Eigenvalues and $[1, n]$-odd factors

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Abstract

Amahashi [1] gave a sufficient and necessary condition for the existence of $[1, n]$-odd factor. In this paper, for the existence of $[1, n]$-odd factors, we obtain some sufficient conditions in terms of eigenvalues. Moreover, we construct some examples which show that those results are best possible.

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1 Introduction

Throughout this paper, let G denote a simple graph of *order* v (the number of vertices) and *size* e (the number of edges). The eigenvalues of G are the eigenvalues λ_i of its adjacency matrix A, indexed so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_v$. If G is k-regular, then it is easy to see that $\lambda_1 = k$ and also, $\lambda_2 < k$ if and only if G is connected. Recall that the Laplacian matrix L, is related to the adjacency matrix A by $L = D - A$, where D is the diagonal matrix of the vertex degrees. The Laplacian matrix L is positive semi-definite with row sum 0. Its eigenvalues will be denoted by $0 = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_v$. For k-regular graphs, we have $\lambda_i + \mu_i = k$ for all $1 \leq i \leq v$.

We use [8] for terminologies and notations not defined here.

Let G be a graph. For two disjoint subsets S, T of $V(G)$, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other in T, and $o(G - S)$ to denote

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the number of components with odd number of vertices in $G - S$. Let $\overline{G}$ denote the complement of a graph G.

Given an odd integer-valued function $f : V(G) \to \{1, 3, 5, \ldots, \}$, a spanning subgraph F of G is called a $(1, f)$-*odd factor* if

$$d_F(x) \in \{1, 3, 5, \ldots, f(x)\} \text{ for all } x \in V(F).$$

Of course, if $f(x) = 1$ for all vertices x, then a $(1, f)$-odd factor is a *1-factor*, i.e., a *perfect matching*. For an odd integer $n \geq 1$, if $f(x) = n$ for all $x \in V(G)$, then a $(1, f)$-odd factor is called a $[1, n]$-*odd factor*. So, a $[1, n]$-odd factor F satisfies

$$d_F(x) \in \{1, 3, 5, \ldots, n\} \text{ for all } x \in V(F).$$

In [2], Brouwer and Haemers gave sufficient conditions for the existence of a 1-factor in a graph in terms of its Laplacian eigenvalues and, for a regular graph, gave an improvement in terms of the third largest adjacency eigenvalue, λ_3. Cioabă and Gregory [3] also studied relations between 1-factors and eigenvalues in regular graphs. Later, Cioabă, Gregory and Haemers [4] found a best upper bound on λ_3 that is sufficient to guarantee that a regular graph G of order v has a 1-factor when v is even, and a matching of order $v - 1$ when v is odd. Motivated by these results, in this paper, we relate the eigenvalues of a connected graph G to the existence of a $[1, n]$-odd factor. We give a sufficient condition in terms of Laplacian eigenvalues for the existence of $[1, n]$-odd factors of graphs, as well as sufficient conditions in terms of eigenvalues for the existence of $[1, n]$-odd factors of regular graphs.

The main tool in our proofs is the following theorem given by Amahashi [1]. It is a sufficient and necessary condition of $[1, n]$-odd factors in a multigraph. Here, a multigraph is a graph that has no loops but may have multiple edges.

Theorem 1.1 (Amahashi [1]) *Let G be a multigraph and $n \geq 1$ be an odd integer. Then G has a $[1, n]$-odd factor if and only if*

$$o(G - S) \leq n|S| \text{ for all } S \subseteq V(G).$$

The set S in Theorem 1.1 may be taken to be empty. The theorem then implies the obvious necessary condition that each component of G have an even number of vertices. It is interesting to note that by taking n sufficiently large, the theorem implies an exercise in [7] which states that a graph with no odd components must contain a spanning subgraph whose vertex degrees are all odd.

2 Graphs

In this section, we investigate the relationship between the Laplacian eigenvalues of a graph G and its $[1, n]$-odd factors. For graphs, we will use an inequality for disconnected vertex sets in graphs, due to Haemers [5].

Two disjoint vertex sets A and B in a graph are called *disconnected* if there are no edges between A and B.

Lemma 2.1 (Haemers, [5]) *If A and B are disconnected vertex sets of a graph with v vertices and Laplacian eigenvalues $0 = \mu_1 \leq \ldots \leq \mu_v$, then*

$$\frac{|A| \cdot |B|}{(v - |A|)(v - |B|)} \leq (\frac{\mu_v - \mu_2}{\mu_v + \mu_2})^2.$$

For 1-factors, Brouwer and Haemers proved that:

Theorem 2.2 (Brouwer and Haemers, [2]) *Let G be a graph with v vertices, and Laplacian eigenvalues $0 = \mu_1 \leq \ldots \leq \mu_v$. If v is even and $\mu_v \leq 2\mu_2$, G has a 1-factor.*

Brouwer and Haemers also gave a technical lemma in the proof of Theorem 2.2.

Lemma 2.3 (Brouwer and Haemers, [2]) *Let $x_1, \ldots, x_n$ be n positive integers such that $\sum_{i=1}^{n} x_i = k \leq 2n - 1$. Then for every integer l, satisfying $0 \leq l \leq k$, there exists a set $I \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in I} x_i = l$.*

We generalize the theorem above to $[1, n]$-odd factors, we have the following theorem. From now on, n will always be assumed to be a positive odd integer.

Theorem 2.4 *Let G be a graph with v vertices, and Laplacian eigenvalues $0 = \mu_1 \leq \ldots \leq \mu_v$. If v is even and $\mu_v \leq (n+1)\mu_2$, G has a $[1, n]$-odd factor.*

Proof. Assume $G = (V, E)$ has no $[1, n]$-odd factor. By Theorem 1.1, there exists an s-vertex-set $S \subset V$, such that $q = o(G - S) > ns$. Since v is even, q and ns have the same parity, hence $q \geq ns + 2$. Then $v \geq (n + 1)s + 2$. There are two cases to consider.

Case 1. $v \leq 2ns + s + 3$.

Since $q = o(G - S) \geq ns + 2$, and $|V(G - S)| = v - s \leq 2ns + 3 < 2q$, it follows from Lemma 2.3 that there exists a pair of disconnected vertex sets A and B with $|A| = \lfloor \frac{v-s}{2} \rfloor$ and $|B| = \lceil \frac{v-s}{2} \rceil$. By Lemma 2.1, we have

$$(\frac{\mu_v - \mu_2}{\mu_v + \mu_2})^2 \geq \frac{|A| \cdot |B|}{vs + |A| \cdot |B|} \geq \frac{(v - s)^2 - 1}{(v + s)^2 - 1}.$$

Since $g(v) = \frac{(v-s)^2-1}{(v+s)^2-1}$ is an increasing function of v on $[(n + 1)s + 2, 2ns + s + 3]$, it follows that

$$(\frac{\mu_v - \mu_2}{\mu_v + \mu_2})^2 \geq g(v) \geq \frac{(ns + s + 2 - s)^2 - 1}{(ns + s + 2 + s)^2 - 1} > (\frac{n}{n + 2})^2.$$

Therefore, $(n+1)\mu_2 < \mu_v$, a contradiction.

Case 2. $v \geq 2ns + s + 4$.

We claim that G must have a pair of disconnected vertex sets A and B with $|A| + |B| = v - s$ and $\min\{|A|, |B|\} \geq ns + 1$.

If $q \geq 2ns + 2$, let A be a union of $ns + 1$ odd components of $G - S$ and B be the complement of A in the vertex set $G - S$, then $\min\{|A|, |B|\} \geq \min\{ns + 1, (2ns + 2) - (ns + 1)\} = ns + 1$. Thus in addition to the previous observation that $q \geq ns + 2$, we may assume that $q \leq 2ns + 1$.

Let $V_1, \ldots, V_{q-1}$ be the vertex sets of $q - 1$ of the q odd components of $G - S$, and let $V_q = V(G - S) - \bigcup_{i=1}^{q-1} V_i$. If the $V_1', \ldots, V_q'$ are nonempty subsets of $V_1, \ldots, V_q$, then

$$q \leq \sum_{i=1}^{q} |V_i'| \leq \sum_{i=1}^{q} |V_i| = v - s.$$

Since $q \leq 2ns + 1$ and $v - s \geq 2ns + 4$, the subset V_i' may be chosen such that $\sum_{i=1}^{q} |V_i'| = 2ns + 3$. As $2q - 1 \geq 2(ns + 2) - 1 = 2ns + 3$, it follows from Lemma 2.3 that there is a subset $I \subseteq [q] = \{1, \ldots, q\}$ such that $\sum_{i \in I} |V_i'| = ns + 1$. Let $J = [q] - I$, we have $\sum_{i \in J} |V_i'| = (2ns + 3) - (ns + 1) > ns + 1$. Therefore, $A = \bigcup_{i \in I} V_i$ and $B = \bigcup_{i \in J} V_i$ are disconnected vertex sets with $|A| + |B| = v - s$ and $\min\{|A|, |B|\} \geq ns + 1$.

So $|A| \cdot |B| \geq (ns + 1)(v - s - ns - 1)$. Then Lemma 2.1 implies

$$\left(\frac{\mu_v - \mu_2}{\mu_v + \mu_2}\right)^2 \geq \frac{|A| \cdot |B|}{vs + |A| \cdot |B|} \geq 1 - \frac{vs}{vs + (ns + 1)(v - s - ns - 1)}.$$

Let

$$f(s) = \frac{vs + (ns + 1)(v - s - ns - 1)}{vs}.$$

By use of $v \geq (2n + 1)s + 4$, we have

$$\begin{aligned} f(s) &\geq 1 + (n + \tfrac{1}{s})(1 - \tfrac{(n+1)s+1}{(2n+1)s+4}) \\ &= 1 + \tfrac{(ns+1)(ns+3)}{(2n+1)s^2+4s} \\ &> 1 + \tfrac{n^2}{2n+1} \\ &= \tfrac{(n+1)^2}{2n+1}. \end{aligned}$$

Thus

$$\left(\frac{\mu_v - \mu_2}{\mu_v + \mu_2}\right)^2 \geq 1 - \frac{1}{f(s)} > \left(\frac{n}{n+1}\right)^2 > \left(\frac{n}{n+2}\right)^2,$$

and hence $(n+1)\mu_2 < \mu_v$, which is a contradiction. ∎

Remark. Theorem 2.4 is sharp. Consider a bipartite graph $K_{a,b}$ with $b > a$. Its Laplacian eigenvalues are $\mu_1 = 0, \mu_2 = \cdots = \mu_b = a, \mu_{b+1} = \cdots = \mu_{v-1} = b, \mu_v = a + b$. When $b = an$, $\mu_v = (n+1)\mu_2$ and $K_{a,an}$ has a $[1, n]$-odd factor. When $b > an$, $\mu_v > (n+1)\mu_2$ and $K_{a,b}$ has no $[1, n]$-odd factor.

3 Regular graphs

For regular graphs, we improve the result in the previous section.

Lemma 3.1 (Cioabă and Gregory, [3]) *For every graph G,*

$$\lambda_1 - \frac{2e}{v} \geq \frac{(\Delta - \delta)^2}{4v\Delta}.$$

In particular, if $v \geq 4$ and $\delta \leq \Delta - 1$, then

$$\lambda_1 - \frac{2e}{v} > \frac{1}{v(\Delta + 2)}.$$

Brouwer, Haemers [2] and Cioabă, Gregory [3] studied the relationship between the existence of 1-factors of a regular graph and its eigenvalue λ_3. Similarly, we investigate the existence of $[1, n]$-odd factors in terms of λ_3, by use of Lemma 3.1. First we'd like to give the following result as a special case.

Theorem 3.2 *Let G be a connected k-regular graph of even order v, where k is even. If n is odd and $2n \geq k$, G has a $[1, n]$-odd factor.*

Proof. Suppose that G contains no $[1, n]$-odd factor. As in the proof of Theorem 2.4, there exists $S \subseteq V(G)$ with $|S| = s$ such that $G - S$ has $q \geq ns + 2$ components of odd order, say $G_1, \ldots, G_q$. Since k is even, $e_G(V(G_i), S) = k|V(G_i)| - \sum_{x \in V(G_i)} d_{G_i}(x)$ is even for $i = 1, \ldots, q$. Since G is k-regular, hence

$$k|S| \geq \sum_{i=1}^{q} e_G(V(G_i), S) \geq 2q \geq 2ns + 4 \geq k|S| + 4,$$

a contradiction. ∎

Theorem 3.3 *Let G be a connected k-regular graph of even order v, $k \geq 3$, and eigenvalues $k = \lambda_1 \geq \ldots \geq \lambda_v$. If one of the following conditions holds, G contains a $[1, n]$-odd factor:*

(1) *k is even, $\lceil \frac{k}{n} \rceil$ is even, and $\lambda_3 \leq k - \frac{\lceil \frac{k}{n} \rceil - 2}{k+1} + \frac{1}{(k+1)(k+2)}$;*

(2) *k is even, $\lceil \frac{k}{n} \rceil$ is odd, and $\lambda_3 \leq k - \frac{\lceil \frac{k}{n} \rceil - 1}{k+1} + \frac{1}{(k+1)(k+2)}$;*

(3) *k is odd, $\lceil \frac{k}{n} \rceil$ is even, and $\lambda_3 \leq k - \frac{\lceil \frac{k}{n} \rceil - 1}{k+2} + \frac{1}{(k+2)^2}$;*

(4) *k is odd,* $\lceil \frac{k}{n} \rceil$ *is odd, and* $\lambda_3 \le k - \frac{\lceil \frac{k}{n} \rceil - 2}{k+2} + \frac{1}{(k+2)^2}$.

Proof. Assume that G contains no $[1,n]$-odd factors. As seen earlier, because v is even, there exists $S \subseteq V(G)$ with $|S| = s$ such that $G - S$ has $q \ge ns + 2$ components of odd order, say $G_1, \dots, G_q$. For each subgraph G_i ($1 \le i \le q$), let t_i be the number of edges between $V(G_i)$ and S, and let v_i, e_i, respectively, be the order and the size of G_i.

We claim that there are at least three odd components, say G_1, G_2, G_3, satisfying $t_j < \lceil \frac{k}{n} \rceil$ for all $1 \le j \le 3$. Otherwise, $e_G(V(G-S), S) \ge \sum_{i=1}^{q} t_i \ge \lceil \frac{k}{n} \rceil (q-2) + 2 \ge \lceil \frac{k}{n} \rceil (n|S| + 2 - 2) + 2 > k|S| = \sum_{x \in S} d_G(x)$, a contradiction.

For each $1 \le i \le 3$, $t_i < \lceil \frac{k}{n} \rceil$. Since vertices in G_i are only adjacent to vertices in S or $V(G_i)$, we deduce that $2e_i = kv_i - t_i \ge kv_i - \lceil \frac{k}{n} \rceil + 1$ if k and $\lceil \frac{k}{n} \rceil$ are of different parities; and $2e_i = kv_i - t_i \ge kv_i - \lceil \frac{k}{n} \rceil + 2$ if k and $\lceil \frac{k}{n} \rceil$ are of the same parity. So,

$$\frac{2e_i}{v_i} \ge \begin{cases} k - \frac{\lceil \frac{k}{n} \rceil - 1}{v_i} & \text{if } k, \lceil \frac{k}{n} \rceil \text{ are of different parities;} \\ k - \frac{\lceil \frac{k}{n} \rceil - 2}{v_i} & \text{if } k, \lceil \frac{k}{n} \rceil \text{ are of the same parity.} \end{cases}$$

Note that $\lceil \frac{k}{n} \rceil \ge 2$. Otherwise $n \ge k$, so G is itself a $[1,n]$-odd factor if k is odd and, by Theorem 3.2, contains a $[1,n]$-odd factor if k is even. This contradicts our assumption at the beginning. Also, $v_i(v_i - 1) \ge 2e_i \ge kv_i - \lceil \frac{k}{n} \rceil + 1 \ge kv_i - k + 1$. Then $v_i \ge k+1$ if k is even and $v_i \ge k+2$ if k is odd.

According to the parity of k and $\lceil \frac{k}{n} \rceil$, there are four cases together. Here, we only argue about the case that k is even and $\lceil \frac{k}{n} \rceil$ is even. Other cases can be dealt with along the same line. Since $k \equiv 0 \pmod 2$, then $\lceil \frac{k}{n} \rceil > 2$; otherwise, G contains a $[1,n]$-odd factor by Theorem 3.2. By Lemma 3.1,

$$\lambda_1(G_i) > \frac{2e_i}{v_i} + \frac{1}{v_i(\Delta+2)} \ge k - \frac{\lceil \frac{k}{n} \rceil - 2}{v_i} + \frac{1}{v_i(\Delta+2)} \ge k - \frac{\lceil \frac{k}{n} \rceil - 2}{k+1} + \frac{1}{(k+1)(k+2)}.$$

It follows from interlacing theorem [6], that

$$\lambda_3(G) \ge \lambda_3(G_1 \cup G_2 \cup G_3) \ge \min_{1 \le i \le 3} \lambda_1(G_i) > k - \frac{\lceil \frac{k}{n} \rceil - 2}{k+1} + \frac{1}{(k+1)(k+2)},$$

a contradiction. This completes the proof. ∎

Remark. Let k be an odd integer and n an integer with $k = an + b$, where $a \ge 4$ is even and $0 < b < n$. Let $H = \overline{M_{(k-a+3)/2} \vee C_{a-1}}$, where $M_{(k-a+3)/2}$ denotes a 1-factor on $k - a + 3$ vertices, and the *join* $H_1 \vee H_2$ denotes the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1 \vee H_2) = E(H_1) \cup E(H_2) \cup \{xy : x \in V(H_1), y \in V(H_2)\}$. Take k copies of H, add an $(a-1)$-vertex-set S and join each vertex of S to a vertex of degree $k-1$ in each H. Then we obtain a new graph G on $k^2 + 2k + a - 1$

vertices. G is k-regular and has no $[1,n]$-odd factors, for $|V(H)| = k+2 \equiv 1 \pmod 2$ and $o(G-S) = k = an+b > n(a-1) = n|S|$. Moreover,

$$\begin{aligned}\lambda_3(G) \geq \lambda_1(H) &= \frac{1}{2}(k-3+\sqrt{(k+3)^2-4(a-1)}) \\ &= k - \frac{a-1}{k+2} + \frac{1}{(k+2)^2} + O(k^{-2}).\end{aligned}$$

It implies that there exist k-regular graphs with no $[1,n]$-odd factor for k and $\lceil \frac{k}{n} \rceil$ odd, even if λ_3 is arbitrarily close to the value given in Theorem 3.3. The upper bound of λ_3 given in Theorem 3.3 is best possible up to order $O(k^{-2})$. Similarly, we can construct graphs for other cases.

In fact, we can restrict on studying a more general eigenvalue λ rather than λ_3. Thus, we obtain two results as follows.

Theorem 3.4 *Let G be a connected k-regular graph of even order v with $k \equiv 0 \pmod 4$. If $\lambda_k \leq k - \frac{2}{k+1} + \frac{1}{(k+1)(k+2)}$, G has a $[1,n]$-odd factor for $n = \frac{k}{2} - 1$.*

Proof. Assume that G has no $[1,n]$-odd factors. As seen earlier, because v is even, there exists $S \subseteq V(G)$ with $|S| = s$ such that $G - S$ has $q \geq ns + 2$ components of odd order, say $G_1, \ldots, G_q$. Let t_i denote the number of edges in G between S and $V(G_i)$, and let v_i and e_i be the number of vertices and edges of G_i, respectively. Because vertices in G_i are adjacent only to vertices in G_i or S, we deduce that $2e_i = kv_i - t_i = k(v_i - 1) + k - t_i$. Since v_i is odd and $k = 2n+2$ is even, it is easy to see t_i is even. That is, $t_i \geq 2$ is even.

The sum of the degrees of the vertices in S is at least the number of edges between S and $\cup_{i=1}^{q} V(G_i)$. Then clearly $ks \geq \sum_{i=1}^{q} t_i$. If $s = 1$, we have $k \geq \sum_{i=1}^{n+2} t_i \geq 2(n+2) > k$ by $t_i \geq 2$, a contradiction. So $s \geq 2$. Suppose that $t_1 \leq t_2 \leq \cdots \leq t_q$.

Claim. $t_{2n+2} \leq 2$.

Otherwise, suppose that $t_{2n+2} > 2$. Since t_i is even, so $t_{2n+2} \geq 4$. Then

$$\begin{aligned}\sum_{i=1}^{q} t_i &= \sum_{i=1}^{2n+1} t_i + \sum_{i=2n+2}^{q} t_i \\ &\geq 2(2n+1) + 4(ns+2-(2n+1)) \\ &= 4ns - 4n + 6 > (2n+2)s = ks,\end{aligned}$$

a contradiction. This completes the claim.

For $1 \leq i \leq 2n+2$, $t_i = 2$. Since $v_i(v_i - 1) \geq 2e_i = kv_i - t_i = kv_i - 2$, then $v_i \geq k + 1 - \frac{2}{v_i}$. Hence, $v_i \geq k+1$ and the average degree $\overline{d_i}$ of G_i satisfies $\overline{d_i} = \frac{2e_i}{v_i} = k - \frac{2}{v_i}$.

Let l_i denote the largest eigenvalue of G_i for $i \in \{1, 2, \ldots, 2n+2\}$. Suppose $l_1 \geq l_2 \geq \cdots \geq l_{2n+2}$. Then, by interlacing in $G_1 \cup \cdots \cup G_{2n+2}$, it follows that $\lambda_{2n+2} \geq l_{2n+2}$.

Thus, according to Lemma 3.1, $\lambda_{2n+2} \geq l_{2n+2} > \overline{d_{2n+2}} + \frac{1}{v_{2n+2}(k+2)} \geq k - \frac{2}{k+1} + \frac{1}{(k+1)(k+2)}$. This is a contradiction. ∎

Remark. Let $k = 2n+2$ and $H = \overline{K_2} \vee K_{k-1}$. Take k copies of H. Add a two-vertex-set S and join each vertex of S to a vertex of degree $k-1$ in each H. This is a connected k-regular graph denoted by G. As H is of odd order, $o(G-S) = k = 2n+2 > 2n = n|S|$ and then G has no $[1,n]$-odd factors. Moreover,

$$\lambda_{2n+2}(G) \geq \lambda_1(H) = \frac{1}{2}(k-2+\sqrt{(k+2)^2-8}) = k - \frac{2}{k+1} + \frac{1}{(k+1)(k+2)} + O(k^{-2}).$$

So the bound of Theorem 3.4 is sharp up to $O(k^{-2})$.

For k even and $k \geq 2n+4$ or k odd, we obtain the following result similar to Theorem 3.4.

Theorem 3.5 *Let G be a connected k-regular graph of even order v, and eigenvalues $k = \lambda_1 \geq \ldots \geq \lambda_v$. If one of the following conditions holds, G contains a $[1,n]$-odd factor:*

(1) *when k is even, $n \geq 3$ and $k \geq 2n+4$, $\lambda_{n+2} \leq k - 1 + \frac{2n+3}{k+1} + \frac{1}{(k+1)(k+2)}$;*

(2) *when k is odd, $\lambda_{n+2} \leq k - 1 + \frac{n+3}{k+2} + \frac{1}{(k+2)^2}$.*

Proof. When k is odd, G has a $[1,n]$-odd factor if $k \leq n$ or, by Theorem 3.3 (4), if $n = 1$. Thus, in part (2) it may be assumed that $n \geq 2$ and $k \geq n+2$.

Assume that G has no $[1,n]$-odd factors. As seen earlier, because v is even, there exists S with $|S| = s$ such that $G - S$ has $q \geq ns+2$ components of odd order, say $G_1, \ldots, G_q$. Let t_i denote the number of edges in G between S and $V(G_i)$, and let v_i and e_i be the number of vertices and edges of G_i, respectively. Because vertices in G_i are adjacent only to vertices in G_i or S, we deduce that $2e_i = kv_i - t_i = k(v_i-1)+k-t_i$. Since v_i is odd, it is easy to see $k - t_i$ is even. That is, t_i has the same parity with k for each $i \in \{1, 2, \ldots, q\}$. Without loss of generality, we suppose $t_1 \leq \ldots \leq t_q$.

The sum of the degrees of the vertices in S is at least the number of edges between S and $\cup_{i=1}^{q} V(G_i)$. Then clearly $ks \geq \sum_{i=1}^{q} t_i$, $s \geq 1$ and $t_i \geq 1$. Hence $t_i < k$ for at least $(n-1)s+3$ values of i.

Claim. If k is odd, then $t_{n+2} \leq k-(n+1)$; else if k is even, then $t_{n+2} \leq k-(2n+2)$.

Conversely, suppose the claim doesn't hold. Firstly we consider that k is odd. Then we have $t_{n+2} \geq k - n + 1$. Note that $t_i \geq 1$. Then, since t_{n+2}, k and n are all odd, we have

$$\begin{aligned} ks &\geq \sum_{i=1}^{n+2} t_i + \sum_{i=n+3}^{q} t_i \\ &\geq k + 2 + (ns - n)(k - n + 1). \end{aligned}$$

If $s = 1$, then $k \geq k + 2$, a contradiction. So we say $s \geq 2$. Then we have $k(n-1) < n(n-1)$, so $k < n$, a contradiction. Now we consider that k is even. Since t_{n+2}, k and $2n$ are even, by assumption, we have $t_{n+2} \geq k - 2n$. Since $t_i \geq 2$ by parity, then

$$\begin{aligned} ks &\geq \sum_{i=1}^{n+2} t_i + \sum_{i=n+3}^{q} t_i \\ &\geq k + 2 + (ns - n)(k - 2n). \end{aligned}$$

If $s = 1$, clearly, we obtain a contradiction. So we say $s \geq 2$. Then we have $k > n(k - 2n)$ and $2n^2 > k(n-1)$. Note $k \geq 2n + 4$ and $n \geq 3$, a contradiction. This completes the claim.

Thus, the average degree $\overline{d_i}$ $(1 \leq i \leq n+2)$ of G_i satisfies the following inequality

$$\overline{d_i} = \frac{2e_i}{v_i} \geq \begin{cases} k - \frac{k-2n-2}{v_i} & \text{if } k \text{ is even,} \\ k - \frac{k-n-1}{v_i} & \text{if } k \text{ is odd.} \end{cases}$$

Let l_i denote the largest eigenvalue of G_i for $i \in \{1, 2, \ldots, n+2\}$. Suppose $l_1 \geq l_2 \geq \cdots \geq l_{n+2}$. Then, by interlacing in $G_1 \cup \cdots \cup G_{n+2}$, it follows that $\lambda_{n+2} \geq l_{n+2}$. Now, since

$$v_{n+2}(v_{n+2} - 1) \geq 2e_{n+2} = kv_{n+2} - t_{n+2} \geq \begin{cases} k(v_{n+2} - 1) + (2n+2) & \text{if } k \text{ is even,} \\ k(v_{n+2} - 1) + (n+1) & \text{if } k \text{ is odd,} \end{cases}$$

then $v_{n+2} \geq k+1$ if k is even and $v_{n+2} \geq k+2$ is k is odd, and hence by Lemma 3.1, we have

$$\lambda_{n+2} \geq l_{n+2} > \begin{cases} \overline{d_{n+2}} + \frac{1}{v_{n+2}(k+2)} \geq k - 1 + \frac{2n+3}{k+1} + \frac{1}{(k+1)(k+2)} & \text{if } k \text{ is even,} \\ \overline{d_{n+2}} + \frac{1}{v_{n+2}(k+2)} \geq k - 1 + \frac{n+3}{k+2} + \frac{1}{(k+2)^2} & \text{if } k \text{ is odd.} \end{cases}$$

This is a contradiction. ∎

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