C_1'\cup C_2')$ induces an even component of $G-(S\cup T)$. $\square$

Lemma 2.3. Let G be a graph, and f a function from $V(G)$ to $\{1,2\}$ with $f\big(V(G)\big)\equiv 0 \pmod 2$. Suppose that there exists a pair of disjoint subsets S and T of $V(G)$ such that

$$q_G(S,T;f)+\sum_{x\in T}\big(f(x)-\deg_{G-S}(x)\big)-f(S)\geq 2. \tag{2.5}$$

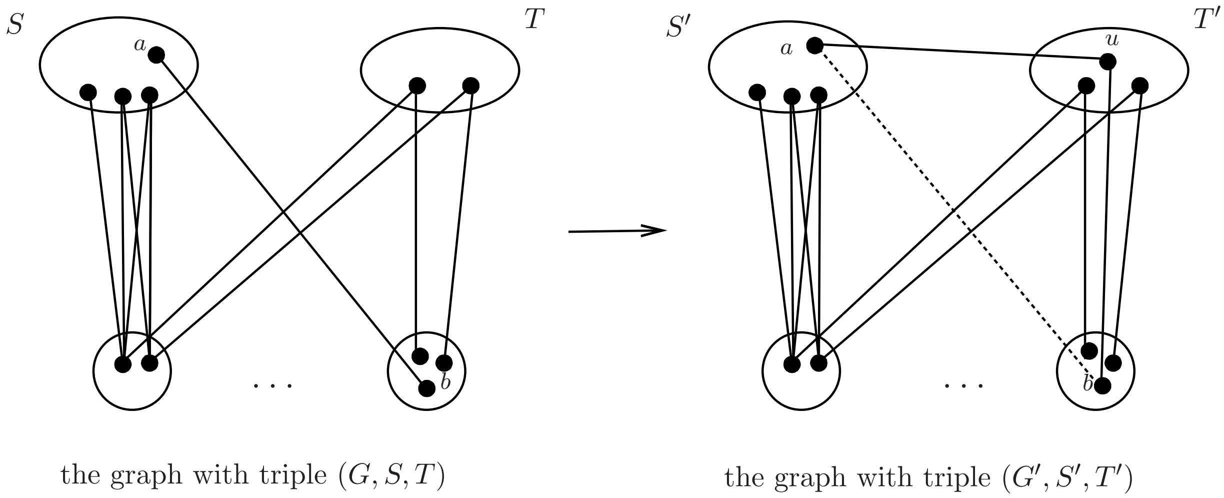


Fig. 2: Location of e in G and that of a,b,u in (G',S',T') as in type (II)

Then

(a) if S is minimal with respect to (2.5), then for any vertex $u\in S$, $\deg_G(u)\ge f(u)+2\ge 3$;

(b) if T is minimal with respect to (2.5), then T is an independent set in G. Moreover, for any vertex $v\in T$, $f(v)=2$, and $e_G\big(\{v\},V(C)\big)\ne 0$ implies that C is an odd component of $G-(S\cup T)$ with $e_G\big(\{v\},V(C)\big)=1$.

Proof. Since $f\big(V(G)\big)\equiv 0 \pmod 2$, it follows from (2.2) that

$$q_G(S,T;f)+\sum_{x\in T}\big(f(x)-\deg_{G-S}(x)\big)-f(S)\equiv 0 \pmod 2.$$

(a) Since S is minimal with respect to (2.5), for any vertex $u\in S$, we have

$$q_G(S-\{u\},T;f)+\sum_{x\in T}\big(f(x)-\deg_{G-(S-\{u\})}(x)\big)-f(S-\{u\})\le 0. \tag{2.6}$$

Combining (2.5) and (2.6), we have $q_G(S,T;f)-q_G(S-\{u\},T;f)+|N_G(u)\cap T|-f(u)\ge 2$. Hence $\deg_G(u)\ge \deg_{G-(S\cup T)}(u)+|N_G(u)\cap T|\ge q_G(S,T;f)-q_G(S-\{u\},T;f)+|N_G(u)\cap T|\ge f(u)+2\ge 3$.

(b) Since T is minimal with respect to (2.5), for any vertex $v\in T$, we have

$$q_G(S,T-\{v\};f)+\sum_{x\in T-\{v\}}\big(f(x)-\deg_{G-S}(x)\big)-f(S)\le 0. \tag{2.7}$$

Combining (2.5) and (2.7), we have $q_G(S,T;f)-q_G(S,T-\{v\};f)+\big(f(v)-\deg_{G-S}(v)\big)\ge 2$. Thus

$$\begin{aligned}\deg_{G-S}(v)&\le q_G(S,T;f)-q_G(S,T-\{v\};f)+f(v)-2 &(2.8)\\ &\le \deg_{G-(S\cup T)}(v). &(2.9)\end{aligned}$$

Therefore, $\deg_{G-S}(v)=\deg_{G-(S\cup T)}(v)$, that is, $|N_G(v)\cap T|=0$. Since v is an arbitrary vertex in T, T is an independent set of G. Moreover, the inequalities in (2.5), (2.7), (2.8) and (2.9) become equalities and thus $\deg_{G-S}(v)=\deg_{G-(S\cup T)}(v)$ implies that $f(v)=2$ and $q_G(S,T;f)-q_G(S,T-\{v\};f)=\deg_{G-(S\cup T)}(v)$. Therefore, if v is adjacent to a component C, then C is an odd component of $G-(S\cup T)$ and $|N_G(v)\cap V(C)|=1$. □

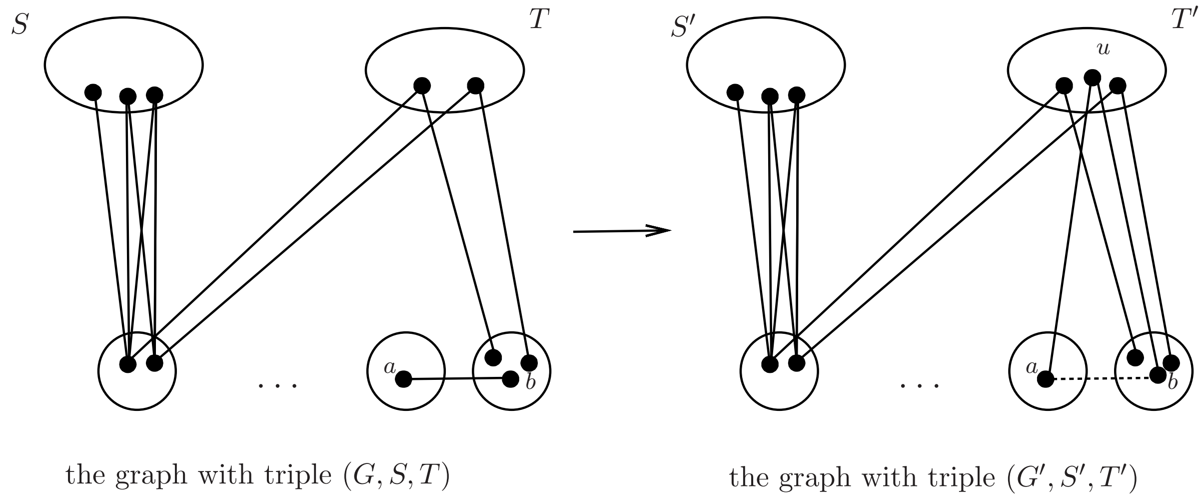


Fig. 3: Location of e in G and that of a, b, u in (G', S', T') as in type (III)

3 Main Results

Suppose G is a non-complete 2-tough graph and $e_1 = a_1b_1, e_2 = a_2b_2$ are two edges of G. The graph G_1 is obtained from G by either subdivision (i.e., inserting a new vertex u_1 on the edge e_1) or deletion of e_1, and G_2 is obtained from G_1 by either subdivision (i.e., inserting a new vertex u_2 on the edge e_2) or deletion of e_2. Consider a function $f : V(G) \to \{1, 2\}$ with $f\big(V(G)\big) \equiv 0 \pmod 2$. If the operation on e_i is deletion, at least one of $f(a_i), f(b_i)$ is equal to 1. Let f_1, f_2 be two functions defined on $V(G_1), V(G_2)$, respectively, with $f_1(v) = f_2(v) = f(v)$ for $v \in V(G)$, $f_1(u_1) = 2$ if the operation on e_1 is subdivision, $f_2(u_i) = 2$ if the operation on e_i is subdivision for $i = 1, 2$. Let $\varepsilon_1, \varepsilon_2$ be two binary variables, corresponding to ε in (2.3) or (2.4) when conducting operations on e_1 and e_2, respectively.

In this paper, we consider the existence of f-factor including/excluding the edges e_1 and e_2. In the proofs of the main theorems, there are several similar arguments in the proofs and so we state the common technique as a lemma below. For convenience, in the following lemma, when u_i $(i = 1, 2)$ is mentioned, it means that the operation on e_i is a subdivision.

Lemma 3.1. Let G, G_1, G_2, f, f_1, f_2 be defined as above. Suppose that G_2 contains no f_2-factor. Then

(1) there exists disjoint subsets $S, T \subseteq V(G)$, $S_1, T_1 \subseteq V(G_1)$, $S_2, T_2 \subseteq V(G_2)$ with $S = S_1 = S_2, T = T_1 - \{u_1\}, T_1 = T_2 - \{u_2\}$ satisfying

$$2 - 2(\varepsilon_1 + \varepsilon_2) \ \le q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - f(S), \tag{3.10}$$

where r is the number of components of $G - (S \cup T)$ which are joined to T, $\varepsilon_1 + \varepsilon_2 \ge 1$ and T is independent with $f(x) = 2$ for all $x \in T$;

(2) if $c\big(G - (S \cup T)\big) = r$ and the inequality (3.10) becomes an equality, then for any component C_i of $G - (S \cup T)$, C_i containing a cut edge e implies $|V(C_i)| = 2$.

Proof. Since G_2 has no f_2-factors, by Tutte's f-Factor Theorem, there exists a pair of disjoint subsets S_2, T_2 of $V(G_2)$ such that

$$q_{G_2}(S_2, T_2; f_2) + \sum_{x \in T_2} \big(f_2(x) - \deg_{G_2 - S_2}(x)\big) - \sum_{x \in S_2} f_2(x) \ge 2. \tag{3.11}$$

Furthermore, assume that S_2 and T_2 are minimal with respect to (3.11), respectively.

By Lemma 2.3, T_2 is independent and $u_i \notin S_2$ as $\deg_{G_2}(u_i) = 2$. Let $S = S_1 = S_2$, $T_1 = T_2 - \{u_2\}$ and $T = T_1 - \{u_1\}$. For every vertex $x \in T$, as $x \in T_2$, $f(x) = f_2(x) = 2$ by Lemma 2.3 again.

Claim 1. T is an independent set in G.

Note that T_2 is independent, so we only need to consider whether e_i lies in $G[T]$ $(i = 1, 2)$. If the operation on e_i is deletion, then there are at least one of $f(a_i)$ and $f(b_i)$, say $f(a_i)$, to be 1. Thus $f_2(a_i) = f(a_i) = 1$. By Lemma 2.3, $a_i \notin T_2$. Then $a_i \notin T$ and hence $e_i \notin G[T]$. Now, consider the case that at least one operation, say on e_2, is a subdivision. Assume T is not independent in G, and $\{a_2, b_2\} \subseteq T$. As T_2 is independent in G_2, $u_2 \notin S_2 \cup T_2$ and so $\{u_2\}$ is an even component of $G_2 - (S_2 \cup T_2)$, a contradiction to the fact that a_2 is only adjacent to odd components of $G_2 - (S_2 \cup T_2)$ (Lemma 2.3 (b)).

By Lemmas 2.1 and 2.2, we have

$$\begin{aligned} q_{G_1}(S_1, T_1; f_1) + \sum_{x\in T_1} \big(f_1(x) - \deg_{G_1 - S_1}(x)\big) &= q_G(S, T; f) + \sum_{x\in T} \big(f(x) - \deg_{G-S}(x)\big) + 2\varepsilon_1; \\ q_{G_2}(S_2, T_2; f_2) + \sum_{x\in T_2} \big(f_2(x) - \deg_{G_2 - S_2}(x)\big) &= q_{G_1}(S_1, T_1; f_1) + \sum_{x\in T_1} \big(f_1(x) - \deg_{G_1 - S_1}(x)\big) + 2\varepsilon_2. \end{aligned}$$

Thus

$$q_{G_2}(S_2, T_2; f_2) + \sum_{x\in T_2} \big(f_2(x) - \deg_{G_2 - S_2}(x)\big) = q_G(S, T; f) + \sum_{x\in T} \big(f(x) - \deg_{G-S}(x)\big) + 2(\varepsilon_1 + \varepsilon_2). \quad (3.12)$$

By (3.11) and (3.12), we have

$$q_G(S, T; f) + \sum_{x\in T} \big(f(x) - \deg_{G-S}(x)\big) - \sum_{x\in S} f(x) \geq 2 - 2(\varepsilon_1 + \varepsilon_2). \quad (3.13)$$

As G is 2-tough, by Theorem 1.2, G has an f-factor and so

$$q_G(S, T; f) + \sum_{x\in T} \big(f(x) - \deg_{G-S}(x)\big) - \sum_{x\in S} f(x) \leq 0.$$

Hence $\varepsilon_1 + \varepsilon_2 \geq 1$, i.e., $\varepsilon_1 + \varepsilon_2 = 1$ or 2, and at least one of $\varepsilon_1, \varepsilon_2$ equals to 1.

Let $H = G - (S \cup T)$. Assume $C_1, C_2, \ldots, C_t$ are the components of H. Let $V_1 = \{v \in V(H) : |N_G(v) \cap T| = 1\}$ and $V_2 = \{v \in V(H) : |N_G(v) \cap T| \geq 2\}$. Suppose that $C_1, C_2, \ldots, C_l$ are the components containing a vertex in V_1. Arbitrarily choose $x_i \in V(C_i)$ for which $x_i \in V_1$ for $i = 1, \ldots, l$, and set $X = \{x_1, x_2, \ldots, x_l\}$. Let $Y = N_G(T) \cap V(H) - X$.

By the definitions of V_1 and V_2, we have

$$|V_1| + 2|V_2| \leq e_G\big(T, V(H)\big). \quad (3.14)$$

Thus $|V_1| + |V_2| \leq e_G\big(T, V(H)\big) - |V_2|$, and $|Y| = |N_G(T) \cap V(H)| - |X| = |V_1| + |V_2| - |X| \leq e_G\big(T, V(H)\big) - |V_2| - |X|$.

Clearly,

$$|V_2| + |X| \geq r, \quad (3.15)$$

where r is the number of components of $G - (S \cup T)$ which are joined to T. So $|Y| \leq e_G\big(T, V(H)\big) - r$, and then $|S| + |Y| \leq |S| + e_G\big(T, V(H)\big) - r$.

By the choice of Y, $c\big(G - (S \cup Y)\big) \geq |T| + c\big(G - (S \cup T)\big) - r$. Assume $|S| + |Y| < 2c\big(G - (S \cup Y)\big)$. Then $c\big(G - (S \cup Y)\big) \leq 1$, otherwise, $S \cup Y$ is a vertex cut of G, contradicting the fact that G is 2-tough. Indeed, $c\big(G - (S \cup Y)\big) = 1$, otherwise $|S| + |Y| < 0$, a contradiction. Since $|S| + |Y| < 2c\big(G - (S \cup Y)\big) = 2$, so $|S| + |Y| \leq 1$; moreover $|S| \leq 1$ and $|T| \leq c\big(G - (S \cup Y)\big) = 1$. If $|T| = 1$, say $T = \{x\}$, then $\deg_{G-S}(x) \leq |Y| + 1 \leq 2$ by the definition of Y. On the other hand, since G is non-complete and

2-tough, $\delta(G) \geq 2t(G) \geq 4$. Then, $\deg_{G-S}(x) \geq 3$ as $|S| \leq 1$, a contradiction. Now we may assume $T = \emptyset$. Since at least one of ε_i $(i = 1, 2)$ is equal to 1, G_2 is obtained by at least one operation on e_i $(i = 1, 2)$ whose location is of type II or type III or type VI. Thus, $\{e_1, e_2\}$ is an edge cut in $G - S$. Since G is 2-tough, it is 4-connected. So $G - S$ is 3-connected and 3-edge-connected. But now, $(G - S) - \{e_1, e_2\}$ is disconnected, a contradiction.

So we may assume

$$|S| + |Y| \geq 2c\big(G - (S \cup Y)\big) \tag{3.16}$$
$$\geq 2|T| + 2c\big(G - (S \cup T)\big) - 2r. \tag{3.17}$$

Then $|S| + e_G\big(T, V(H)\big) - r \geq |S| + |Y| \geq 2|T| + 2c\big(G - (S \cup T)\big) - 2r$, i.e., $e_G\big(T, V(H)\big) \geq 2|T| + 2c\big(G - (S \cup T)\big) - r - |S|$.

Therefore

$$\begin{aligned} 2 - 2(\varepsilon_1 + \varepsilon_2) &\leq q_G(S, T; f) + \sum_{x \in T} \big(f(x) - \deg_{G-S}(x)\big) - f(S) \\ &= q_G(S, T; f) + f(T) - \sum_{x \in T} \deg_{G-S-T}(x) - f(S) \\ &= q_G(S, T; f) + 2|T| - e_G\big(T, V(H)\big) - f(S) \\ &\leq q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - f(S). \end{aligned}$$

This completes the proof of assertion (1).

Next, suppose that $c\big(G - (S \cup T)\big) = r$ and $2 - 2(\varepsilon_1 + \varepsilon_2) = q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - f(S)$. Then all inequalities above (in the proof) become equalities. So $c\big(G - (S \cup Y)\big) = |T| + c\big(G - (S \cup T)\big) - r = |T|$. By (3.17), $|S| + |Y| = 2c\big(G - (S \cup Y)\big) = 2|T|$. Moreover, the following assertions hold:

(a) for any $v \in V(H)$, $|N_G(v) \cap T| \leq 2$ (by (3.14));

(b) if C_j $(1 \leq j \leq t)$ contains a vertex in V_1, then C_j contains no vertex in V_2; if C_j contains a vertex in V_2, then $|V_2 \cap V(C_j)| = 1$ (by (3.15)).

Consider an arbitrary component C_i of $G - (S \cup T)$.

Claim 2. Either $|N_G(v) \cap T| = 1$ for any $v \in V(C_i)$ or $|V(C_i)| = 1, e_G\big(V(C_i), T\big) = 2$.

As $c\big(G - (S \cup T)\big) = r$, $e_G\big(V(C_i), T\big) \neq 0$; that is, C_i contains at least one vertex either in V_1 or V_2. If C_i contains a vertex in V_2, then $|V(C_i)| = 1$ by (a), (b) and the fact that $c\big(G - (S \cup Y)\big) = |T|$. Next assume that C_i contains a vertex in V_1. According to (a), for any $v \in V(C_i)$, $|N_G(v) \cap T| \leq 1$. Let $W_i = \{v \in V(C_i) : |N_G(v) \cap T| = 0\}$. If $W_i = \emptyset$, we are done. Otherwise, since $c\big(G - (S \cup Y)\big) = |T|$, W_i is joined to $x_i \in X \cap V(C_i)$. Thus, $G - (S \cup Y \cup \{x_i\})$ contains at least $c\big(G - (S \cup Y)\big) + 1$ components. As $|S| + |Y| = 2c\big(G - (S \cup Y)\big)$, we have $|S| + |Y| + |\{x_i\}| = 2c\big(G - (S \cup Y)\big) + 1 < 2c\big(G - \big(S \cup Y \cup \{x_i\}\big)\big)$, contradicting the 2-toughness of G.

We continue to prove assertion (2) by contradiction. Suppose C_i contains an edge $e = ab$ such that $C_i - e$ is disconnected, but $|V(C_i)| > 2$. Then there exists a vertex v_0 different from a, b. Without loss of generality, we may assume v_0 is not adjacent to a, because $e = ab$ is a cut edge of C_i. Now $|V(C_i)| \geq 2$, and by Claim 2, $|N_G(v) \cap T| = 1$ for any $v \in V(C_i)$. Thus $e_G(a, T) = 1$ and $e_G(v_0, T) = 1$. Select $x_i = a$. (This is possible because vertices of X are choosen arbitrarily at the beginning.) Since $a \in X$, $a \notin Y$. Then $c\big(G - ((S \cup Y) - \{v_0\})\big) = c\big(G - (S \cup Y)\big) = |T|$ for $av_0 \notin E(G)$. As $|S| + |Y| = 2c\big(G - (S \cup Y)\big)$, $2c\big(G - ((S \cup Y) - \{v_0\})\big) = 2c\big(G - (S \cup Y)\big) = |S \cup Y| > |(S \cup Y) - \{v_0\}|$, contradicting the 2-toughness of G. This completes the proof. $\square$

Now we are ready to state and prove our main theorems. The first result shows the existence of f-factors including two edges.

Theorem 3.1. Let G be a 2-tough graph on at least five vertices and let f be a function with $f : V(G) \to \{1, 2\}$ such that $f\big(V(G)\big) \equiv 0 \pmod 2$. If $e_1 = a_1b_1, e_2 = a_2b_2$ are two edges of G with $f(a_i) = 2$ and $f(b_i) = 2$ $(i = 1, 2)$, then G has an f-factor containing e_1 and e_2.

Proof. If G is a complete graph on at least five vertices, the assertion is trivial. Now, we may assume that G is non-complete. For convenience of applying Lemma 2.1, we consider the operations on e_1, e_2 consecutively.

Let G_1 be the graph obtained from G by subdivision of e_1 (inserting a new vertex u_1 on the edge e_1), and let G_2 be the graph obtained from G_1 by subdivision of e_2 (inserting a new vertex u_2 on the edge e_2). Define two functions $f_1 : V(G_1) \to \{1, 2\}, f_2 : V(G_2) \to \{1, 2\}$ as follows:

$$f_1(v) = \begin{cases} 2, & \text{if } v = u_1; \\ f(v), & \text{if } v \in V(G); \end{cases}$$

and

$$f_2(v) = \begin{cases} 2, & \text{if } v = u_2; \\ f_1(v), & \text{if } v \in V(G_1). \end{cases}$$

Then G has an f-factor containing e_1 and e_2 if and only if G_2 contains an f_2-factor.

Suppose that G_2 contains no f_2-factors. Then, by Lemma 3.1, there exist $S, T \subseteq V(G)$, $S_i, T_i \subseteq V(G_i)$ $(i = 1, 2)$ satisfying $S = S_1 = S_2, T = T_1 - \{u_1\}, T_1 = T_2 - \{u_2\}$ and

$$2 - 2(\varepsilon_1 + \varepsilon_2) \leq q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - f(S), \tag{3.18}$$

where r is the number of components of $G - (S \cup T)$ which are joined to T, $\varepsilon_1 + \varepsilon_2 \geq 1$ and T is an independent set of G with $f(x) = 2$ for all $x \in T$.

If $\varepsilon_1 + \varepsilon_2 = 1$, i.e., either $\varepsilon_1 = 1, \varepsilon_2 = 0$ or $\varepsilon_1 = 0, \varepsilon_2 = 1$, then by Lemma 2.1, there exists exactly one edge e_i ($i = 1$ or 2) whose location together with u_i in (G_i, S_i, T_i) is of type I, or II, or III. Since $q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - f(S) \leq 0$, the inequality in (3.18) becomes an equality, and then $f(S) = |S|$, i.e., $f(x) = 1$ for all $x \in S$. Now $\{a_1, a_2, b_1, b_2\} \cap S = \emptyset$. So, the location cannot be of type I or II and it must be of type III. As $q_G(S, T; f) = c\big(G - (S \cup T)\big) = r$, $G - (S \cup T)$ has no even components. But a location of type III requires an even component, so the operation on the edge which is located as in type III is conducted in step two. That is, the locations of a_2, b_2, u_2 in (G_2, S_2, T_2) are of type III. The first step (i.e., subdivision of e_1) produces an even component required in step two. As $a_1, b_1 \notin S$, we deduce that $u_1 \in T_1$ and $e_1 \in E_G\big(V(C_{\text{odd}}), V(C_{\text{even}})\big)$, where C_{odd} is an odd component of $G_1 - (S_1 \cup T_1)$ and C_{even} is an even component of $G_1 - (S_1 \cup T_1)$. Moreover, C_{even} is the very component to which e_2 belongs. That is, $e_2 \in E(C_{\text{even}})$.

Since $e_1 \in E_G\big(V(C_{\text{odd}}), V(C_{\text{even}})\big)$, $C'_0 = C_{\text{odd}} \cup C_{\text{even}} \cup \{e_1\}$ corresponds to an odd component of $G - (S \cup T)$ and e_1 is a cut edge of C'_0. As $e_1, e_2 \in E(C'_0)$, so $|V(C'_0)| > 2$, a contradiction to Lemma 3.1 (2).

If $\varepsilon_1 + \varepsilon_2 = 2$, then a_i, b_i, u_i in (G_i, S_i, T_i) $(i = 1, 2)$ are located as in type I, or II, or III. Note that the operation on an edge $e = ab$ (i.e., subdividing $e = ab$ by a vertex u) does not produce an even component in the new graph when vertices a, b, u are located as in type I, or II, or III; and the original graph contains at least one even component when a, b, u are located as in type II or III.

(1) If both locations are of type I, then $\{a_1, a_2, b_1, b_2\} \subseteq S$. Since $f(a_i) = f(b_i) = 2$ for $i = 1, 2$, so we have

$$q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - \sum_{x \in S} f(x) \leq -3,$$

a contradiction to (3.18).

(2) If two locations are of types I and II (or types I and III), respectively, then there exists exactly one edge e_i ($i = 1$ or 2) lying in $G[S]$ and thus $|S| - f(S) \leq -2$. Assume $e_1 \in E(G[S])$, then

$c\big(G-(S\cup T)\big)=c\big(G_1-(S_1\cup T_1)\big)$ and $q_G(S,T;f)=q_{G_1}(S_1,T_1;f_1)$. Since the location of type II (or type III) requires an even component in $G_1-(S_1\cup T_1)$, we have $q_G(S,T;f)-c\big(G-(S\cup T)\big)=q_{G_1}(S_1,T_1;f_1)-c\big(G_1-(S_1\cup T_1)\big)\le -1$. Therefore,

$$q_G(S,T;f)+r-2c\big(G-(S\cup T)\big)+|S|-\sum_{x\in S}f(x)\le -1-2=-3,$$

a contradiction to (3.18).

The case of $e_2\in E(G[S])$ can be discussed similarly.

(3) If both locations are of type II, then $G-(S\cup T)$ has at least two even components and $\{a_1,a_2,b_1,b_2\}\cap S\ne\emptyset$. Therefore,

$$q_G(S,T;f)+r-2c\big(G-(S\cup T)\big)+|S|-\sum_{x\in S}f(x)\le -2-1=-3,$$

a contradiction to (3.18).

(4) If two locations are of types II and III, respectively, then $G-(S\cup T)$ has at least two even components and $\{a_1,a_2,b_1,b_2\}\cap S\ne\emptyset$. We obtain a contradiction similarly as in (3).

(5) If both locations are of type III, note that each of the two operations requires an even component as type III requires, so $G-(S\cup T)$ contains at least two even components and thus $q_G(S,T;f)-c(G-S-T)\le -2$. If $q_G(S,T;f)-c\big(G-(S\cup T)\big)<-2$ or $r<c\big(G-(S\cup T)\big)$, then

$$q_G(S,T;f)+r-2c\big(G-(S\cup T)\big)+|S|-\sum_{x\in S}f(x)\le -3,$$

a contradiction to (3.18). Finally, we may assume $q_G(S,T;f)+2=c\big(G-(S\cup T)\big)=r$. Again, the inequality (3.18) becomes an equality and $f(x)=1$ for any $x\in S$. Suppose $e_1\in E(C_0')$. Since the locations of a_1,b_1,u_1 in (G_1,S_1,T_1) are of type III, $e_1=a_1b_1$ is a cut edge of C_0', where C_0' is an even component of $G-(S\cup T)$ and $V(C_0')$ induces two odd components in $G_1-(S_1\cup T_1)$. Since $f(a_1)=f(b_1)=2$, there exists a vertex $v\in V(C_0')$ distinct from a_1 and b_1, or $|V(C_0')|\ge 3$. But by Lemma 3.1 (2), $|V(C_0')|=2$, a contradiction. □

The next theorem shows the existence of f-factors excluding two edges under the condition of 2-toughness.

Theorem 3.2. Let G be a 2-tough graph on at least five vertices, and f a function with $f:V(G)\to\{1,2\}$ and $f\big(V(G)\big)\equiv 0 \pmod 2$. If $e=a_1b_1, e=a_2b_2$ are two distinct edges of G with $f(a_i)=f(b_i)=1$ $(i=1,2)$ or $f(a_1)=f(a_2)=1$, $f(b_i)\in\{1,2\}$ and $f(b_1)\ne f(b_2)$, then G contains an f-factor excluding e_1 and e_2.

Proof. Similarly, we only need to consider the case that G is non-complete. For convenience of applying Lemma 2.2, we consider the operations on e_1,e_2 consecutively.

Let $G_1=G-\{e_1\}$ and $G_2=G_1-\{e_2\}$. Suppose that G_2 contains no f-factors. Since $f(a_i)=1$ $(i=1,2)$, then by Lemma 3.1, there exist $S,T\subseteq V(G)$ and $S_i,T_i\subseteq V(G_i)$ $(i=1,2)$ such that $S=S_1=S_2, T=T_1=T_2$ and

$$2-2(\varepsilon_1+\varepsilon_2)\le q_G(S,T;f)+r-2c\big(G-(S\cup T)\big)+|S|-f(S), \tag{3.19}$$

where r denotes the number of components of $G-(S\cup T)$ which are joined to T, $\varepsilon_1+\varepsilon_2\ge 1$, and T is independent in G with $f(x)=2$ for all $x\in T$.

There are two cases to consider according to the values of $\varepsilon_1+\varepsilon_2$. Note that the operation on an edge $e=ab$ (i.e., deleting e) does not produce an even component in the new graph when vertices a, b are located as in type IV, or V, or VI; and the original graph contains at least one even component when a, b are located as in type V or VI.

If $\varepsilon_1 + \varepsilon_2 = 1$, then $2 - 2(\varepsilon_1 + \varepsilon_2) = q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - f(S) = 0$ and there exists exactly one edge e_i ($i = 1$ or 2) whose location is of type V or VI. Here no edge of e_i is located as in type IV for $f(a_1) = f(a_2) = 1$. Since $q_G(S, T; f) = c\big(G - (S \cup T)\big) = r$, $G - (S \cup T)$ contains no even components. But type V or VI requires an even component. So the location of e_2 is of type V or VI; and the first step (i.e., deletion of e_1) produces an even component which the operation on e_2 requires. It is not hard to see that either $e_1 \in E_G\big(V(C_{\text{odd}}), V(C_{\text{even}})\big)$ or $e_1 \in E_G\big(V(C_{\text{even}}), T\big)$, where C_{odd} (resp. C_{even}) is an odd (resp. even) component of $G_1 - (S_1 \cup T_1)$. Moreover, C_{even} is the very component that operation on e_2 requires, and hence $a_2 \in C_{\text{even}}$.
If $e_1 \in E_G\big(V(C_{\text{odd}}), V(C_{\text{even}})\big)$, then $C'_0 = C_{\text{odd}} \cup C_{\text{even}} \cup \{e_1\}$ is an odd component of $G - (S \cup T)$ and e_1 is a cut edge of C'_0. If $e_1 \in E_G\big(V(C_{\text{even}}), T\big)$, then $f(b_1) = 2$ and $C'_0 = C_{\text{even}}$ is an odd component of $G - (S \cup T)$. As $f(b_2) \neq f(b_1)$, $b_2 \in V(C'_0)$ and the location of e_2 must be of type VI. Hence e_2 is a cut edge of C'_0. In both cases, C'_0 contains a cut edge and $|V(C'_0)| \geq 3$, a contradiction to Lemma 3.1 (2).

Next consider the case $\varepsilon_1 + \varepsilon_2 = 2$. If the location of $e_i = a_i b_i$ is of type IV, then $a_i, b_i \in T$ and thus $f(a_i) = f(b_i) = 2$, which is impossible. Thus the locations of both e_1 and e_2 are of types V or VI. Furthermore, operation on the edge which is located as in type V happens at most once, because $f(b_1) \neq f(b_2)$. Since both types V and VI require even components, the operations on e_1 and e_2 require even components. Then $G - (S \cup T)$ has at least two even components. Thus $q_G(S, T; f) - c\big(G - (S \cup T)\big) \leq -2$. If $q_G(S, T; f) - c\big(G - (S \cup T)\big) < -2$ or $r < c\big(G - (S \cup T)\big)$, then

$$q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - \sum_{x \in S} f(x) \leq -3,$$

a contradiction to (3.19). Suppose $q_G(S, T; f) + 2 = c(G - S - T) = r$. Similar to the discussion of case $\varepsilon_1 + \varepsilon_2 = 1$ above, the inequality (3.19) becomes an equality. Without loss of generality, assume that the location of e_1 is of type VI and $f(b_1) = 1$. Suppose $e_1 \in E(C'_0)$, where C'_0 is an even component of $G - (S \cup T)$. So $V(C'_0)$ induces two odd components of $G_1 - (S_1 \cup T_1)$ and e_1 is a cut edge of C'_0. Since $f(a_1) = f(b_1) = 1$, there is a vertex $v \in V(C'_0)$ distinct from a_1 and b_1, or $|V(C'_0)| \geq 3$, a contradiction to Lemma 3.1 (2). □

Remark 1. The condition that $f(a_i) = 1$ ($i = 1, 2$) and at least one of $f(b_1)$ and $f(b_2)$ equal to 1 in Theorem 3.2 is best possible, as there exists a class of 2-tough graphs in which after deletion of e_1 and e_2 the resulting graph does not have f-factors, when at least one end of e_i ($i = 1, 2$) equal to 2. For example, in Fig. 4, let $G = S \cup T \cup \{x, y\}$, where T is an independent set, $|S| = 2|T|$ with every vertex $s \in S$ being adjacent to every vertex of $V(G) - s$ and $ax, by \in E(G)$, and let $f(t) = 2$ for all $t \in T$, $f(s) = 1$ for all $s \in S$ and $f(x) = f(y) = 1$. Then G contains no f-factors excluding edges ax, by. For another example, in Fig. 5, $G = S \cup T \cup \{x, y\}$, where T is an independent set, $|S| = 2|T| - 1$ with every vertex $s \in S$ being adjacent to all vertices of $G - s$ and $ax, ay, by \in E(G)$, and let $f(s) = 1$ for every vertex $s \in S$, and for every vertex $t \in T$, $f(t) = 2$, and $f(x) = 1, f(y) = 2$. Of course, G contains no f-factors excluding ax and by.

Our final result deals with the existence of f-factors including an edge and excluding another edge.
Theorem 3.3. Let G be a 2-tough graph on at least five vertices, and f a function with $f : V(G) \to \{1, 2\}$ and $f\big(V(G)\big) \equiv 0 \pmod 2$. If $e = a_1 b_1, e = a_2 b_2$ are two edges of G with $f(a_2) = f(b_2) = 2$ and at least one of $\{f(a_1), f(b_1)\}$ is equal to 1, then G has an f-factor F excluding e_1 and including e_2.

Proof. The assertion clearly holds for the case that G is a non-complete graph on at least five vertices. From now on, we consider the case that G is non-complete. For convenience of applying Lemmas 2.1 and 2.2, we consider the operations on e_1 and e_2 consecutively.

Let G_1 be the graph obtained from G by deletion of e_1, and let G_2 be the graph obtained from G_1 by subdivision of e_2 (inserting a new vertex u_2 on the edge e_2). Define a function $f_2 : V(G_2) \to \{1, 2\}$

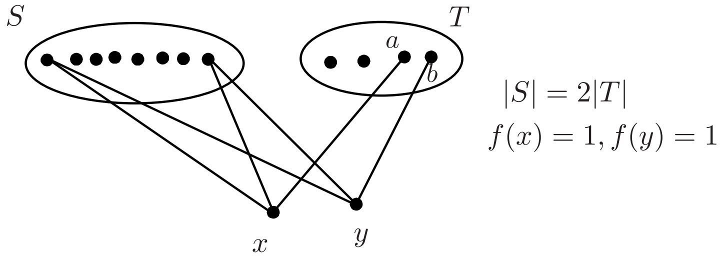


Fig. 4: A graph has no $\{1,2\}$-factors excluding edges ax, by

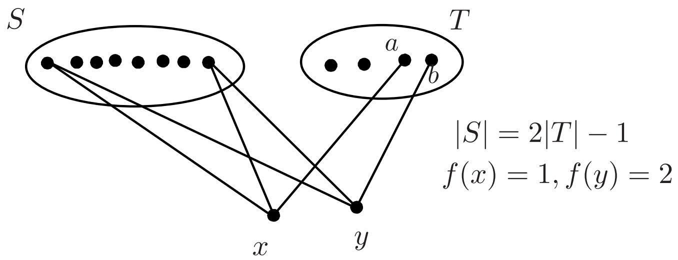


Fig. 5: A graph has no $\{1,2\}$-factors excluding edges ax, by

as follows:

$$f_2(v) = \begin{cases} 2, & \text{if } v = u_2; \\ f(v), & \text{if } v \in V(G_1) = V(G). \end{cases}$$

Then G has an f-factor containing e_2 and excluding e_1 if and only if G_2 contains an f_2-factor.

Suppose that G_2 contains no f_2-factors, then by Lemma 3.1, there exist $S, T \subseteq V(G)$ and $S_i, T_i \subseteq V(G_i)$ $(i = 1, 2)$ such that $S = S_1 = S_2, T = T_1 = T_2 - \{u_2\}$ and

$$2 - 2(\varepsilon_1 + \varepsilon_2) \le q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - f(S), \tag{3.20}$$

where r denotes the number of components of $G - (S \cup T)$ which are joined to T, $\varepsilon_1 + \varepsilon_2 \ge 1$ and T is independent in G with $f(x) = 2$ for all $x \in T$.

If $\varepsilon_1 + \varepsilon_2 = 1$, then the inequality (3.20) becomes an equality, and either the location of e_1 is of one of types IV–VI or the location of e_2 is of one of types I – III. Moreover $f(S) = |S|$, i.e., $f(x) = 1$ for all $x \in S$. So $\{a_2, b_2\} \cap S = \emptyset$. Since T is independent, $\{a_1, b_1\} \not\subseteq T$; so this location is not of type I, II, IV. As $q_G(S, T; f) = c\big(G - (S \cup T)\big) = r$, $G - (S \cup T)$ contains no even components. However the types V and VI require even components and both types can only occur in the step one (deletion of e_1 from G), so the location must be type III and it occurs in the step two. That is, the locations of a_2, b_2, u in (G_2, S_2, T_2) are of type III; and deletion of e_1 produces an even component that type III requires. So we see that either $e_1 \in E_G\big(V(C_{\text{odd}}), V(C_{\text{even}})\big)$ or $e_1 \in E_G\big(T, V(C_{\text{even}})\big)$, where C_{odd} is an odd component of $G_1 - (S_1 \cup T_1)$, and C_{even} is an even component of $G_1 - (S_1 \cup T_1)$. Moreover, in both cases, C_{even} is the very component operation on e_2 requires. In other words, $e_2 \in E(C_{\text{even}})$.

Now C_{even} is an even component of $G_1 - (S_1 \cup T_1)$, and $C_{\text{even}} - \{e_2\}$ corresponds to two odd components of $G_2 - (S_2 \cup T_2)$. Moreover, $C_0' = C_{\text{odd}} \cup C_{\text{even}} \cup \{e_1\}$ (or $C_0' = C_{\text{even}}$) is an odd component of $G - (S \cup T)$ and e_2 is a cut edge of C_0'. Since $|N_G(a_2) \cap T| = 1$ and $u_2 \in T_2$, we have $|N_{G_2}(a_2) \cap T_2| = 2$. As $f(a_2) = 2$, the odd component of $G_2 - (S_2 \cup T_2)$ that contains a_2 is not a singleton. Therefore $|V(C_0')| \geq 3$. But by Lemma 3.1 (2), $|V(C_0')| = 2$, a contradiction.

If $\varepsilon_1 + \varepsilon_2 = 2$, then the location of e_1 is of type V or VI, and the location of e_2 is of type I or II or III. As T is independent, type IV never occur.

As argued above, the operation on an edge $e = ab$ does not produce an even component in the new graph when the location of a, b is of one of types I–VI; and the original graph requires an even component when the locations of a, b are of type II, or III, or V, or VI.

(1) If the two locations are of types V and I respectively, then e_2 lies in $G[S]$ and thus $|S| - f(S) \leq -2$. Since deletion of e_1 requires an even component in $G - (S \cup T)$, $q_G(S, T; f) - c\big(G - (S \cup T)\big) \leq -1$. Therefore

$$q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - f(S) \leq -1 - 2 = -3,$$

a contradiction to (3.20).

(2) If the two locations are of types VI and I respectively, then $\{a_2, b_2\} \subseteq S$ and $G - (S \cup T)$ contains at least one even component as type VI requires. Therefore,

$$q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - \sum_{x \in S} f(x) \leq -1 - 2 = -3,$$

a contradiction to (3.20).

(3) If the two locations are of types II and V (or types II and VI) respectively, then $|\{a_2, b_2\} \cap S| = 1$. Since both locations of types II and V (or types II and VI) require even components but produce none, we have

$$q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - \sum_{x \in S} f(x) \leq -2 - 1 = -3,$$

a contradiction to (3.20).

(4) If the two locations are of types V and III (or types VI and III) respectively, then $G - (S \cup T)$ has at least two even components. Thus $q_G(S, T; f) - c\big(G - (S \cup T)\big) \leq -2$. If $q_G(S, T; f) - c\big(G - (S \cup T)\big) < -2$ or $r < c\big(G - (S \cup T)\big)$, then

$$q_G(S, T; f) + r - 2c\big(G - (S \cup T)\big) + |S| - \sum_{x \in S} f(x) \leq -3,$$

a contradiction to (3.20). Now we may assume $q_G(S, T; f) + 2 = c\big(G - (S \cup T)\big) = r$. As discussed in the case $\varepsilon_1 + \varepsilon_2 = 1$, the inequality (3.20) becomes an equality. Suppose $e_2 \in E(C_0')$. Then e_2 is a cut edge of C_0'. Note that deletion of e_1 produces no even component because the location of e_1 is of type V or type VI. It follows from the location of e_2 being of type III that C_0' is an even component of $G - (S \cup T)$ and $V(C_0')$ induces two odd components of $G_2 - (S_2 \cup T_2)$. Since $f(a_2) = f(b_2) = 2$, by parity argument, we see that there exists a vertex $v \in V(C_0')$ distinct from a_2 and b_2, or $|V(C_0')| \geq 3$. On the other hand, $|V(C_0')| = 2$ by Lemma 3.1 (2) and the fact that e_2 is a cut edge of C_0'.

This completes the proof. □

Remark 2. The condition that at least one of $f(a_1)$ and $f(b_1)$ is equal to 1 in Theorem 3.3 is necessary. For instance, let G be a graph with vertex-set $S \cup T \cup \{x, y\}$, where $|S| = 2|T| - 2$ and $|T| \geq 4$, and $G[T]$ has only one edge $e_1 = a_1 b_1$. Moreover, every vertex of S is adjacent to all other vertices of G, each of $\{x, y\}$ has only one neighbor in T, and their neighbors are distinct (see Fig. 6). Select an edge $e_2 = a_2 b_2$ from $G[S]$ and let $f(v) = 2$ for all $v \in T \cup \{x, y, a_2, b_2\}$, and $f(v) = 1$ for all $v \in S - \{a_2, b_2\}$. Then G contains no f-factors excluding e_1 and including e_2.

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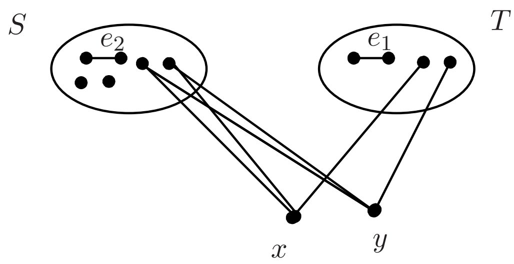

Fig. 6: A graph has no $\{1,2\}$-factors including e_2 and excluding e_1

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