Ratio Monotonicity of Polynomials Derived from Nondecreasing Sequences

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Abstract

The ratio monotonicity of a polynomial is a stronger property than log-concavity. Let $P(x)$ be a polynomial with nonnegative and nondecreasing coefficients. We prove the ratio monotone property of $P(x+1)$, which leads to the log-concavity of $P(x+c)$ for any $c \geq 1$ due to Llamas and Martínez-Bernal. As a consequence, we obtain the ratio monotonicity of the Boros-Moll polynomials obtained by Chen and Xia without resorting to the recurrence relations of the coefficients.

Keywords: log-concavity, ratio monotonicity, Boros-Moll polynomials.

1 Introduction

This paper is concerned with the ratio monotone property of polynomials derived from nonnegative and nondecreasing sequences. A sequence $\{a_k\}_{0\leq k\leq m}$ of positive real numbers is said to be unimodal if there exists an integer $r \geq 0$ such that

$$a_0 \leq \cdots \leq a_{r-1} \leq a_r \geq a_{r+1} \geq \cdots \geq a_m,$$

and it is said to be spiral if

$$a_m \leq a_0 \leq a_{m-1} \leq a_1 \leq \cdots \leq a_{[\frac{m}{2}]}, \tag{1.1}$$

where $[\frac{m}{2}]$ stands for the largest integer not exceeding $\frac{m}{2}$. We say that a sequence $\{a_k\}_{0\leq k\leq m}$ is log-concave if for any $1 \leq k \leq m-1$,

$$a_k^2 - a_{k+1}a_{k-1} \geq 0,$$

or equivalently,

$$\frac{a_0}{a_1} \leq \frac{a_1}{a_2} \leq \cdots \leq \frac{a_{m-1}}{a_m}.$$

It is easy to see that either log-concavity or the spiral property implies unimodality, while a log-concave sequence is not necessarily spiral, and vice versa.

A stronger property, which implies both log-concavity and the spiral property, was introduced by Chen and Xia [6] and is called the ratio monotonicity. A sequence of positive real numbers $\{a_k\}_{0\leq k\leq m}$ is said to be ratio monotone if

$$\frac{a_m}{a_0} \leq \frac{a_{m-1}}{a_1} \leq \cdots \leq \frac{a_{m-i}}{a_i} \leq \cdots \leq \frac{a_{m-[\frac{m-1}{2}]}}{a_{[\frac{m-1}{2}]}} \leq 1 \tag{1.2}$$

and

$$\frac{a_0}{a_{m-1}} \leq \frac{a_1}{a_{m-2}} \leq \cdots \leq \frac{a_{i-1}}{a_{m-i}} \leq \cdots \leq \frac{a_{[\frac{m}{2}]-1}}{a_{m-[\frac{m}{2}]}} \leq 1. \tag{1.3}$$

Given a polynomial $P(x) = a_0 + a_1x + \cdots + a_mx^m$ with positive coefficients, we say that $P(x)$ is log-concave (or ratio monotone) if $\{a_k\}_{0\leq k\leq m}$ is log-concave (resp., ratio monotone).

Assume that $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [3] proved the unimodality of $P(x+1)$ which implies the unimodality of the Boros-Moll polynomials. They posed the conjecture that the Boros-Moll polynomials are log-concave, which was confirmed by Kauers and Paule [8]. Alvarez et al. [1] showed that $P(x+n)$ is also unimodal for any positive integer n. Wang and Yeh [12] obtained a stronger result that $P(x+c)$ is unimodal for $c > 0$. Llamas and Martínez-Bernal [9] proved that $P(x+c)$ is log-concave for $c \geq 1$.

In this paper, we prove that if $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then $P(x+1)$ is ratio monotone. This property implies the log-concavity of $P(x+1)$. Note that by a criterion for log-concavity due to Brenti [5], the log-concavity of $P(x+1)$ leads to the log-concavity of $P(x+c)$ for $c \geq 1$, as established by Llamas and Martínez-Bernal [9]. The ratio monotonicity of $P(x+1)$ serves as a simple proof of the ratio monotonicity of the Boros-Moll polynomials obtained by Chen and Xia [7] without resorting to the recurrence relations of the coefficients.

2 The ratio monotone property

The main result of this paper is given below.

Theorem 2.1 *If $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then $P(x+1)$ is ratio monotone.*

To prove Theorem 2.1, we need three lemmas. The first lemma is a special case of [6, Lemma 2.1].

Lemma 2.2 *Suppose that a, b, c, d, e, f are positive real numbers satisfying*

$$\frac{a}{b} \le \frac{c}{d} \le \frac{e}{f}.$$

Then

$$\frac{a+c}{b+d} \le \frac{e+c}{f+d}.$$

Lemma 2.3 *If $B(x)$ is a ratio monotone polynomial, so is $(x+1)B(x)$.*

Proof. Let

$$B(x) = \sum_{k=0}^{m} a_k x^k \quad \text{and} \quad (x+1)B(x) = \sum_{k=0}^{m+1} b_k x^k.$$

For each k we have $b_k = a_{k-1} + a_k$, where a_{-1} and a_{m+1} are set to 0.

When $m = 2n$, the ratio monotonicity of $B(x)$ states that

$$\frac{a_{2n}}{a_0} \le \frac{a_{2n-1}}{a_1} \le \cdots \le \frac{a_{2n-i}}{a_i} \le \cdots \le \frac{a_{n+1}}{a_{n-1}} \le 1 \tag{2.1}$$

and

$$\frac{a_0}{a_{2n-1}} \le \frac{a_1}{a_{2n-2}} \le \cdots \le \frac{a_{i-1}}{a_{2n-i}} \le \cdots \le \frac{a_{n-1}}{a_n} \le 1. \tag{2.2}$$

In order to show that $(x+1)B(x)$ is ratio monotone, we need to verify that

$$\frac{b_{2n+1}}{b_0} \le \frac{b_{2n}}{b_1} \le \cdots \le \frac{b_{2n+1-i}}{b_i} \le \cdots \le \frac{b_{n+1}}{b_n} \le 1 \tag{2.3}$$

and

$$\frac{b_0}{b_{2n}} \le \frac{b_1}{b_{2n-1}} \le \cdots \le \frac{b_i}{b_{2n-i}} \le \cdots \le \frac{b_{n-1}}{b_{n+1}} \le 1. \tag{2.4}$$

We first consider (2.3). Since

$$\frac{a_{2n}}{a_0} \le \frac{a_{2n-1}}{a_1},$$

we see that

$$\frac{a_{2n}}{a_0} \le \frac{a_{2n-1}+a_{2n}}{a_1+a_0},$$

that is,

$$\frac{b_{2n+1}}{b_0} \le \frac{b_{2n}}{b_1}.$$

For $1 \le i \le n-1$, from (2.1) we deduce that

$$\frac{a_{2n+1-i}}{a_{i-1}} \le \frac{a_{2n-i}}{a_i} \le \frac{a_{2n-i-1}}{a_{i+1}}.$$

By Lemma 2.2, we obtain

$$\frac{a_{2n+1-i}+a_{2n-i}}{a_i+a_{i-1}} \le \frac{a_{2n-i}+a_{2n-i-1}}{a_{i+1}+a_i},$$

or equivalently,

$$\frac{b_{2n+1-i}}{b_i} \leq \frac{b_{2n-i}}{b_{i+1}}.$$

In light of (2.1), we see that $a_{n+1} \leq a_{n-1}$, and thus we have

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1} + a_n}{a_n + a_{n-1}} \leq 1.$$

Next, we proceed to prove (2.4). Since $\frac{a_0}{a_{2n-1}} \leq \frac{a_1}{a_{2n-2}}$, we get that

$$\frac{a_0}{a_{2n-1} + a_{2n}} \leq \frac{a_1 + a_0}{a_{2n-2} + a_{2n-1}},$$

that is,

$$\frac{b_0}{b_{2n}} \leq \frac{b_1}{b_{2n-1}}.$$

For $2 \leq i \leq n-1$, in view of (2.2) we find that

$$\frac{a_{i-2}}{a_{2n-i+1}} \leq \frac{a_{i-1}}{a_{2n-i}} \leq \frac{a_i}{a_{2n-i-1}}.$$

By Lemma 2.2, we have

$$\frac{a_{i-1} + a_{i-2}}{a_{2n-i+1} + a_{2n-i}} \leq \frac{a_i + a_{i-1}}{a_{2n-i} + a_{2n-i-1}},$$

which can be expressed as

$$\frac{b_{i-1}}{b_{2n-i+1}} \leq \frac{b_i}{b_{2n-i}}.$$

From (2.2) it is clear that $a_{n-2} \leq a_{n+1}$ and $a_{n-1} \leq a_n$, and hence

$$\frac{b_{n-1}}{b_{n+1}} = \frac{a_{n-1} + a_{n-2}}{a_{n+1} + a_n} \leq 1.$$

The case $m = 2n+1$ can be dealt with in the same manner. This completes the proof. ∎

The third lemma is concerned with an inequality of increasing positive sequences.

Lemma 2.4 *For any nondecreasing positive sequence* $\{a_k\}_{0\leq k\leq m}$*, we have*

$$\frac{m(m+1)}{2}a_m^2 + a_m a_{m-1} \geq \left(\sum_{k=0}^{m-2}(m-1-k)\,a_k\right)a_{m-1} + \left(\sum_{k=0}^{m} a_k\right)a_{m-2}.$$

Proof. Since $0 < a_0 \le a_1 \le \cdots \le a_{m-1} \le a_m$, we have

$$\begin{aligned} &\frac{m(m+1)}{2} a_m^2 + a_m a_{m-1} - \left(\sum_{k=0}^{m-2} (m-1-k)\, a_k \right) a_{m-1} - \left(\sum_{k=0}^{m} a_k \right) a_{m-2} \\ \ge\ & \frac{m(m+1)}{2} a_m^2 + a_m a_{m-1} - \sum_{k=0}^{m-2} (m-1-k) a_m^2 - \sum_{k=1}^{m} a_m^2 - a_m a_{m-1}, \end{aligned}$$

which simplifies to zero, as desired. ∎

Proof of Theorem 2.1. We use induction on the degree m of $P(x)$. Let

$$P(x) = \sum_{k=0}^{m} a_k x^k,$$

where $0 < a_0 \le a_1 \le \cdots \le a_{m-1} \le a_m$.

When $m = 2$, we have

$$P(x+1) = a_2 x^2 + (a_1 + 2a_2)x + a_0 + a_1 + a_2.$$

Note that $a_2 \le a_0 + a_1 + a_2$, $a_0 + a_1 + a_2 \le a_1 + 2a_2$. Therefore, the theorem holds for $m = 2$.

Now assume that the theorem holds for polynomials of degree $m-1$. We need to show that it is also true for polynomials $P(x)$ of degree m. Suppose that

$$P(x+1) = \sum_{k=0}^{m} a_k (x+1)^k = \sum_{k=0}^{m} d_k x^k. \tag{2.5}$$

We wish to prove that

$$\frac{d_m}{d_0} \le \frac{d_{m-1}}{d_1} \le \cdots \le \frac{d_{m-i}}{d_i} \le \cdots \le \frac{d_{m-[\frac{m-1}{2}]}}{d_{[\frac{m-1}{2}]}} \le 1 \tag{2.6}$$

and

$$\frac{d_0}{d_{m-1}} \le \frac{d_1}{d_{m-2}} \le \cdots \le \frac{d_{i-1}}{d_{m-i}} \le \cdots \le \frac{d_{[\frac{m}{2}]-1}}{d_{m-[\frac{m}{2}]}} \le 1. \tag{2.7}$$

Let

$$Q(x) = \sum_{k=0}^{m-1} a_{k+1} x^k.$$

Then

$$P(x+1) = a_0 + (x+1)Q(x+1).$$

By the induction hypothesis and Lemma 2.3, we deduce that the polynomial

$$(x+1)Q(x+1) = d_0 - a_0 + \sum_{k=1}^{m} d_k x^k$$

is ratio monotone. It follows that

$$\frac{d_m}{d_0 - a_0} \leq \frac{d_{m-1}}{d_1} \leq \cdots \leq \frac{d_{m-i}}{d_i} \leq \cdots \leq \frac{d_{m-[\frac{m-1}{2}]}}{d_{[\frac{m-1}{2}]}} \leq 1 \tag{2.8}$$

and

$$\frac{d_0 - a_0}{d_{m-1}} \leq \frac{d_1}{d_{m-2}} \leq \cdots \leq \frac{d_{i-1}}{d_{m-i}} \leq \cdots \leq \frac{d_{[\frac{m}{2}]-1}}{d_{m-[\frac{m}{2}]}} \leq 1. \tag{2.9}$$

Clearly, (2.6) follows from (2.8). To prove (2.7), it remains to show that

$$\frac{d_0}{d_{m-1}} \leq \frac{d_1}{d_{m-2}}.$$

From (2.5), we see that

$$d_0 = \sum_{k=0}^{m} a_k, \qquad d_{m-1} = a_{m-1} + m a_m,$$

and

$$d_1 = \sum_{k=0}^{m} k a_k, \qquad d_{m-2} = a_{m-2} + (m-1)a_{m-1} + \binom{m}{2} a_m.$$

Consequently, it suffices to show that

$$\frac{\sum_{k=0}^{m} a_k}{a_{m-1} + m a_m} \leq \frac{\sum_{k=0}^{m} k a_k}{a_{m-2} + (m-1)a_{m-1} + \binom{m}{2} a_m},$$

or equivalently,

$$\left(\sum_{k=0}^{m} k a_k\right) a_{m-1} + \left(\sum_{k=0}^{m} m k a_k\right) a_m - \left(\sum_{k=0}^{m} a_k\right) a_{m-2} - \left(\sum_{k=0}^{m} (m-1) a_k\right) a_{m-1} - \left(\sum_{k=0}^{m} \binom{m}{2} a_k\right) a_m \geq 0.$$

The left hand side of the above inequality can be simplified to

$$\left(\sum_{k=0}^{m} \frac{2k-m+1}{2} a_k\right) m a_m + \left(\sum_{k=0}^{m} (k-m+1)\, a_k\right) a_{m-1} - \left(\sum_{k=0}^{m} a_k\right) a_{m-2},$$

which can be rewritten as a sum of

$$\left(\sum_{k=0}^{m-1} \frac{2k-m+1}{2} a_k\right) m a_m \tag{2.10}$$

and

$$\frac{m(m+1)}{2}a_m^2 + a_m a_{m-1} - \left(\sum_{k=0}^{m-2}(m-1-k)\,a_k\right)a_{m-1} - \left(\sum_{k=0}^{m}a_k\right)a_{m-2}. \tag{2.11}$$

By Lemma 2.4, the sum in (2.11) is nonnegative. The sum in (2.10) is also nonnegative, since

$$\begin{aligned}\sum_{k=0}^{m-1}\frac{2k-m+1}{2}a_k &= \sum_{k=[\frac{m-1}{2}]+1}^{m-1}\frac{2k-m+1}{2}a_k - \sum_{k=0}^{[\frac{m-1}{2}]}\frac{m-1-2k}{2}a_k\\ &= \sum_{k=0}^{m-2-[\frac{m-1}{2}]}\frac{m-1-2k}{2}a_{m-1-k} - \sum_{k=0}^{[\frac{m-1}{2}]}\frac{m-1-2k}{2}a_k\\ &= \sum_{k=0}^{[\frac{m-1}{2}]}\frac{m-1-2k}{2}(a_{m-1-k}-a_k),\end{aligned}$$

which is nonnegative, and thus the proof is complete. ∎

Theorem 2.1 leads to the following result of Llamas and Martínez-Bernal [9], since the ratio monotonicity implies log-concavity of $P(x+1)$ and the log-concavity of $P(x+1)$ implies the log-concavity of $P(x+c)$ for $c \geq 1$ by a criterion of Brenti [4, 5].

Corollary 2.5 *If $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then for any $c \geq 1$ the polynomial $P(x+c)$ is log-concave and has no internal zero coefficients.*

Theorem 2.1 also serves as a simple proof of the ratio monotonicity of the Boros-Moll polynomials $P_m(x)$, which were introduced by Boros and Moll [2] in their study of the following quartic integral

$$\int_0^{+\infty}\frac{1}{(t^4+2xt^2+1)^{m+1}}dt = \frac{\pi}{2^{m+3/2}(x+1)^{m+1/2}}P_m(x).$$

Let

$$c_k(m) = 2^{-2m+k}\binom{2m-2k}{m-k}\binom{m+k}{k}.$$

Boros and Moll showed that

$$P_m(x) = \sum_{k=0}^{m}c_k(m)(x+1)^k. \tag{2.12}$$

They also observed that, for $0 \leq k \leq m-1$,

$$\frac{c_k(m)}{c_{k+1}(m)} = \frac{(2m-2k-1)(k+1)}{(m-k)(m+k+1)} < 1.$$

Thus, $P_m(x-1)$ is a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [2] proved that $P_m(x)$ is unimodal for any $m \geq 0$, and Moll [10] conjectured that $P_m(x)$ is log-concave for any m. This conjecture was confirmed by Kauers and Paule [8]. The ratio monotonicity of $P_m(x)$ was established by Chen and Xia and the proof is quite involved and heavily depends on inequalities on the coefficients. The proof of Theorem 2.1 shows that the log-concavity and ratio monotonicity only depend on the nondecreasing property of the coefficients of $P_m(x-1)$.

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