# A scale-free network with limiting on vertices

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**Abstract.** We propose and analyze a random graph model which explains a phenomena in the economic company network in which company may not expand its business at some time due to the limiting of money and capacity etc.. The random graph process is defined as follows: at any time-step $t$, (i) with probability $\alpha(k)$ and independently of other time-step, each vertex $v_i$ $(i \leq t-1)$ is inactive which means it cannot being connected by more edges, where $k$ is the degree of $v_i$ at the time-step $t$; (ii) a new vertex $v_t$ is added along with $m$ edges incident with $v_t$ at one time and its neighbors are chosen in the manner of preferential attachment. We prove that the degree distribution $P(k)$ of this random graph process satisfies $P(k) \propto C_1 k^{-\frac{3-\alpha_0}{1-\alpha_0}}$ if $\alpha(\cdot)$ is a constant $\alpha_0$; and $P(k) \propto C_2 k^{-3}$ if $\alpha(\ell) \downarrow 0$ as $\ell \uparrow \infty$, where $C_1, C_2$ are two positive constants. The analytical result is found to be in good agreement with that obtained by numerical simulations. Furthermore, we get the degree distributions in this model with $m$-varying functions by simulation.

**PACS:** 02.50.Ey, 89.75.-k, 05.10.-a;

**Keywords:** Scale free; Random graph process, Inactive.

## 1 Introduction

In 1959, Erdös and Rényi [1] introduced a random graph model $G(n,p)$, whose vertices are $[n]$ and every two vertices are connected independently with probability $p$. The degree sequence of $G(n,p)$ is of Poisson distribution. However, people find that most networks are scale-free from the empirical evidences in the Internet and WWW or other complex networks. So in order to model real-world networks, Barabási and Albert [2] proposed the classical Barabási-Albert model at the end of last century. Since then, there has been a flourish in modeling real-world networks. It is well-known that many real-world networks such as economic companies, social networks, and the World Wide Web (internet) etc. can be modeled by different random evolving graphs. These networks have many similar properties such as power law degree (which is one of basic properties of real-world networks), small world phenomena and clustering. See [2-6] for the degree sequences in the Barabási-Albert model, [7-8] for the degree distribution in models including the mechanisms of random deletion of vertices. For other types of degree distributions on random graphs, see [9-11].

Our research is mainly motivated by the following observations. There are two mechanisms in the classical Barabási-Albert model or some other random graph processes: successive additions of new vertices and certain preference in linking to existing nodes during the evolving process. Note that in these existing network models, the vertices being still in network can always be connected by more edges at any time step. However, it may be considered more. In fact, we notice that there is a phenomena in some real-world networks in which the vertices' degrees cannot be increased at some

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time $t$ due to some reasons. Take company network for example. In this network, every company is denoted as a vertex, if every two companies do business with each other, then the two vertices are connected. At some time $t$, a company cannot expand the business (i.e., the corresponding vertex cannot receive more edges) due to some reasons such as limiting of the money, capacity etc.; while at next time, it can receive more edges again as people always try to make some effects to improve the company. As far as we know, there is no random graph process to model such a phenomena. In this paper, we introduce the following random graph processes $G_t$ to model this phenomena.

Assume graph $G_t = (V_t, E_t)$, $e_t = |E_t|$. For simplification, we start the process at the time-step 3. Take $G_1$ consisting of vertices $v_1, v_2$ with $2m$ edges between them. At time-step $t \geq 3$ :

(i) Every vertex $w \in G_{t-1}$ may be active or inactive. The inactive means its degree cannot be increased. Suppose vertex $w$ is inactive with probability $\alpha(d_w(t-1))$ independently of its status and other points' statuses before time-step $t$. Here $d_w(t-1)$ denotes the degree of $w$ in $G_{t-1}$, $\alpha(d_w(t-1))$ is a non-increasing function with respect to $d_w(t-1)$. The reason that we use the non-increasing function is that in real-world network the vertex of higher degree is likely to be more active.

(ii) A new vertex $v_t$ is added to $G_{t-1}$ along with $m$ edges incident with $v_t$ at one time. The random neighbors $w_1, w_2, \cdots, w_m$ are chosen independently conditioned on $w_i$s are active with probability

$$P(w_i v_t \in E_t) = \frac{d_{w_i}(t-1)}{2e_{t-1}}.$$

Otherwise the edges of $v_t$ are connected to itself. We take the denominator $2e_{t-1} = \sum_{v \in G_{t-1}} d_v(t-1)$ rather than $\sum_{\substack{v\ is\ active \\ v \in G_{t-1}}} d_v(t-1)$ due to that in some economic society, the companies which cannot expand their business at some time $t-1$ would also affect the decision of the new set-up company (namely the vertex $v_t$). For example, in an investment network, although some companies cannot expand its business at some time $t-1$, the capital or the market share of the companies cannot be ignored by the new company that enters the investment market at the time $t$. Because the capital or market share of just mentioned companies at time $t-1$ will affect the new company's decision obviously.

**Remark. (i)** When $\alpha(\cdot) \equiv 0$, the above model is just the well-known Barabási-Albert model.

**(ii)** Fix $\alpha \equiv \alpha_0$. From the model, we have

$$\sum_i P(w_i v_t \in E_t) = \sum_i \frac{(1-\alpha_0) d_{w_i}(t-1)}{2e_{t-1}} = 1 - \alpha_0.$$

Thus the new vertex connected to itself with probability $\alpha_0$, which can be understood as follows. When the inactive probability $\alpha_0$ is large, it means that the market is bad. In this case, it is much difficult to do business for the new company. So the new set-up company will not do business with other companies. To connect to itself means to make itself stronger to fight in the market in future. The larger $\alpha_0$ is, the worse the market is.

**(iii)** The authors [12-14] consider Barabási-Albert model with $m$-varying functions, which are of interests as empirical evidences have shown that the number of edges grows faster than the number of vertices in many networks. Note that our model is just Barabási-Albert model when $\alpha = 0$. So as a contrast, we simulated our model with $m$-varying functions. For $m$ varying in time $t$ or following a certain distribution, we simulate some special cases (see Figure 5-8).

# 2 Degree distribution

We use the master equation approach which was introduced by Dorogovtsev, Mendes and Samukhin [15] to get the degree distributions. It is easy to see that at any time step $t$, the number of vertices and edges in the network are $t$ and $mt$ respectively. Let $p(k;t_i,t)$ be the probability of vertex $v$ (which is added at time $t_i$) is of degree $k$ at the time-step $t$.

## 2.1. The case $\alpha(\cdot) \equiv \alpha_0$.

For the case $\alpha \equiv \alpha_0 < 1$, from the definition of the model, when a new node with $m$ edges enters the system, the degree of active node $w_i$ of degree $k$ at time step $t-1$ increases with 1 with probability $\frac{k}{2t}(1-\alpha_0)$, and stays the same with probability $\left(1-\frac{k}{2t}\right)(1-\alpha_0)$. Thus we have

$$p(k;t_i,t+1) = \frac{k-1}{2t}(1-\alpha_0)p(k-1;t_i,t) + \left(1-\frac{k}{2t}\right)(1-\alpha_0)p(k;t_i,t) + \alpha_0 p(k;t_i,t).$$

Namely,

$$p(k;t_i,t+1) = \frac{k-1}{2t}(1-\alpha_0)p(k-1;t_i,t) + \left(1-\frac{k}{2t}(1-\alpha_0)\right)p(k;t_i,t). \tag{2.1}$$

Let

$$P(k) = \lim_{t\to\infty}\frac{\sum_{t_i} p(k;t_i,t)}{t}.$$

Then (2.1) implies that $P(k)$ is the solution to the following recursive equation:

$$P(k) = \begin{cases} \frac{(k-1)(1-\alpha_0)}{k(1-\alpha_0)+2}P(k-1), & k \geq m+1; \\ \frac{2}{m(1-\alpha_0)+2}, & k = m. \end{cases} \tag{2.2}$$

Therefore,

$$\begin{aligned} P(k) &= \frac{k-1}{k+2/(1-\alpha_0)}P(k-1) \\ &= \prod_{i=m+1}^{k}\frac{i-1}{i+2/(1-\alpha_0)}P(m) \\ &= \frac{\Gamma(k)\Gamma(m+2/(1-\alpha_0))}{\Gamma(k+1+2/(1-\alpha_0)))\Gamma(m-1)}P(m), \end{aligned}$$

where $\Gamma(t)$ denotes the Gamma function, i.e.,

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx,\ t \in [0,+\infty);$$

and we have used the recursion of formula

$$\Gamma(t+1) = t\Gamma(t),\ t \in [0,+\infty).$$

Notice that

$$\frac{\Gamma(t+a)}{\Gamma(t)} = t^a(1+O(1/t)).$$

It is easy to obtain that for some positive constant $C$,

$$P(k) \propto Ck^{-1-2/(1-\alpha_0)} \propto Ck^{-\frac{3-\alpha_0}{1-\alpha_0}}. \tag{2.3}$$

## 2.2. The case $\alpha(\ell) \downarrow 0$ as $\ell \uparrow \infty$.

As vertices of higher degree are likely to be less inactive in real-world networks, we consider the case $\alpha(\ell) \downarrow 0$ as $\ell \uparrow \infty$ in this subsection.

Similarly to (2.1),

$$p(k;t_i,t+1) = \frac{k-1}{2t}(1-\alpha(k-1))p(k-1;t_i,t) + \left(1-\frac{k}{2t}\right)(1-\alpha(k))p(k;t_i,t) + \alpha(k)p(k;t_i,t).$$

Thus we have that

$$P(k) = \begin{cases} \frac{(k-1)(1-\alpha(k-1))}{k(1-\alpha(k))+2} P(k-1), & k \geq m+1; \\ \frac{2}{m(1-\alpha(m))+2}, & k = m. \end{cases} \tag{2.4}$$

Let $f(k) = k - k\alpha(k)$, and $C(m) = \prod_{i=1}^{m-1} \frac{2+f(i)}{f(i)}$. Then by (2.4), we have

$$P(k) = \prod_{i=m}^{k} \frac{f(i-1)}{2+f(i)} = \frac{C(m)}{f(k)} \prod_{i=1}^{k} \frac{f(i)}{2+f(i)} = \frac{C(m)}{f(k)} \prod_{i=1}^{k} \left(1+\frac{2}{f(i)}\right)^{-1}.$$

Since as $k \to \infty$, $\alpha(k) \to 0$ and

$$\log \prod_{i=1}^{k} \left(1+\frac{2}{f(i)}\right)^{-1} \sim -\sum_{i=1}^{k} \frac{2}{f(i)} = -\sum_{i=1}^{k} \frac{2}{i - i\alpha(i)} \sim -2\log k,$$

we obtain

$$\prod_{i=1}^{k} \left(1+\frac{2}{f(i)}\right)^{-1} \sim k^{-2}.$$

Therefore

$$P(k) \propto \frac{C(m)}{f(k)} k^{-2} \propto C(m) k^{-3}. \tag{2.5}$$

# 3 Numerical Analysis

In order to check the analytical results, numerical simulations were performed. In all figures, that $P(k)$ vs $k$ is in log-log plot.

In Figure 1, we simulate the case $m = 1$ and $\alpha = 0.3$ for $t = 50000, 100000, 150000$ and $200000$ to see whether the number of vertices (when it is large) affects the degree distribution or not. We can see that when $t$ is large enough, the degree distribution is independent of $t$.

In Figure 2, we simulate the cases $\alpha = 0.3$ and $t = 200000$ to see whether the concrete $m$ (constant) affects the degree distribution or not. From the simulation, we come to a conclusion that the concrete $m$ (constant) does not affect the degree distribution. Thus we fix $m = 1$ in the following Figures 3-4.

In Figure 3, we simulate the cases of constants $\alpha_0 = 0.01,\ 0.1,\ 0.3, 0.9$. For the case of $\alpha = 0.9$, by noting the dots around the black line in the right figure, we can see that it is scale-free. The figure for the case of $\alpha = 0.9$ is different from those of $\alpha = 0.01, 0.1, 0.3$. We think it is due to the fact that the number of vertices is relatively too small (note the exponent is 21).

In Figure 4, we do the cases with $\alpha(k) \downarrow 0$ as $k \uparrow \infty$, in which we choose $\alpha(k) = \frac{1}{k^2}, \frac{1}{k^{0.5}}, \frac{1}{e^k}, \frac{1}{1+\log^2 k}$ respectively.

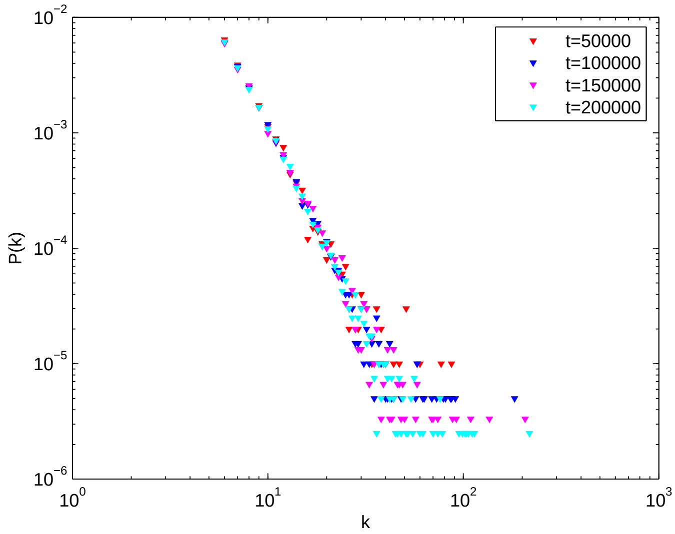


Figure 1: The $P(k)$ vs $k$ in log-log plot for different $t$ with $m = 1$ and $\alpha = 0.3$.

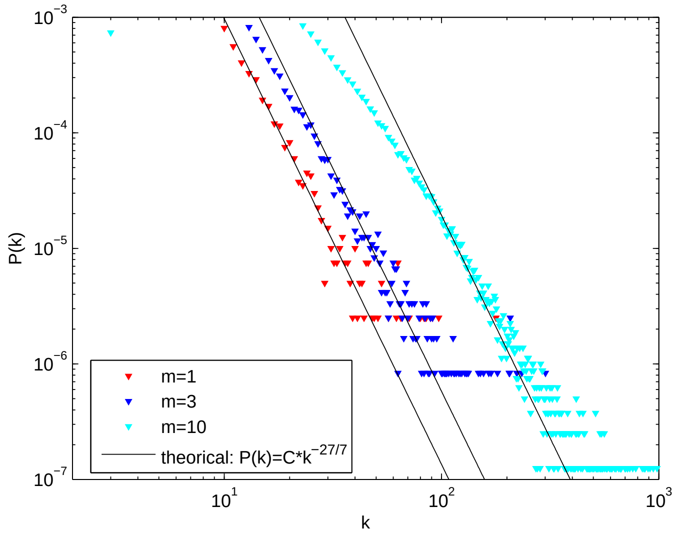


Figure 2: The $P(k)$ vs $k$ in log-log plot for different constant $m$. Numerical simulations by choosing $\alpha = 0.3$ and $t = 200000$. The slope of the dashed line is $-\frac{27}{7}$ from (2.3). Data are averaged over 10 independent runs.

We see that the numerical simulations are well consistent with analytical results (see (2.3), (2.5)).

The authors [12-14] consider Barabási-Albert model with $m$-varying functions, which are of interests as empirical evidences have shown that the number of edges grows faster than the number of vertices in many networks. And as far as we know, in $m$-vary Barabási-Albert model, when $m$ varies with respect to $t$ or follows a certain distribution, there have been no analysis results for the degree

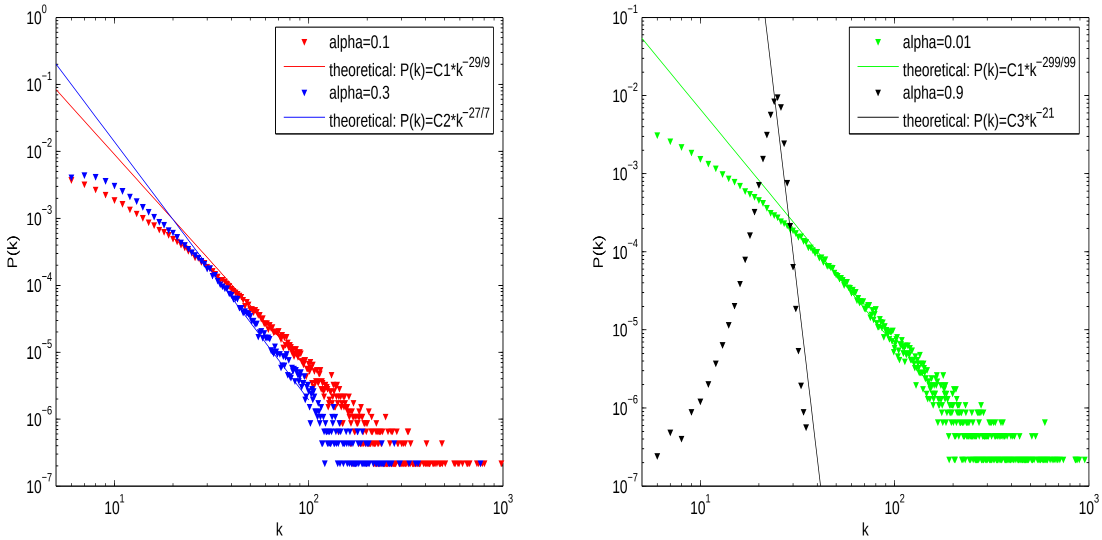


Figure 3: We simulate the cases $m = 1$, $t = 200000$ for different $\alpha$. We take $\alpha = 0.01,\ 0.1,\ 0.3,\ 0.9$. The $P(k)$ vs $k$ in log-log plot. The case of $\alpha = 0.9$ is due to that the number of vertices is not enough for the exponent 21.

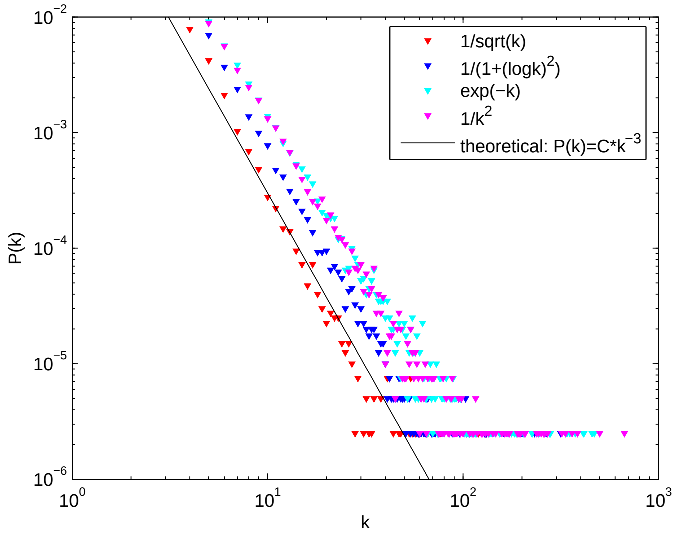


Figure 4: We simulate the cases $m = 1$, $t = 200000$ for different functions $\alpha(k)$. We take $\alpha(k) = \frac{1}{k^2}, \frac{1}{k^{0.5}},\ \frac{1}{e^k},\ \frac{1}{1+\log^2 k}$. The slope of the line is $-3$ from (2.5). Data are averaged over 5 independent runs. The $P(k)$ vs $k$ in log-log plot.

distribution except the special $m$-varying functions $m = \lceil A \cdot t^a \rceil$ and $m = \lceil B \cdot \ln t \rceil$, where $A$, $B$ and $a$ are three positive numbers and $\lceil x \rceil$ denote the smallest natural number larger than $x$. It seems very difficult to analyze these cases, as we can see that it is even not easy to get the number of edges $G_t$ when $m$ is a function with respect to $t$ or follows a certain distribution. Note that our model is just Barabási-Albert model when $\alpha = 0$. So as a contrast, we simulated our model with $m$-varying functions. i.e. we simulate the following process: at first there are two vertices and two edges between them, then the process evolving just like step $(i)$ and $(ii)$, where $m$ is a function.

For the Figures 5-8, we fix $\alpha = 0.3$, $t = 200000$ and take $m = \left[t^{0.1}\right], \left[t^{0.2}\right], [\log t]$ or follows normal distribution $N\left(\mu, \sigma^2\right)$, Poisson distribution $Poi(\lambda)$ respectively. It seems that, in these cases, the corresponding networks are also scale-free (It may be non-stationary [13, 14]). Let us talk something more about the simulation results.

In Figure 5, we think about whether the increasing rate of $m$ with respect to $t$ can affect the degree distribution or not. It seems that it does from the simulation. This can be understood intuitively from [13, 14], noting that in our model the case when $\alpha = 0$ is just the $m$-vary model in [13, 14].

In Figure 7, we consider the cases when $m$ follows normal distribution $N(\mu, \sigma^2)$. First, we fixed $\mu$ to see whether $\sigma^2$ can affect the degree distribution or not. Secondly, we checked $\mu$. From the results, it seems $\mu$ and $\sigma^2$ cannot affect the degree distribution.

In Figure 8, it seems that the exponent $\lambda$ in Poisson distribution $Posi(\lambda)$ cannot affect the degree distribution.

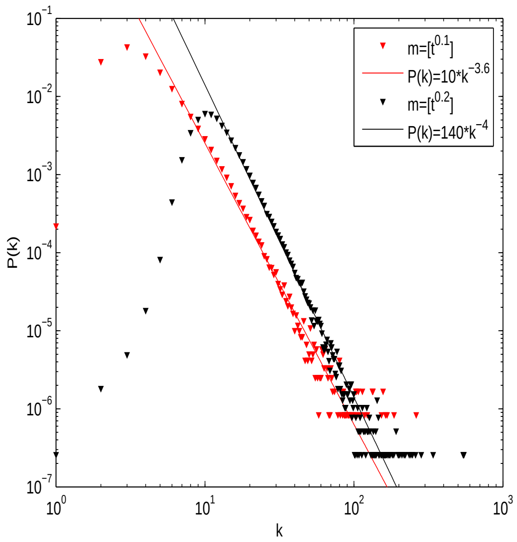


Figure 5: The cases $m = \left[t^{0.1}\right], \left[t^{0.2}\right]$

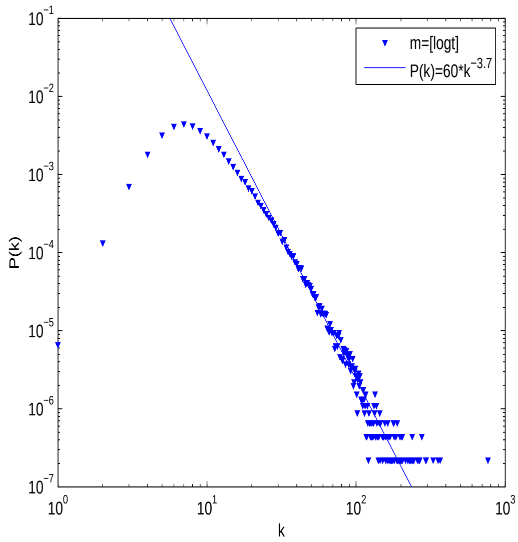


Figure 6: The case $m = [\log t]$

# 4 Conclusion

In order to model the limiting on vertices in the network during the evolving process, we treat the limiting as being inactive and introduced a random graph process model. By the master equation, we prove the corresponding network is scale-free.

Our model is simple as we only consider that the vertices are always added in the evolving process. In fact, in real-world networks, vertices can also be preferentially deleted. Our next step is to study evolving networks including the mechanisms that at any time vertices not only have limiting but also may disappear; see [16].

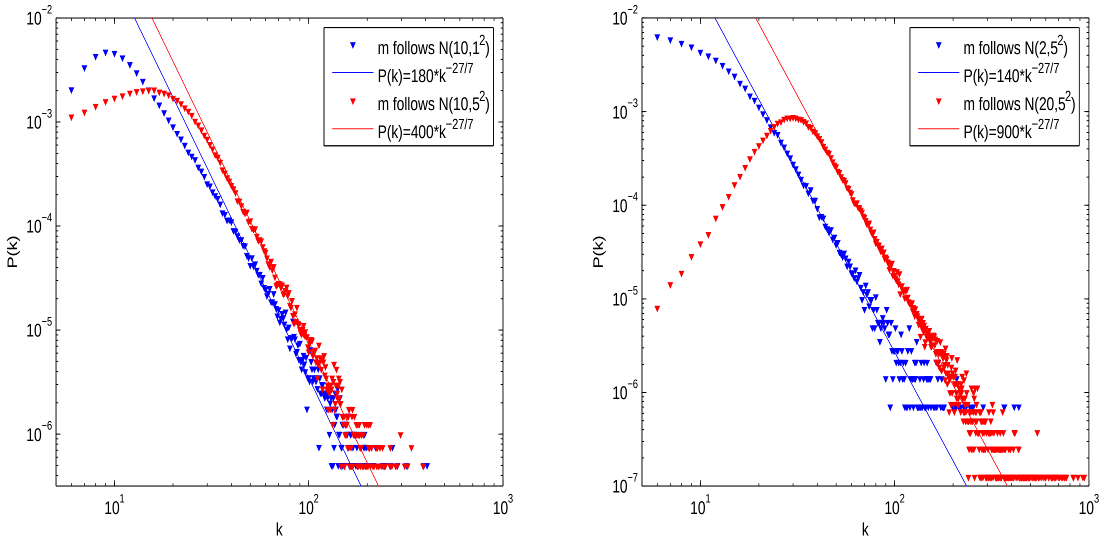


Figure 7: We simulate the case $m$ follows normal distribution $N(\mu, \sigma^2)$. In the left Figure we fixed $\mu$ to see whether $\sigma$ can effect the degree distribution or not. And in the right Figure we fixed $\sigma$ to see whether $\mu$ can effect the degree distribution or not. The slopes of the lines are both $-\frac{27}{7}$. The $P(k)$ vs $k$ in log-log plot.

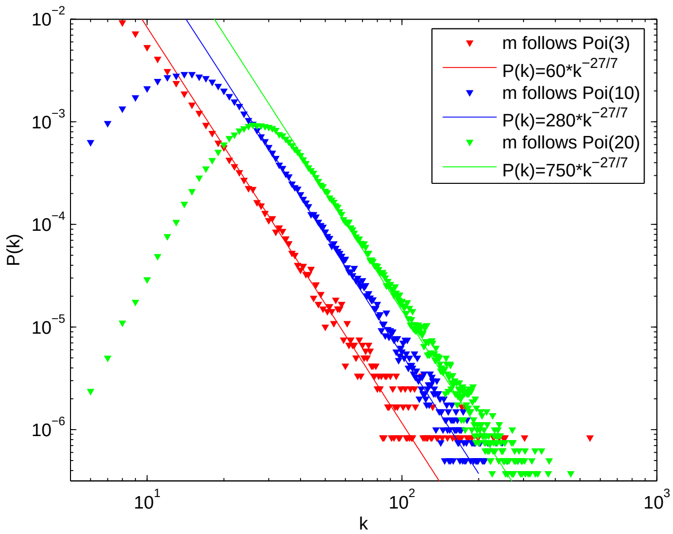


Figure 8: We simulate the case $m$ follows Poisson distribution $Poi(\lambda)$. We take $\lambda = 3, 10, 20$ respectively. The slopes of all the lines are $-\frac{27}{7}$. The $P(k)$ vs $k$ in log-log plot.

## Acknowledgement

We thank the anonymous referees and Prof. Xiang Kai-Nan for their careful reading and valuable comments which improve the presentation of this manuscript greatly. This research was supported partially by CNNSF(No.10971106) and PCSIRT.

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