

On the Rainbow k -Connectivity of Complete Graphs*

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Abstract

A path in an edge-colored graph G , where adjacent edges may be colored the same, is called a rainbow path if no two edges of the path are colored the same. For a κ -connected graph G and an integer k with $1 \leq k \leq \kappa$, the rainbow k -connectivity $rc_k(G)$ of G is defined as the minimum integer j for which there exists a j -edge-coloring of G such that every two distinct vertices of G are connected by k internally disjoint rainbow paths. This paper is to investigate the rainbow k -connectivity of complete graphs. We improve the upper bound of $f(k)$ from $(k+1)^2$ by Chartrand et al. to $ck^{\frac{3}{2}} + C(k)$, where c is a constant, $C(k) = o(k^{\frac{3}{2}})$, and $f(k)$ is the integer such that if $n \geq f(k)$ then $rc_k(K_n) = 2$. Recently, we saw that Dellamonica et al. got the best possible upper bound $2k$, which is linear in k . However, our proof is more structural or constructive.

Keywords: edge-colored graph, rainbow path, rainbow k -connectivity, complete (multipartite) graph

AMS Subject Classification 2000: 05C15, 05C40

1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let G be a nontrivial connected graph with an edge coloring $c : E(G) \rightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is called *rainbow* if no two edges of it are colored the same. A well-known result shows that in every κ -connected graph G with $\kappa \geq 1$, there are k internally disjoint $u - v$ paths connecting any two distinct

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vertices u and v for every integer k with $1 \leq k \leq \kappa$. Chartrand et al. [2] defined the *rainbow k -connectivity* $rc_k(G)$ of G to be the minimum integer j for which there exists a j -edge-coloring of G such that for every two distinct vertices u and v of G , there exist at least k internally disjoint $u - v$ rainbow paths. It is clearly well-defined.

The concept of rainbow k -connectivity has application in transferring information of high security in communication networks. For details we refer to [2] and [4].

By the definition of rainbow k -connectivity $rc_k(G)$, we know that it is almost impossible to derive exact value or a nice bound of the rainbow k -connectivity for a general graph G . So one investigates the rainbow k -connectivity of some classes of special graphs, such as complete graphs, complete multipartite graphs (see [2]). In [2], Chartrand et al. studied the rainbow k -connectivity of the complete graph K_n for various pairs k, n of integers, and they derived the following result:

Theorem 1.1 ([2]) *For every integer $k \geq 2$, there exists an integer $f(k)$ such that if $n \geq f(k)$, then $rc_k(K_n) = 2$. ■*

They obtained an upper bound $(k + 1)^2$ for $f(k)$, namely $f(k) \leq (k + 1)^2$. This paper is to continue their investigation, and the following result is derived:

Theorem 1.2 *For every integer $k \geq 2$, there exists an integer $f(k) = ck^{\frac{3}{2}} + C(k)$ where c is a constant and $C(k) = o(k^{\frac{3}{2}})$ such that if $n \geq f(k)$, then $rc_k(K_n) = 2$. ■*

From Theorem 1.2, we can obtain an upper bound $ck^{\frac{3}{2}} + C(k)$ for $f(k)$, where c is a constant and $C(k) = o(k^{\frac{3}{2}})$, that is, we improve the upper bound of $f(k)$ from $O(k^2)$ to $O(k^{\frac{3}{2}})$, a considerable improvement. Recently, Dellamonica et al. [3] got the best possible upper bound $2k$, which is linear in k . However, our proof is more structural or constructive, and informative.

For notation and terminology not defined here, we refer to [1].

2 Proof of Theorem 1.2

In [2], the authors derived the following two propositions:

Proposition 2.1 ([2]) *For $n \geq 4$, $rc_2(K_n) = 2$ and for $n \geq 5$, $rc_3(K_n) = 2$. ■*

Proposition 2.2 ([2]) *Let $n \geq 6$ be an integer. Then*

$$rc_4(K_n) = \begin{cases} 3 & \text{if } n = 6, 7 \\ 2 & \text{if } n \geq 8. \end{cases}$$

■

It is easy to show that Theorem 1.2 holds for the case $2 \leq k \leq 4$ by the above two propositions, so we assume $k \geq 5$.

Let $K_{\ell[r]}(r \geq 2)$ denote the complete ℓ -partite graph each part of which contains r elements and $\ell_0 = \lceil \max\{\sqrt{\frac{k}{2}} + 1, \frac{k-1}{r} + 2\} \rceil (\geq 3)$. Let $V = \bigcup_{s=1}^{\ell} V_s$ where $V_s = \{u_{s,1}, u_{s,2}, \dots, u_{s,r}\}$ ($1 \leq s \leq \ell$) is the vertex set of each part, and $U_j = \{u_{1,j}, u_{2,j}, \dots, u_{\ell,j}\}$ ($1 \leq j \leq r$). We will give our result by the following four steps:

Step 1. We will show that for every integer $k \geq 5$, if $\xi \geq \ell_0$, then $rc_k(K_{\xi^2[r]}) = 2$ (Proposition 2.3), which gives a result on the rainbow k -connectivity of complete ℓ -partite graph $K_{\ell[r]}$ where the number of parts $\ell = \xi^2$ is a square number.

Step 2. We will obtain a similar result for the general complete ℓ -partite graph $K_{\ell[r]}$ using Proposition 2.3: for every integer $k \geq 5$, if $\ell \geq \ell_0^2$, then $rc_k(K_{\ell[r]}) = 2$ (see Lemma 2.4). Here the number of parts ℓ is not always a square number.

Step 3. Let G' be a complete $(\ell + 1)$ -partite graph with ℓ parts of order r and a part of order p where $0 \leq p \leq r - 1$, that is, G' is obtained from $K_{\ell[r]}$ by adding a new part with p vertices. We will obtain a similar result: for every integer $k \geq 5$, if $\ell \geq \ell_0^2$, then $rc_k(G') = 2$ (Proposition 2.5). We can see that Lemma 2.4 is a special case (when $p = 0$) of Proposition 2.5.

Step 4. We derive Theorem 1.2 from Proposition 2.5.

We first consider the graph $G = K_{\xi^2[r]}$ where $\xi \geq \ell_0 \geq 3$ is an integer. Now $V = \bigcup_{s=1}^{\xi^2} V_s$ where $V_s = \{u_{s,1}, u_{s,2}, \dots, u_{s,r}\}$ ($1 \leq s \leq \xi^2$). Let $U_{i,j} = \{u_{(i-1)\xi+1,j}, \dots, u_{i\xi,j}\}$, $\overline{V}_i = \bigcup_{t=1}^{\xi} V_{(i-1)\xi+t}$, where $1 \leq i \leq \xi$, $1 \leq j \leq r$. Then the subgraph of G induced by $U_{i,j}$, say $G_{i,j}$, is the complete graph K_{ξ} ; the subgraph of G induced by U_j , say G_j , is the complete graph K_{ξ^2} , and $G_{i,j}$ is a subgraph of G_j . See Figure 2.1 for the case $k = 5$, $\xi = 3$, $r = 4$, the vertex subsets $U_1, U_{1,1}, V_1$ and \overline{V}_2 are shown in the figure and the subgraph $G_{1,1}$ is K_3 , G_1 is K_9 .

Proposition 2.3 *For every integer $k \geq 5$, if $\xi \geq \ell_0$, then $rc_k(K_{\xi^2[r]}) = 2$, where $r \geq 2$.*

Proof. At first, we give G a 2-edge-coloring. Similar to the edge coloring in the proof of Theorem 1.1 in [2], we first give a 2-edge-coloring of each G_j ($1 \leq j \leq r$) as follows: We

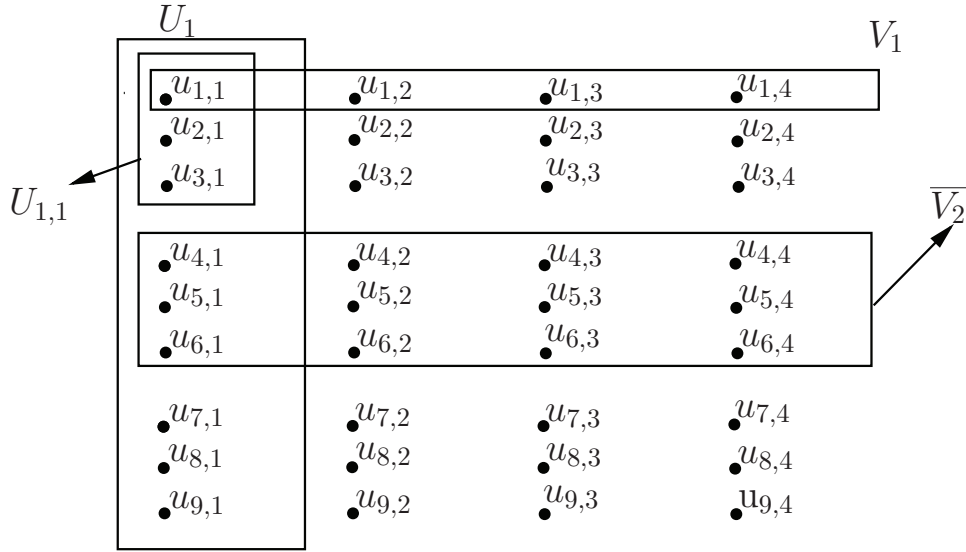


Figure 2.1 The figure for the case $k = 5$, $\xi = 3$, $r = 4$ (we omitting the edges).

assign the edge uv of G the color 1 if either $uv \in E(G_{i,j})$ for some $i(1 \leq i \leq \xi)$ or if $uv = u_{(i_1-1)\xi+t,j} u_{(i_2-1)\xi+t,j}$ for some i_1, i_2, t with $1 \leq i_1, i_2, t \leq \xi$ and $i_1 \neq i_2$. All other edges of G_j are assigned the color 2. For example, in the graph G of Figure 2.1, we choose $i_1 = 1$, $i_2 = 2$, $t = 2$, $j = 1$, $\xi = 3$, then edge $u_{(i_1-1)\xi+t,j} u_{(i_2-1)\xi+t,j} = u_{2,1}u_{5,1}$ receives the color 1, all edges with color 1 in G_1 are shown in Figure 2.2. With a similar argument to Theorem 1.1 in [2], we can obtain that

There are $\xi - 1$ disjoint rainbow $u - v$ paths for any two vertices u, v in the subgraph G_j including one path of length 1 and $\xi - 2$ paths of length 2. (*)

For other edges, that is, the edges between distinct G_j s, we use the above two colors as follows: Let $uv = u_{i_1,j_1}u_{i_2,j_2}$ where $u_{i_1,j_1} \in G_{j_1}, u_{i_2,j_2} \in G_{j_2}$, we give it the color different from the color of edge $u_{i_1,j_1}u_{i_2,j_1}$ of graph G_{j_1} or $u_{i_1,j_2}u_{i_2,j_2}$ of graph G_{j_2} (since edges $u_{i_1,j_1}u_{i_2,j_1}$ and $u_{i_1,j_2}u_{i_2,j_2}$ have the same color by the above coloring). For example, we choose $i_1 = 1$, $i_2 = 5$, $j_1 = 1$, $j_2 = 2$, edges $u_{i_1,j_1}u_{i_2,j_1} = u_{1,1}u_{5,1}$ and $u_{i_1,j_2}u_{i_2,j_2} = u_{1,2}u_{5,2}$ get the same color 2 as shown above, so edges $u_{i_1,j_1}u_{i_2,j_2} = u_{1,1}u_{5,2}$ and $u_{i_1,j_2}u_{i_2,j_1} = u_{1,2}u_{5,1}$ get color 1.

To complete our proof, it suffices to show that there are at least k disjoint rainbow paths connecting any two vertices u and v of G . So we now count the number of disjoint rainbow $u - v$ paths for any two vertices u, v of G . Without loss of generality, let $u = u_{1,1}$. We consider four cases:

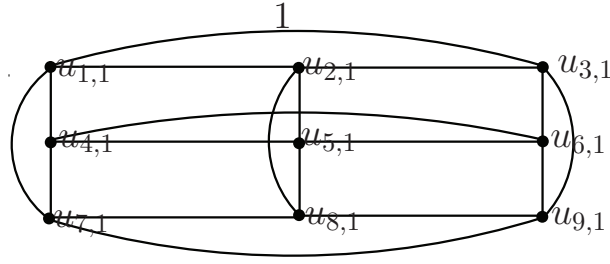


Figure 2.2 The figure for the edges with color 1 in G_1 .

Case 1. $v = u_{1,j}$ ($2 \leq j \leq r$), that is, u and v are in the same part. An example of $v = u_{1,2}$ is shown in Figure 2.1.

By the coloring, edges $u_{1,1}u_{s,j}$ and $u_{s,j}u_{1,j}$ have distinct colors where $2 \leq s \leq \xi^2$, and so path $u_{1,1}, u_{s,j}, u_{1,j}$ is a rainbow $u - v$ path. Similarly, path $u_{1,1}, u_{s,1}, u_{1,j}$ is a rainbow $u - v$ path for each $2 \leq s \leq \xi^2$. Clearly any two such rainbow paths are disjoint, so there are at least $2(\xi^2 - 1) \geq k$ disjoint rainbow $u - v$ paths in G .

Case 2. $v = u_{s,1}$ ($2 \leq s \leq \xi^2$), that is, u and v are in the same G_j (here $j = 1$). An example of $v = u_{5,1}$ is shown in Figure 2.1.

By (*), in subgraph G_1 there are $\xi - 1$ disjoint rainbow $u - v$ paths; in each subgraph G_j where $2 \leq j \leq r$, since there are $\xi - 2$ disjoint rainbow $u_{1,j} - u_{s,j}$ paths $u_{1,j}, y, u_{s,j}$ of length 2 for some ys where $y \in G_j$, and the color of $u_{1,j}y(yu_{s,j})$ is different from that of $u_{1,1}y(yu_{s,1})$, we obtain $\xi - 2$ disjoint rainbow $u_{1,1} - u_{s,1}$ paths $u_{1,1}, y, u_{s,1}$. Then there are at least $r(\xi - 2) + 1 \geq k$ disjoint rainbow $u - v$ paths in G .

Case 3. $v = u_{s,j_0}$ where $2 \leq s \leq \xi, 2 \leq j_0 \leq r$, that is, u and v belong to the same \overline{V}_i (here $i = 1$) but in distinct parts and distinct G_{j_s} . Without loss of generality, let $j_0 = 2$, an example of $v = u_{2,2}$ is shown in Figure 2.1.

At first, the edge $u_{1,1}u_{s,2}$ is a rainbow $u - v$ path. Next we consider the monochromatic $u_{1,1} - u_{s,1}$ path $u_{1,1}, y, u_{s,1}$ in G_1 , then each path $u_{1,1}, y, u_{s,2}$ is a rainbow $u_{1,1} - u_{s,2}$ path of length 2. In U_1 , we choose $y = u_{(i-1)\xi+t,1}$ where $1 \leq i \leq \xi, 2 \leq t \leq \xi$ and $t \neq s$, there are $\xi(\xi - 2)$ such paths. Similarly, there are $\xi(\xi - 2)$ disjoint monochromatic $u_{1,2} - u_{s,2}$ paths $u_{1,2}, y, u_{s,2}$ where $y \in U_2$ and we can obtain another $\xi(\xi - 2)$ disjoint rainbow $u - v$ paths. In each subgraph G_j where $3 \leq j \leq r$, there are $\xi - 2$ disjoint rainbow $u_{1,j} - u_{s,j}$ paths $u_{1,j}, y, u_{s,j}$ of length 2, then paths $u_{1,1}, y, u_{s,2}$ are disjoint rainbow $u - v$ paths since the color of $u_{1,j}y(yu_{s,j})$ is different from that of $u_{1,1}y(yu_{s,2})$.

So the number of disjoint rainbow $u-v$ paths in G is at least $1+2\xi(\xi-2)+(r-2)(\xi-2) = 1 + (2\xi + r - 2)(\xi - 2) \geq 1 + r(\xi - 2) \geq k$.

Case 4. $v = u_{(i_0-1)\xi+t_0,j_0}$ where $2 \leq i_0 \leq \xi, 1 \leq t_0 \leq \xi, 2 \leq j_0 \leq r$, this is, v and u are in distinct \overline{V}_i s and G_j s (here v is not in \overline{V}_1 and G_1).

Subcase 4.1. $t_0 = 1$, an example of $v = u_{4,2}$ is shown in Figure 2.1.

Edge uv is a rainbow $u-v$ path. In subgraph G_1 , we find the monochromatic $u - u_{(i_0-1)\xi+1,1}$ path $u, y, u_{(i_0-1)\xi+1,1}$. We choose any vertex of $U_1 \setminus \{U_{1,1} \cup U_{i,1}\}$ to be y . Since the color of edge $yu_{(i_0-1)\xi+1,1}$ is different from that of yv , each path u, y, v is rainbow, so we get $\xi(\xi - 2)$ disjoint rainbow $u - v$ paths. Similarly, in subgraph G_{j_0} , we can find monochromatic paths $u_{1,j_0}, y, u_{(i_0-1)\xi+1,j_0}$, and get another $\xi(\xi - 2)$ disjoint rainbow $u - v$ paths. In each subgraph G_j where $j \neq 1, j_0$, we find the disjoint rainbow $u_{1,j} - u_{(i_0-1)\xi+1,j}$ paths $u_{1,j}, y, u_{(i_0-1)\xi+1,j}$ of length 2. Since the color of $u_{1,j}y(yu_{(i_0-1)\xi+1,j})$ is different from that of $u_{1,1}y(yu_{(i_0-1)\xi+1,1})$, respectively, there are $\xi - 2$ disjoint rainbow $u - v$ paths $u_{1,1}, y, u_{(i_0-1)\xi+1,1}$.

So the number of disjoint rainbow $u-v$ paths in G is at least $1+2\xi(\xi-2)+(r-2)(\xi-2) = 1 + (2\xi + r - 2)(\xi - 2) > 1 + r(\xi - 2) \geq k$.

Subcase 4.2. $t_0 \neq 1$, an example of $v = u_{5,2}$ is shown in Figure 2.1.

Edge uv is a rainbow $u - v$ path. In subgraph G_1 , we find the monochromatic $u - u_{(i_0-1)\xi+t_0,1}$ path $u, y, u_{(i_0-1)\xi+t_0,1}$. Let $y = u_{(i-1)\xi+t,1}$ where $i \neq 1, i_0, t \neq 1, t_0$ or $y = u_{t_0,1}$ or $u_{(i_0-1)\xi+1,1}$. Since the color of edge $yu_{(i_0-1)\xi+t_0,1}$ is different from that of yv , each path u, y, v is rainbow, and so there are $(\xi - 2)(\xi - 2) + 2$ disjoint rainbow $u - v$ paths. Similarly, in subgraph G_{j_0} , we can find monochromatic paths $u_{1,j_0}, y, u_{(i_0-1)\xi+1,j_0}$, and get other $(\xi - 2)(\xi - 2) + 2$ disjoint rainbow $u - v$ paths. In each subgraph G_j where $j \neq 1, j_0$, we find the disjoint rainbow $u_{1,j} - u_{(i_0-1)\xi+t_0,j}$ paths $u_{1,j}, y, u_{(i_0-1)\xi+t_0,j}$ of length 2. Since the color of $u_{1,j}y$ and $yu_{(i_0-1)\xi+t_0,j}$ is different from that of $u_{1,1}y$ and $yu_{(i_0-1)\xi+t_0,j_0}$, respectively, there are $\xi - 2$ disjoint rainbow $u - v$ paths $u_{1,1}, y, u_{(i_0-1)\xi+t_0,j_0}$ (here $y \in G_j$ where $j \neq 1, j_0$). So the number of disjoint rainbow $u - v$ paths in G is $5 + 2(\xi - 2)(\xi - 2) + (r - 2)(\xi - 2) = 5 + (2\xi + r - 6)(\xi - 2) \geq 5 + r(\xi - 2) > k$. ■

Next we will consider the case for general complete multipartite graph $G \cong K_{\ell[r]}$ with equal parts, where $\ell \geq \ell_0^2$. Let ξ be the integer satisfying $\xi^2 \leq \ell \leq (\xi + 1)^2$ where $\xi \geq \ell_0$. Then by Proposition 2.3, we only need to consider the case $1 \leq q = \ell - \xi^2 \leq 2\xi$. Let G be obtained from $K_{\xi^2[r]}$ by adding q new parts P_i ($1 \leq i \leq q$). Let the vertex set of each corresponding new part be $V(P_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,r}\}$ ($1 \leq i \leq q$). We now **update** \overline{V}_i as follows: If $1 \leq q \leq \xi$, let the new \overline{V}_i be the union of the old \overline{V}_i and $V(P_i)$; If $\xi < q \leq 2\xi$,

let the new \overline{V}_i be the union of the old \overline{V}_i and $V(P_i) \cup V(P_{\xi+i})$ (if there exists). Similarly, we **update** $U_{i,j}, U_j$ ($1 \leq i \leq \xi, 1 \leq j \leq r$) (and also $G_j, G_{i,j}$). See Figure 2.3 for the case $k = 5, \xi = 3, r = 4$ and $\ell = 13$, the vertex subsets $V_1, V(P_1)$ and new vertex sets $U_1, U_{1,1}, \overline{V}_2$ are shown in the figure and the new subgraph $G_{1,1}$ is K_5, G_1 is K_{13} .

By a similar argument (but a little more complicated) to the proof of Proposition 2.3, we derive the following lemma which will be used in the sequel:

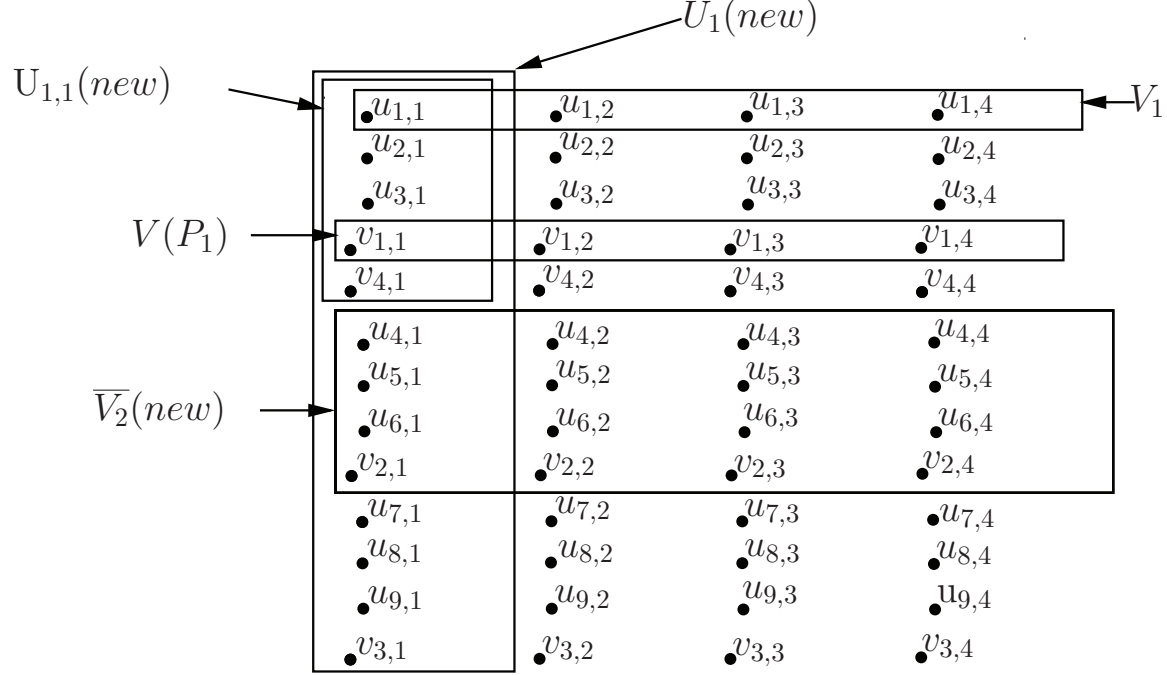


Figure 2.3 The figure for the case $k = 5, \xi = 3, r = 4$ and $\ell = 13$ (we omitting the edges).

Lemma 2.4 For every integer $k \geq 5$, if $\ell \geq \ell_0^2$, then $rc_k(K_{\ell[r]}) = 2$, where $r \geq 2$.

Proof. At first, we give a 2-edge-coloring to graph G as follows: For each G_j ($1 \leq j \leq r$) (similar to the proof of Theorem 1.1 in [2]), we assign the edge uv of G the color 1 if $uv \in E(G_{i,j})$ for some i ($1 \leq i \leq \xi$) or $uv = u_{(i_1-1)\xi+t,j} u_{(i_2-1)\xi+t,j}$ or $uv = v_{i_1,j} v_{i_2,j}$ or $uv = v_{\xi+i_1,j} v_{\xi+i_2,j}$ for some i_1, i_2, t with $1 \leq i_1, i_2, t \leq \xi$ and $i_1 \neq i_2$. For example, all edges with color 1 in the subgraph G_1 of graph $K_{13[4]}$ are shown in Figure 2.4. All other edges of G_j are assigned the color 2. For other edges of G , that is, edges between distinct G_j s, by a similar way to the proof of Proposition 2.3, we use the above two colors as follows: If $uv = u_{i_1,j_1} u_{i_2,j_2}$ where $u_{i_1,j_1} \in G_{j_1}, u_{i_2,j_2} \in G_{j_2}$, we give it the color different from that of edge $u_{i_1,j_1} u_{i_2,j_1}$ of graph G_{j_1} or $u_{i_1,j_2} u_{i_2,j_2}$ of graph G_{j_2} (since edges $u_{i_1,j_1} u_{i_2,j_1}$ and $u_{i_1,j_2} u_{i_2,j_2}$ have the same color by the coloring); Similarly, we color the edges for the

cases that $uv = u_{i_1, j_1} v_{i_2, j_2}$ and $uv = v_{i_1, j_1} v_{i_2, j_2}$.

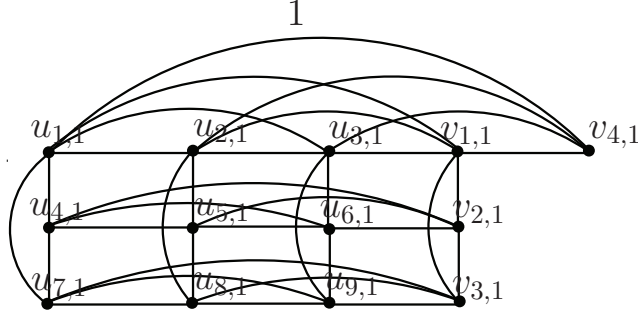


Figure 2.4 The figure for the edges with color 1 in subgraph G_1 .

By the proof of our Proposition 2.3, there are at least k disjoint rainbow paths connecting any two vertices $u, v \in V(G) \setminus \bigcup_{i=1}^q V(P_i)$. We need to count the number of disjoint rainbow paths between u and v where $u \in \bigcup_{i=1}^q V(P_i)$ and $v \in V(G)$. Without loss of generality, let $u = v_{1,1}$, and we consider four cases, which are similar to the four cases in the proof of our Proposition 2.3.

Case 1. $v \in V(P_1)$, that is, u and $v = v_{1, j_0}$ ($2 \leq j_0 \leq r$) are in the same part (here the part is P_1). An example of $v = v_{1,2}$ is shown in Figure 2.3.

G contains the disjoint rainbow $u-v$ paths $v_{1,1}, y, v_{1, j_0}$ where $y \in U_1 \cup U_{j_0} \setminus \{v_{1,1}, v_{1, j_0}\}$. So the number of these paths is at least $2\xi^2 \geq 2\ell_0^2 > 2(\ell_0^2 - 1) \geq k$.

Case 2. $v \in U_1$, that is, v and u are in the same subgraph G_1 . An example of $v = v_{2,1}$ is shown in Figure 2.3.

In G_1 , there are at least $\xi - 1$ disjoint rainbow $u - v$ paths. In each G_j ($2 \leq j \leq r$) there are at least $\xi - 2$ disjoint rainbow $u' - v'$ paths u', y, v' of length 2 where u', v' are in the same part with u, v , respectively. So we get at least $\xi - 2$ disjoint rainbow $u - v$ paths u, y, v , where $y \in G_j$ ($2 \leq j \leq r$). Then the number of disjoint rainbow $u - v$ paths is at least $1 + r(\xi - 2) \geq 1 + r(\ell_0 - 2) \geq k$.

Case 3. $v \in \overline{V_1}$ but not in $U_1 \cup V(P_1)$. Without loss of generality, let $v \in U_{j_0}$.

At first, edge uv is a $u - v$ path.

Subcase 3.1. $v = v_{\xi+1, j_0}$. An example of $v = v_{4,2}$ is shown in Figure 2.3.

Similar to Case 3 in the proof of Proposition 2.3, we find the monochromatic $u - v_{\xi+1,1}$ path of length 2 in subgraph G_1 and get at least ξ^2 disjoint rainbow $u - v$ paths. Similarly, there are other ξ^2 disjoint rainbow $u - v$ paths by finding the monochromatic $v_{1, j_0} - v_{\xi+1, j_0}$

paths in G_{j_0} . In each G_j ($j \neq 1, j_0$), similar to Case 3 in the proof of Proposition 2.3, we get at least $\xi - 2$ disjoint rainbow $u - v$ paths of length 2 by finding rainbow $v_{1,j} - v_{\xi+1,j}$ paths in G_j . So the number of disjoint rainbow $u - v$ paths is at least $1 + 2\xi^2 + (r - 2)(\xi - 2) > 1 + 2\xi(\xi - 2) + (r - 2)(\xi - 2) = 1 + (2\xi + r - 2)(\xi - 2) \geq 1 + (2\ell_0 + r - 2)(\ell_0 - 2) > 1 + r(\ell_0 - 2) \geq k$.

Subcase 3.2. $v = u_{s_0, j_0}$ where $1 \leq s_0 \leq \xi, 2 \leq j_0 \leq r$. An example of $v = u_{3,2}$ is shown in Figure 2.3.

By a similar procedure to Subcase 3.1, we can get at least $1 + 2\xi(\xi - 1) + (r - 2)(\xi - 2) > 1 + (2\xi + r - 2)(\xi - 2) \geq 1 + (2\ell_0 + r - 2)(\ell_0 - 2) > 1 + r(\ell_0 - 2) \geq k$ disjoint rainbow $u - v$ paths in graph G .

Case 4. v is not in $\overline{V_1}$ and U_1 , without loss of generality, let $v \in \overline{V_2} \setminus U_1$. We consider two subcases similar to Case 4 in the proof of Proposition 2.3.

Case 4.1. $v = v_{2, j_0}$ where $2 \leq j_0 \leq r$. An example of $v = v_{2,2}$ is shown in Figure 2.3.

Similar to Subcase 4.1 in the proof of Proposition 2.3, we can get at least $1 + 2\xi(\xi - 2) + (r - 2)(\xi - 2) = 1 + (2\xi + r - 2)(\xi - 2) \geq 1 + (2\ell_0 + r - 2)(\ell_0 - 2) > 1 + r(\ell_0 - 2) \geq k$ disjoint rainbow $u - v$ paths in G .

Case 4.2. $v \neq v_{2, j_0}$. An example of $v = u_{4,2}$ is shown in Figure 2.3.

Similar to Subcase 4.2 in the proof of Proposition 2.3, we can get at least $1 + 2(\xi - 1)(\xi - 2) + (r - 2)(\xi - 2) = 1 + (2\xi + r - 4)(\xi - 2) \geq 1 + r(\xi - 2) \geq 1 + r(\ell_0 - 2) \geq k$ (since $\xi \geq 2$) disjoint rainbow $u - v$ paths in G . ■

From Lemma 2.4, we derive the following result.

Proposition 2.5 *Let G' be a complete $(\ell + 1)$ -partite graph with ℓ parts of order r and a part of order p where $0 \leq p < r$. Then for every integer $k \geq 5$, if $\ell \geq \ell_0^2$, then $rc_k(G') = 2$.*

Proof. For the case $p = 0$, the conclusion clearly holds by Lemma 2.4, so we assume $p \geq 1$. We know that G' can be obtained from $G \cong K_{\ell[r]}$ (graph of Lemma 2.5) by adding p new vertices: w_1, w_2, \dots, w_p ($1 \leq p \leq r - 1$) and edge vw_j where $1 \leq j \leq p$ and $v \in V(G)$. For example, let graph G' be a complete 14-partite graph with 13 parts of order $r = 4$ and a part of order $p = 2$, then G' can be obtained from the graph G in Figure 2.3 by adding a new part with $p = 2$ vertices.

We color the new edges as follows: Give color 2 to edges between U_j and w_j , and color 1 to the other edges.

Now we count the number of disjoint rainbow $u - v$ paths between any two vertices of G . By Lemma 2.4, we need to consider the case that $u = w_j, v \in V(G')$ ($1 \leq j \leq p$). Without loss of generality, let $u = w_1$.

Case 1. $v \in V(G)$.

Subcase 1.1. $v \in U_1$, without loss of generality, let $v = u_{1,1}$.

Then G contains the $u - v$ path u, v as well as the $u - v$ rainbow paths $w_1, y, u_{1,1}$ of length 2 where $y \in \overline{V_1} \setminus \{u_{1,1}, \dots, u_{1,r}\}$. So the number of disjoint rainbow $u - v$ paths is at least $1 + r(\xi - 1) > 1 + r(\ell_0 - 2) \geq k$.

Subcase 1.2. v is not in U_1 , without loss of generality, let $v = u_{1,2}$.

Then G contains the $u - v$ path u, v as well as the $u - v$ rainbow paths $w_1, y, u_{1,2}$ where $y \in (U_1 \cup U_2) \setminus (U_{1,1} \cup U_{1,2} \cup \{u_{(i-1)\xi+1, j_0}\}_{i=1}^{\xi})$ where $j_0 = 1, 2$, or $y \in \{u_{(i-1)i_1+1, j_1}\}_{i=2}^{\xi}$ ($j_1 \neq 1, 2$), or $y = u_{s,j}$ where $2 \leq s \leq \xi, j \neq 1, 2$. So the number of disjoint rainbow $u - v$ paths is at least $1 + 2(\xi - 1)(\xi - 1) + 2(\xi - 1)(r - 2) = 1 + (\xi - 1)(2\xi + 2r - 6) > 1 + (\xi - 2)(2\xi + 2r - 6) \geq 1 + (\xi - 2)r \geq k$ (here we let $r \geq 2$). The proof for the case that $v = v_{1,2}$ or $v_{\xi+1,2}$ is similar.

Case 2. $v = w_j$ where $j \neq 1$.

Then G contains the $u - v$ rainbow paths w_1, y, w_j where $y \in U_1 \cup U_j$. So the number of disjoint rainbow $u - v$ paths is at least $2\xi^2 \geq 2\ell_0^2 > 2(\ell_0^2 - 1) \geq k$. ■

We know that the graph in Proposition 2.5 is a spanning subgraph of the complete graph K_n where $n = \ell r + p, \ell \geq \ell_0^2, 0 \leq p < r$. On the other hand, We choose $r = r_0 = \lceil \sqrt{2k} \rceil$, then $\ell_0 = \lceil \max\{\sqrt{\frac{k}{2}} + 1, \frac{k-1}{r} + 2\} \rceil = \lceil \max\{\sqrt{\frac{k}{2}} + 1, \frac{k-1}{\lceil \sqrt{2k} \rceil} + 2\} \rceil = \lceil \frac{k-1}{\lceil \sqrt{2k} \rceil} \rceil + 2$. Let G'' be the complete graph K_n where $n = \ell_3 r_0 + p, \ell_3 \geq \ell_0^2, 0 \leq p < r_0$, then G'' contains a spanning subgraph G' , where G' is a complete $(\ell_3 + 1)$ -partite graph with ℓ_3 parts of order r_0 and a part of order p where $0 \leq p < r_0$. From Proposition 2.5, Theorem 1.2 follows.

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