KERNEL METHOD AND SYSTEMS OF FUNCTIONAL EQUATIONS WITH SEVERAL CONDITIONS

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Abstract

We generalize the kernel method to equation systems in which the number of unknowns is allowed to exceed the number of equations. With this generalization, we derive the generating functions for several kinds of sequences and generating trees whose recursions are dependent on the parity of the indices.

Keywords: kernel method, generate functions, sequences with two indices, generating trees

2000 Mathematics Subject Classification: 65Q05, 05A15

1. Introduction

The kernel method is a powerful tool in solving equations of generating functions. The standard model deals with the case of a functional equation of the form

$$K(x,y)F_u(x,y) = A(x,y)G_u(x) + B(x,y),$$

where $F_u(x,y)$ and $G_u(x)$ are unknown functions. It was generalized to the following equation system in [7] (for another kind of equation system see [11]):

$$\mathbf{K}(x,y)\mathbf{F}_u(x,y) = \mathbf{A}(x,y)\mathbf{G}_u(x) + \mathbf{B}(x,y), \tag{1.1}$$

where $\mathbf{K}(x,y) = (K_{i,j}(x,y))$ and $\mathbf{A}(x,y) = (A_{i,j}(x,y))$ are $n \times n$ matrices, $\mathbf{B}(x,y) = (B_i(x,y))$ is a column vector, and $\mathbf{F}_u(x,y) = (F_i(x,y)), \mathbf{G}_u(x) = (G_i(x))$ are unknown column vectors.

Notice that in the above system, we require that $\mathbf{F}_u$ and $\mathbf{G}_u$ have the same dimension. In fact, we can remove this restriction and consider the more general form

$$\mathbf{K}(x,y)\mathbf{F}_u(x,y) = \mathbf{A}(x,y)\mathbf{G}_u(x) + \mathbf{B}(x,y), \tag{1.2}$$

where $\mathbf{K}(x,y) = (K_{i,j}(x,y))$ and $\mathbf{A}(x,y) = (A_{i,j}(x,y))$ are $n \times n$ and $n \times \ell$ matrices, respectively. The case of $n = 1$ has been discussed for combinatorial structures. For example, Elizalde showed that a class of permutations are enumerated by such a functional equation, see [6, Equation (28)]. Bousquet-Mélou and Jehanne [3] provided a strategy for solving a polynomial equation involving several unknown formal power series. In this paper, we provide two methods for solving the system for general n.

Recently, the second author used the kernel method to solve several recurrence relations with two indices; see [5,8,9]. In this paper, we show how to apply the new methods to obtain explicit formulas for other kinds of recurrence relations with two indices where we fix the parity.

This paper is organized as follows. In Section 2, we describe the induction method and the elimination method for solving equation systems of the form (1.2). Using these methods, we derive formal power series solutions $\mathbf{F}_u(x,y)$ and $\mathbf{G}_u(x)$. In Section 3, we provide several equation systems to illustrate the power of our methods. We derive the generating functions for several sequences and generating trees whose recursions are dependent on the parity of the indices.

2. Solving (1.2)

With a slight modification to the methods given in [7], we can solve the equation system (1.2) as follows.

2.1. **The induction method.**

Step 1. Multiply both sides of (1.2) by a suitable matrix so that the system becomes

$$\mathbf{D}(x,y)\mathbf{F}_u(x,y) = \mathbf{A}^{(1)}(x,y)\mathbf{G}_u(x) + \mathbf{B}^{(1)}(x,y), \tag{2.1}$$

where the entries of $\mathbf{D}, \mathbf{A}^{(1)}, \mathbf{B}^{(1)}$ are all polynomials and $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix.

Step 2. Find a solution $(G_1(x), \ldots, G_\ell(x))$ to the system (2.1) by induction on the dimension n.

Case $n = 1$. For each root of $d_1(x,y)$ of the form $y_0(x) \in x\mathbb{C}[[x]]$ with multiplicity m, we have a system of linear equations on $G_1(x), \ldots G_\ell(x)$:

$$H(x,y)\Big|_{y=y_0(x)} = 0, \; \frac{\partial H(x,y)}{\partial y}\Big|_{y=y_0(x)} = 0, \; \ldots, \; \frac{\partial^{m-1} H(x,y)}{\partial y^{m-1}}\Big|_{y=y_0(x)} = 0, \tag{2.2}$$

where

$$H(x,y) = \sum_{j=1}^{\ell} A_{1,j}^{(1)}(x,y)G_j(x) + B_1^{(1)}(x,y).$$

Combining these equations for all such roots, we get a larger system of linear equations on $G_1(x), \ldots, G_\ell(x)$. Solving this linear system, we obtain a solution $(G_1(x), \ldots, G_\ell(x))$.

Case $n > 1$. As above, by substituting the roots of $d_1(x,y)$ into the first equation, we have a system of linear equations on $G_1(x), \ldots, G_\ell(x)$. Solving the system leads us to an equivalent linear system:

$$G_1(x) = \sum_{j=2}^{\ell} c_{1,j}(x)G_j(x), \tag{2.3}$$

$$\sum_{j=2}^{\ell}(c_{2,j} - c_{1,j})G_j(x) = 0, \ldots, \sum_{j=2}^{\ell}(c_{r,j} - c_{r-1,j})G_j(x) = 0. \tag{2.4}$$

Now substituting (2.3) into (2.1) and deleting the first equation, we obtain a new equation system of dimension $n-1$:

$$\mathbf{D}'(x,y)\mathbf{F}'_u(x,y) = \mathbf{A}'(x,y)\mathbf{G}'_u(x) + \mathbf{B}'(x,y), \tag{2.5}$$

where

$$D'(x,y) = \operatorname{diag}(d_2(x,y),\dots,d_n(x,y))$$

is an $(n-1)\times(n-1)$ submatrix of $D(x,y)$. By induction, (2.5) is solvable. Let $(G_2(x),\dots,G_\ell(x))$ be a solution to (2.5) and $G_1(x)$ be given by (2.3). Then $(G_1(x),\dots,G_\ell(x))$ is a solution to the system (2.1).

Step 3. Substitute $\mathbf{G}_u(x)$ into the original equation system (1.2) and multiply both sides by $\mathbf{K}^{-1}(x,y)$. We obtain the vector $\mathbf{F}_u(x,y)$.

We also have the elimination method for obtaining algebraic equations that the solutions satisfy.

2.2. **The elimination method.**

Step 1. Transform the system to

$$\mathbf{D}(x,y)\mathbf{F}_u(x,y) = \mathbf{A}^{(1)}(x,y)\mathbf{G}_u(x) + \mathbf{B}^{(1)}(x,y),$$

as in the induction method.

Step 2. For each root of $d_i(x,y)$ of the form $y_0(x)\in x\mathbb{C}[[x]]$ with multiplicity m, set up a system of linear equations on $G_1(x),\dots,G_\ell(x)$:

$$H(x,y)\Big|_{y=y_0(x)} = 0,\quad \frac{\partial H(x,y)}{\partial y}\Big|_{y=y_0(x)} = 0,\ \dots,\ \frac{\partial^{m-1} H(x,y)}{\partial y^{m-1}}\Big|_{y=y_0(x)} = 0, \tag{2.6}$$

where

$$H(x,y) = \sum_{j=1}^{\ell} A^{(1)}_{i,j}(x,y)G_j(x) + B^{(1)}_i(x,y).$$

Adding the new equation $d_i(x,y_0)=0$ to the system (2.6), we obtain a polynomial system on $G_1(x),\dots,G_n(x)$ and y_0. We then use Gröbner basis theory [1,4] or Wu's method [15] to reduce the system by eliminating the variable y_0. Thus, we obtain an algebraic equation $P_i(G_1,\dots,G_\ell)=0$.

Step 3. Apply polynomial elimination to the system $P_1=\cdots=P_n=0$ to find the algebraic equation that $G_i(x)$ satisfies for each $1\le i\le \ell$.

As a special case, we can solve the equation system of the form

$$\mathbf{K}(x,v)\mathbf{F}_u(x,v) = \sum_{i=1}^{\ell}\mathbf{A}_i(x,v)\mathbf{F}_u(x,z_i) + \mathbf{B}(x,v),$$

where $\mathbf{K},\mathbf{A}$ are matrices of size $n\times n$ and z_i is a real or complex number. In fact, let $G_{n(i-1)+j}(x) = F_j(x,z_i)$ for $1\le j\le n, 1\le i\le \ell$. Then, the system becomes

$$\mathbf{K}(x,v)\mathbf{F}_u(x,v) = (\mathbf{A}_1(x,v),\mathbf{A}_2(x,v),\dots,\mathbf{A}_\ell(x,v))\mathbf{G}_u(x) + \mathbf{B}(x,v),$$

which falls into our framework.

For $n = 1$, the system becomes

$$K(x,y)F_u(x,y) = \sum_{i=1}^{\ell} a_i(x,y)F_i(x) + b(x,y).$$

Thus, we can immediately solve the equation that appears in the enumeration of the column-convex polynominoes [3, P. 2]:

$$F(t,u) = \frac{ut^2}{1-ut} + \frac{t^3u^2F(t,u)}{(1-ut)^2} + 2\frac{t^2u^2}{1-ut}\frac{F(t,u)-F(t,1)}{u-1} + ut\frac{uF(t,u)-uF(t,1)-(u-1)F'_u(t,1)}{(u-1)^2}.$$

There is an important application of the methods. Suppose that we are given an equation system of form (1.2). We eliminate $F_2(x,y),\dots,F_n(x,y)$ to obtain an equation on $F_1(x,y)$ and $G_1(x),\dots,G_n(x)$. Then, we can use the above methods to find $G_1(x),\dots,G_n(x)$ directly, which reduces the computation. We have implemented the two methods in Maple [10] and will illustrate them in the following section.

3. Applications

In this section, we give two examples for our methods. The first one deals with solving two kinds of recurrence relations with two indices, and the second one deals with a special kind of generating tree with two labels.

3.1. **Recurrence relations with two indices.**

A *sequence with k indices* is a function $a : A^k \to B$, and is denoted by $\{a_{n_1,\dots,n_k}\}_{n_1,\dots,n_k\in A}$ or $\{a_{\mathbf{n}}\}_{\mathbf{n}\in A^k}$, where $A \subseteq \mathbb{N}$. The element $a_{\mathbf{n}}$ of a sequence $\{a_{\mathbf{n}}\}_{\mathbf{n}\in A}$ is called the $\mathbf{n}$*-th term*, and the vector $\mathbf{n}$ of integers is the *sequence vector of indices*. A *recurrence relation* is an equation that defines a sequence recursively, that is, each term of the sequence is defined as a function of the preceding terms, together with *initial conditions*. The initial conditions are necessary for ensuring the uniqueness of the sequence. The aim of this subsection is to use our methods to solve two kinds of recurrence relations with two indices (standard kinds of recurrence relations and the kernel method can be found in [5, 8, 9]).

Example 3.1. (Recurrence relations with two indices) *Let $\{a_{n,k}\}$ be a sequence with two indices defined by $a_{1,1} = 1$ and*

$$a_{n,i} = \begin{cases} \sum_{j=1}^{(i+1)/2} a_{n-1,2j-1}, & \text{if } i < n \text{ and } i \text{ is odd},\\ \sum_{j=1}^{n-1} a_{n-1,j}, & i = n,\\ 0, & \text{otherwise}. \end{cases}$$

In order to solve this recurrence relation we define $A_n(v) = \sum_{i=1}^{n} a_{n,i}v^{i-1}$. Rewriting the above recurrence relation in terms of polynomials $A_n(v)$, we get that

$$\begin{aligned} A_{2n+1}(v) &= \tfrac{1}{1-v^2}(A_{2n}(v) - v^{2n}A_{2n}(1)) - \tfrac{v^{2n-1}}{1+v}A_{2n-1}(1) + v^{2n}A_{2n}(1),\\ A_{2n}(v) &= \tfrac{1}{1-v^2}(A_{2n-1}(v) - v^{2n}A_{2n-1}(1)) + v^{2n-1}A_{2n-1}(1). \end{aligned}$$

Now, we define

$$OA(x,v)=\sum_{n\geq 0}A_{2n+1}(v)x^{2n+1} \text{ and } EA(x,v)=\sum_{n\geq 1}A_{2n}(v)x^{2n}.$$

Then,

$$OA(x,v)=\frac{x}{1-v^2}EA(x,v)+\left(x-\frac{x}{1-v^2}\right)EA(xv,1)-\frac{x^2}{1+v}OA(xv,1)+x,$$

$$EA(x,v)=\frac{x}{1-v^2}OA(x,v)+\left(x-\frac{xv}{1-v^2}\right)OA(xv,1).$$

Eliminating $EA(x,v)$, yields

$$\left(1-\frac{x^2}{(1-v^2)^2}\right)OA(x,v)=-\frac{x^2v^3}{(1-v^2)^2}OA(xv,1)-\frac{xv^2}{1-v^2}EA(xv,1)+x.$$

By substituting in $x=x/v$, our Maple program yields

$$x^4f^3+3x^3f^2+(3x^2-1)f+x=0,$$
$$x^4g^3+(3x^4-2x^2)g^2+(3x^4-4x^2+1)g+x^4-2x^2=0,$$

where $f=OA(x,1)$ and $g=EA(x,1)$. The first few terms are

$$OA(x,1)=x+3x^3+12x^5+55x^7+273x^9+\cdots,$$

and

$$EA(x,1)=2x^2+7x^4+30x^6+143x^8+728x^{10}+\cdots.$$

Note that the above example can be generalized as follows. Let p be any positive integer, and let $\{a_{n,k}\}$ be the sequence defined by $a_{1,1}=1$ and

$$a_{n,i}=\begin{cases}a_{n-1,1}+a_{n-1,p+1}+\cdots+a_{n-1,i}, & \text{if } i<n \text{ and } i\equiv 1 \pmod p,\\ \sum_{j=1}^{n-1}a_{n-1,j}, & i=n,\\ 0, & \textit{otherwise}.\end{cases} \tag{3.1}$$

Define $A_n(v)=\sum_{i=1}^n a_{n,i}v^{i-1}$ and $A^{(j)}(x,v)=\sum_{n\geq 0}A_{pn+j}(v)x^{pn+j}$, for all $j=1,2,\ldots,p$. By a similar discussion, we derive that

$$\begin{aligned}A^{(1)}(x,v) &= \frac{x}{1-v^p}A^{(p)}(x,v)+\left(x-\frac{x}{1-v^p}\right)A^{(p)}(xv,1)\\ &\quad -\frac{x^2(1-v)}{1-v^p}A^{(p-1)}(xv,1)+x,\\ A^{(2)}(x,v) &= \frac{x}{1-v^p}A^{(1)}(x,v)+\left(x-\frac{xv^{p-1}}{1-v^p}\right)A^{(1)}(xv,1),\\ A^{(s)}(x,v) &= \frac{x}{1-v^p}A^{(s-1)}(x,v)+\left(x-\frac{xv^{p+1-s}}{1-v^p}\right)A^{(s-1)}(xv,1)\\ &\quad -\frac{x^2(1-v^{p+2-s})}{1-v^p}A^{(s-2)}(xv,1), \qquad s=3,4,\ldots,p.\end{aligned}$$

Eliminating $A^{(2)}(x,v),\dots,A^{(p)}(x,v)$, we finally derive that

$$\left(1-\frac{x^p}{(1-v^p)^p}\right)A^{(1)}(x,v)=-\sum_{j=1}^{p}\frac{x^{p+1-j}v^{2p-j}}{(1-v^p)^{p+1-j}}A^{(j)}(xv,1)+x.$$

For the case $p>2$, our Maple program cannot help much since it runs out of memory after hours of computation..

More generally, for any integer r, we can define a sequence $\{b_{n,k}\}$ by $b_{1,1}=1$ and

$$b_{n,i}=\begin{cases} b_{n-1,r}+b_{n-1,p+r}+\cdots+b_{n-1,i}, & \text{if } i<n \text{ and } i\equiv r \pmod p,\\ \sum_{j=1}^{n-1}b_{n-1,j}, & i=n,\\ 0, & \textit{otherwise}.\end{cases} \tag{3.2}$$

However, it can be proved by induction that these sequences are just shifts of the sequence defined by (3.1). More precisely, we have $b_{n+r-1,i+r-1}=a_{n,i}$ for $1\le r\le p$, where $\{a_{n,i}\},\{b_{n,i}\}$ are the sequences defined by (3.1) and (3.2), respectively.

Example 3.2. (System of recurrence relations with two indices) *Now, we consider two interacting sequences with two indices.*

Let $\{a_{n,i}\}$ and $\{b_{n,i}\}$ be two sequences defined by $a_{1,1}=1$, $b_{1,1}=0$, and

$$a_{n,i}=\begin{cases} b_{n-1,i}+\sum_{j=1}^{i}a_{n-1,j}, & \textit{if } i<n \textit{ and } i \textit{ is even},\\ \sum_{j=1}^{n-1}a_{n-1,j}, & i=n,\\ 0, & \textit{otherwise}.\end{cases}$$

$$b_{n,i}=\begin{cases} a_{n-1,i}+\sum_{j=1}^{i}b_{n-1,j}, & \textit{if } i<n \textit{ and } i \textit{ is even},\\ \sum_{j=1}^{n-1}b_{n-1,j}, & i=n,\\ 0, & \textit{otherwise}.\end{cases}$$

In order to solve these recurrence relations, we define $A_n(v)=\sum_{i=1}^{n}a_{n,i}v^{i-1}$ and $B_n(v)=\sum_{i=1}^{n}b_{n,i}v^{i-1}$. Rewriting the above recurrence relations in terms of polynomials $A_n(v)$ and $B_n(v)$ yields

$$\begin{aligned}
A_{2n+1}(v) &= v^{2n}A_{2n}(1)+\tfrac{1}{1-v^2}(A_{2n}(v)-v^{2n+1}A_{2n}(1))+B_{2n}(v),\\
A_{2n+2}(v) &= \tfrac{1}{1-v^2}(A_{2n+1}(v)-v^{2n+3}A_{2n+1}(1))-\tfrac{v^{2n}}{1+v}A_{2n}(1)+B_{2n+1}(v)-v^{2n}B_{2n}(1),\\
B_{2n+1}(v) &= v^{2n}B_{2n}(1)+\tfrac{1}{1-v^2}(B_{2n}(v)-v^{2n+1}B_{2n}(1))+A_{2n}(v),\\
B_{2n+2}(v) &= \tfrac{1}{1-v^2}(B_{2n+1}(v)-v^{2n+3}B_{2n+1}(1))-\tfrac{v^{2n}}{1+v}B_{2n}(1)+A_{2n+1}(v)-v^{2n}A_{2n}(1).
\end{aligned}$$

If we define

$$\begin{aligned}
OA(x,v)&=\textstyle\sum_{n\ge0}A_{2n+1}(v)x^{2n+1}, \quad EA(x,v)=\sum_{n\ge1}A_{2n}(v)x^{2n},\\
OB(x,v)&=\textstyle\sum_{n\ge0}B_{2n+1}(v)x^{2n+1}, \quad EB(x,v)=\sum_{n\ge1}B_{2n}(v)x^{2n},
\end{aligned}$$

then we have

$$OA(x,v) = x + xEA(xv,1) + \frac{x}{1-v^2}(EA(x,v) - vEA(xv,1)) + xEB(x,v), \tag{3.3}$$

$$EA(x,v) = \frac{x}{1-v^2}(OA(x,v) - v^2 OA(xv,1)) - \frac{x^2}{1+v}(1 + EA(xv,1)) + xOB(x,v) - x^2 EB(xv,1), \tag{3.4}$$

$$OB(x,v) = xEB(xv,1) + \frac{x}{1-v^2}(EB(x,v) - vEB(xv,1)) + xEA(x,v), \tag{3.5}$$

$$EB(x,v) = \frac{x}{1-v^2}(OB(x,v) - v^2 OB(xv,1)) - \frac{x^2}{1+v}EB(xv,1) + x(OA(x,v) - x) - x^2 EA(xv,1). \tag{3.6}$$

Although we can deal with the above equation system directly, the computation time is unacceptably long. Therefore, we solve the system by using the following process which is much faster.

Taking the difference of equations (3.3) *and* (3.5), (3.4) *and* (3.6), *respectively, we obtain a system of equations on* $f_1(x,v) = OA(x,v) - OB(x,v)$ *and* $g_1(x,v) = EA(x,v) - EB(x,v)$*:*

$$\begin{aligned} f_1(x,v) + \left(x - \frac{x}{1-v^2}\right) g_1(x,v) &= \left(x - \frac{xv}{1-v^2}\right) g_1(xv,1) + x, \\ \left(x - \frac{x}{1-v^2}\right) f_1(x,v) + g_1(x,v) &= -\frac{xv^2}{1-v^2} f_1(xv,1) + \left(x^2 - \frac{x^2}{1+v}\right) g_1(xv,1) \\ &\quad + \left(x^2 - \frac{x^2}{1+v}\right). \end{aligned}$$

Eliminating $g_1(x,v)$ *and substituting* $x = x/v$ *in the above equations, our Maple program yields*

$$f_1(x,1) = \frac{x}{1-x^2}, \quad g_1(x,1) = \frac{x^2}{1-x^2}. \tag{3.7}$$

Now taking the summation of equations (3.3) *and* (3.5), (3.4) *and* (3.6), *respectively, we obtain a system of equations on* $f_2(x,v) = OA(x,v) + OB(x,v)$ *and* $g_2(x,v) = EA(x,v) + EB(x,v)$*:*

$$\begin{aligned} f_2(x,v) - \left(x + \frac{x}{1-v^2}\right) g_2(x,v) &= \left(x - \frac{xv}{1-v^2}\right) g_2(xv,1) + x, \\ -\left(x + \frac{x}{1-v^2}\right) f_2(x,v) + g_2(x,v) &= -\frac{xv^2}{1-v^2} f_2(xv,1) - \left(x^2 + \frac{x^2}{1+v}\right) g_2(xv,1) \\ &\quad - \left(x^2 + \frac{x^2}{1+v}\right). \end{aligned}$$

Our Maple program yields

$$(1-3x^2)f_2^3+(4x+x^3)f_2^2-(1+2x^2)f_2+x=0, \tag{3.8}$$

$$(1-3x^2)g_2^3-(2+7x^2)g_2^2+(1-5x^2)g_2-x^2=0, \tag{3.9}$$

where $f_2 = f_2(x,1)$ *and* $g_2 = g_2(x,1)$. *From equations* (3.7), (3.8) *and* (3.9), *we can easily derive the equations that* $OA(x,1), EA(x,1), OB(x,1)$ *and* $EB(x,1)$ *satisfy. For example, substituting* $f_2(x,1) = 2OA(x,1) - x/(1-x^2)$ *into* (3.8), *we obtain*

$$(-12x^8+40x^6-48x^4+24x^2-4)f^3+(2x^9-16x^7+24x^5-8x^3-2x)f^2 + (-5x^4+4x^2+1)f-x=0,$$

where $f = OA(x,1)$. *The few first terms are*

$$OA(x,1) = x+2x^3+13x^5+134x^7+1617x^9+21106x^{11}+\cdots.$$

Similarly, $g = EA(x,1)$ *satisfies*

$$(24x^8-80x^6+96x^4-48x^2+8)g^3+(64x^8-160x^6+120x^4-16x^2-8)g^2 + (56x^8-104x^6+54x^4-8x^2+2)g+16x^8-20x^6+8x^4-2x^2=0,$$

and a few first terms are

$$EA(x,1) = x^2+4x^4+35x^6+400x^8+5071x^{10}+68268x^{12}+\cdots.$$

3.2. **Generating trees.** A generating tree is an infinite rooted tree, which is essentially a process for generating labels from a single label of the root by successively applying certain rules. Formally speaking, a generating tree consists of the label of the root and the succession rules. The use of generating tree in combinatorial problems, which was systematized in [13,14], has received much attention in the last decade; see [2] and references therein. For instance, the number t_n of permutations $\pi = a_1a_2\cdots a_n$ of length n such that there is no $a_i < a_j < a_k$ with $1 \le i < j < k \le n$ is equal to the number of vertices at level n in the generating tree

$$\begin{array}{l}(1)\\ (p) \to (2)(3)\cdots(p+1).\end{array}$$

A standard method of finding an explicit formula for the number of vertices in the n-th level of the tree is given by translating the rules to functional equations that the generating functions satisfy. In our example, we define $t_{n,p}$ to be the number of vertices labelled with p in the n-th level of the tree, and let

$$f(x,v) = \sum_{n,p\ge 0} t_{n,p}x^nv^p = \sum_{p\ge 1} f_p(x)v^p.$$

Then, the above rule gives

$$f(x,v) = v + x\sum_{p\ge 1} f_p(x)(v^2+v^3+\cdots+v^{p+1}) = v + \frac{xv^2}{1-v}\sum_{p\ge 1} f_p(x)(1-v^p),$$

which implies that

$$f(x,v) = v + \frac{xv^2}{1-v}(f(x,1)-f(x,v)).$$

This equation can be solved systemically using the kernel method [7]. In this case by assuming $v = \frac{1-\sqrt{1-4x}}{2x}$, we obtain

$$f(x,1) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n.$$

That is, $t_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number; see [12, Page 219 and Exercise 6.19]. In this subsection, we present the following example in which our methods are used to find the number of vertices in the n-th level of the generating tree.

Example 3.3. *Let T be the following generating tree:*

$$\begin{array}{l} (0) \\ (kp+j) \rightarrow (j+1)(k+j+1)\cdots(kp+j+1), \quad j = 0,1,\ldots,k-2, \\ (kp+k-1) \rightarrow (1)(2)\cdots(kp+k). \end{array}$$

Define $A(x,u) = \sum_{q\geq 0} A_q(x)u^q$ to be the generating function for the number of vertices labelled with q in the n-th level of T. Transforming the above rules to ones in terms of generating functions we arrive at

$$A(x,u) = 1 + x\sum_{j=0}^{k-2}\sum_{p\geq 0} A_{kp+j}\frac{u^{j+1}-u^{kp+k+j+1}}{1-u^k} + x\sum_{p\geq 0} A_{kp+k-1}\frac{u-u^{kp+k+1}}{1-u}.$$

If we denote the generating function

$$\sum_{p\geq 0} A_{kp+j}(x)u^{kp+j}$$

by $A_j(x,u)$ for all $j = 0,1,\ldots,k-1$, we obtain

(3.10)

$$A(x,u) = 1 + \frac{x}{1-u^k}\sum_{j=0}^{k-2}(u^{j+1}A_j(x,1) - u^{k+1}A_j(x,u)) + \frac{x}{1-u}(uA_{k-1}(x,1) - u^2A_{k-1}(x,u)).$$

For $k = 1$ $(A(x,u) = A_0(x,u))$, (3.10) becomes

$$A(x,u) = 1 + \frac{xu}{1-u}A(x,1) - \frac{xu^2}{1-u}A(x,u).$$

Substituting $u = u/x$, our Maple program obtains

$$A(x,1) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n.$$

For $k = 2$, we obtain

$$A(x,u) = 1 + \frac{x}{1-u^2}(uA_0(x,1) - u^3A_0(x,u)) + \frac{x}{1-u}(uA_1(x,1) - u^2A_1(x,u)),$$

i.e.,

$$\left(1 + \frac{xu^3}{1-u^2}\right)A_0(x,u) + \left(1 + \frac{xu^2}{1-u}\right)A_1(x,u) = \frac{xu}{1-u^2}A_0(x,1) + \frac{xu}{1-u}A_1(x,1) + 1.$$

Substituting $-u$ for u in the above equation, we obtain another equation:

$$\left(1 - \frac{xu^3}{1-u^2}\right)A_0(x,u) - \left(1 + \frac{xu^2}{1+u}\right)A_1(x,u) = -\frac{xu}{1-u^2}A_0(x,1) - \frac{xu}{1+u}A_1(x,1) + 1.$$

Substituting u/x for u, our Maple program yields

$$x^2f^3 + xf^2 - (x+1)f + 1 = 0, \quad x^2g^3 + (x^2-2x)g^2 + (1-x)g - x = 0,$$

where $f = A_0(x,1)$ and $g = A_1(x,1)$. The few first terms are

$$\begin{aligned} A_0(x,1) &= 1 + x^2 + x^3 + 4x^4 + 8x^5 + 25x^6 + 64x^7 + 191x^8 + \cdots, \\ A_1(x,1) &= x + x^2 + 3x^3 + 6x^4 + 17x^5 + 43x^6 + 123x^7 + 343x^8 + 1004x^9 + \cdots. \end{aligned}$$

For the case $p > 2$, our Maple program can not help much since it runs out of memory after hours of computation.

Acknowledgments. we are grateful to the referees for their detailed and thoughtful comments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

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