ON SECOND AND EIGHTH ORDER MOCK THETA FUNCTIONS

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Abstract. Mock theta functions have been deeply studied in the literature. Historically, there are many forms of representations for mock theta functions: Eulerian forms, Hecke-type double sums, Appell-Lerch sums, and Fourier coefficients of meromorphic Jacobi forms. In this paper, we first establish Hecke-type double sums for the second and eighth order mock theta functions by Bailey's lemma and a Bailey pair given by Andrews and Hickerson. Meanwhile, we give different proofs of the generalized Lambert series for the mock theta functions $A(q)$, $U_0(q)$, and $U_1(q)$. Furthermore, using Ramanujan's ${}_1\psi_1$ summation formula and a ${}_2\psi_2$ transformation formula due to Bailey, we prove some identities related to these mock theta functions.

1. introduction

Throughout this paper, let q denote a complex number with $|q| < 1$. Here and in what follows, we adopt the standard q-series notation [11]. For $n > 0$,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$
$$(a;q)_{-n} := \frac{q^{\binom{n}{2}}(-q/a)^n}{(q/a;q)_n}, \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a_1,a_2,\cdots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

For convenience, we use $(a)_n$ to denote $(a;q)_n$. Jacobi's triple product identity is stated as follows.

$$j(x;q) := (x)_\infty(q/x)_\infty(q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}} x^n. \tag{1.1}$$

In his last letter to G.H. Hardy, Ramanujan [18] first introduced mock theta functions which were divided into four classes: one class of third order, two of fifth order, and one of seventh order. In that letter, he listed seventeen functions which are defined by q-series convergent for $|q| < 1$ with a complex variable q, and called these functions as "mock theta functions". He stated that they have certain asymptotic properties as q approaches a root of unity, which are similar to theta functions, but they are not really theta functions. In addition, Ramanujan gave some relations holding in each family and these relations were proved in [20, 21]. Furthermore, other mock theta functions with

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orders 2, 6, 8, and 10 were studied in [5, 12, 15, 19]. For more on mock theta functions, see [13]. In particular, Gordon and McIntosh [12] defined the following eighth order mock theta functions.

$$S_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n},\qquad S_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n},$$
$$T_0(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},\qquad T_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$
$$U_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n},\qquad U_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}},$$
$$V_0(q)=-1+2\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}=-1+2\sum_{n=0}^{\infty}\frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}},\tag{1.2}$$
$$V_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}=\sum_{n=0}^{\infty}\frac{q^{n+1}(-q;q)_{2n}}{(-q^2;q^4)_{n+1}}.\tag{1.3}$$

Later, McIntosh [15] studied the second order mock theta functions.

$$A(q)=\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2},\tag{1.4}$$
$$B(q)=\sum_{n=0}^{\infty}\frac{q^{n}(-q;q^2)_n}{(q;q^2)_{n+1}}=\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2},\tag{1.5}$$
$$\mu(q)=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}.$$

Notice that $\mu(q)$ appeared in Ramanujan's "Lost" notebook [19]. In addition, Gordon and McIntosh [12] and McIntosh [15] established some identities related to these mock theta functions aforementioned.

Historically, there are many forms of representations for mock theta functions: Eulerian forms, Hecke-type double sums, Appell-Lerch sums, and Fourier coefficients of meromorphic Jacobi forms. The Hecke-type double sums are defined as follows.

Definition 1.1. *Let $x,\ y\in\mathbb{C}^*:=\mathbb{C}\backslash\{0\}$ and define sg(r) $:=1$ for $n\geq 0$ and sg(r) $:=-1$ for $n<0$. Then*

$$f_{a,b,c}(x,y,q):=\sum_{sg(r)=sg(s)}sg(r)(-1)^{r+s}x^ry^sq^{a\binom{r}{2}+brs+c\binom{s}{2}}.$$

It is clear that

$$f_{a,b,a}(x,y,q)=f_{a,b,a}(y,x,q).$$

The definition of Appell-Lerch sums is stated as follows.

Definition 1.2. *Let x, $z \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ with neither z nor xz an integral power of q. Then*

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}.$$

Andrews [2] first established Hecke-type double sums for the fifth and seventh order mock theta functions by applying Bailey's lemma and Bailey pairs. Then Andrews and Hickerson [5] gave Hecke-type double sums for the sixth order mock theta functions. Around 2000, Choi [8, 9] discussed the tenth order mock theta functions. Moreover, Hecke-type double sums for the third order mock theta functions are found in [3, 7, 10, 14, 17]. The first object of this paper is to establish Hecke-type double sums for the second and eighth order mock theta functions.

Theorem 1.3. *We have*

$$j(q;q^2)A(q) = qf_{1,3,1}(q^5,q^3,q^2), \tag{1.6}$$

$$j(q;q^2)B(q) = f_{1,3,1}(q^3,q^3,q^2), \tag{1.7}$$

$$j(-q^2;q^8)\mu(q) = f_{1,3,1}(-q^2,-q^2,q^2) - qf_{1,3,1}(-q^4,-q^4,q^2). \tag{1.8}$$

Theorem 1.4. *We have*

$$j(q;-q)S_0(q) = 2f_{1,3,1}(q,-q,-q), \tag{1.9}$$

$$j(q;-q)S_1(q) = 2f_{1,3,1}(q^2,-q^2,-q), \tag{1.10}$$

$$j(q^2;q^4)T_0(q) = q^5f_{1,3,1}(q^{10},q^{12},q^4) - q^4f_{1,3,1}(q^8,q^{14},q^4), \tag{1.11}$$

$$j(q^2;q^4)T_1(q) = f_{1,3,1}(q^4,q^6,q^4) + q^7f_{1,3,1}(q^{12},q^{14},q^4), \tag{1.12}$$

$$j(q^2;-q^2)U_0(q) = 2f_{1,3,1}(q^2,-q^2,-q^2) + 2qf_{1,3,1}(q^4,-q^4,-q^2), \tag{1.13}$$

$$j(q;-q^2)U_1(q) = qf_{1,3,1}(-q^3,q^5,-q^2), \tag{1.14}$$

$$j(q;q^4)V_0(q) = f_{1,3,1}(-q,-q^2,-q), \tag{1.15}$$

$$2j(q;q^4)V_1(q) = qf_{1,3,1}(-q^2,-q^3,-q). \tag{1.16}$$

Notice that Garvan [10, Eq. (1.25)] provided another proof of (1.8) by using universal mock theta functions. Furthermore, with the aid of Ramanujan's ${}_1\psi_1$ summation formula and a ${}_2\psi_2$ transformation formula due to Bailey [6], we prove the following identities for these mock theta functions. In [12, 13], Gordon and McIntosh provided different proofs of (1.17)-(1.19).

Theorem 1.5. *We have*

$$B(q) + B(-q) = 2(-q^2;q^2)_\infty^4(q^4;q^4)_\infty, \tag{1.17}$$

$$V_0(q) + V_0(-q) = 2(-q^2;q^4)_\infty^4(q^8;q^8)_\infty, \tag{1.18}$$

$$V_1(q) - V_1(-q) = 2q(-q^2;q^2)_\infty(-q^4;q^4)_\infty^2(q^8;q^8)_\infty, \tag{1.19}$$

$$2S_0(q^2) + 4T_0(q^2) = U_0(q) + U_0(-q) + 2U_1(q) + 2U_1(-q), \tag{1.20}$$

$$2qS_1(q^2) + 4qT_1(q^2) = U_0(q) - U_0(-q) + 2U_1(q) - 2U_1(-q). \tag{1.21}$$

2. Proofs of Theorems 1.3 and 1.4

In this section, applying the Bailey pair due to Andrews and Hickerson [5], we establish Hecke-type double sums for the second and eighth order mock theta functions. In addition, we give different proofs for the generalized Lambert series for $A(q)$, $U_0(q)$, and $U_1(q)$.

Definition 2.1. *The sequences (α_n, β_n) are called a Bailey pair relative to (a, q) if (α_n, β_n) satisfy*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

Bailey's lemma says that if (α_n, β_n) is a Bailey pair relative to a, then

$$\sum_{n\geq 0}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n = \frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty}\sum_{n\geq 0}\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n, \quad (2.1)$$

provided both sums converge absolutely.

Lemma 2.2. *[5, Theorem 2.3] Let a, b, c, and q be complex numbers with $a \neq 1$, $b \neq 0$, $c \neq 0$, $q \neq 0$, and none of a/b, a/c, qb, and qc of the form q^{-k} with $k \geq 0$. For $n \geq 0$, define*

$$A'_n(a,b,c,q) = \frac{q^{n^2}(bc)^n(1-aq^{2n})(a/b)_n(a/c)_n}{(1-a)(qb)_n(qc)_n}\sum_{j=0}^{n}\frac{(-1)^j(1-aq^{2j-1})(a)_{j-1}(b)_j(c)_j}{q^{j(j-1)/2}(bc)^j(q)_j(a/b)_j(a/c)_j} \quad (2.2)$$

and

$$B'_n(a,b,c,q) = \frac{1}{(qb)_n(qc)_n}. \quad (2.3)$$

Then the sequences $\{A'_n(a,b,c,q)\}$ and $\{B'_n(a,b,c,q)\}$ form a Bailey pair relative to a.

The limiting case of $\{A'_n(a,b,c,q)\}$ with $n \geq 1$ in Lemma 2.2 when $a = 1$ was given in [5, Eq. (2.16)].

$$\begin{aligned} A'_n(1,b,c,q) &= \frac{(-1)^n q^{\binom{n+1}{2}}(1+q^n)(1-b)(1-c)}{(1-q^n b)(1-q^n c)} \\ &+ \frac{q^{n^2}(bc)^n(1-q^{2n})(1/b)_n(1/c)_n(bc-1)}{(qb)_n(qc)_n} \\ &\times\left(\frac{1}{(1-b)(1-c)} + \sum_{j=1}^{n-1}\frac{(-1)^j(b)_j(c)_j(1+q^j)}{q^{\binom{j}{2}}(bc)^{j+1}(1/b)_{j+1}(1/c)_{j+1}}\right) \end{aligned} \quad (2.4)$$

and

$$A'_0(1,b,c,q) = 1. \quad (2.5)$$

Based on Lemma 2.2, we derive the following results.

Lemma 2.3. *The following sequences form Bailey pairs:*

$$A'_n(q^4,-q,q,q^2)=\frac{(-1)^nq^{2n^2+2n}(1-q^{4n+4})}{(1-q^4)}\sum_{j=0}^{n}q^{-j^2-j}, \tag{2.6}$$

$$B'_n(q^4,-q,q,q^2)=\frac{1}{(-q^3;q^2)_n(q^3;q^2)_n}; \tag{2.7}$$

$$A'_n(q^2,-1,-q,q^2)=\frac{q^{2n^2+n}(1-q^{2n+1})}{1-q}\sum_{j=-n}^{n}(-1)^jq^{-j^2}, \tag{2.8}$$

$$B'_n(q^2,-1,-q,q^2)=\frac{1}{(-q^2;q^2)_n(-q^3;q^2)_n}; \tag{2.9}$$

$$A'_n(q^2,-1,0,q^2)=\frac{q^{3n^2+n}(1-q^{4n+2})}{1-q^2}\sum_{j=-n}^{n}(-1)^jq^{-2j^2}, \tag{2.10}$$

$$B'_n(q^2,-1,0,q^2)=\frac{1}{(-q^2;q^2)_n}; \tag{2.11}$$

$$A'_n(q^4,-q,0,q^2)=\frac{q^{3n^2+4n}(1+q)(1-q^{4n+4})}{(1-q^2)(1-q^4)}\sum_{j=-n-1}^{n}(-1)^jq^{-2j^2-3j}, \tag{2.12}$$

$$B'_n(q^4,-q,0,q^2)=\frac{1}{(-q^3;q^2)_n}; \tag{2.13}$$

$$A'_n(q^2,-q,0,q^2)=\frac{q^{3n^2+2n}(1+q)(1-q^{2n+1})}{1-q^2}\sum_{j=-n}^{n}(-1)^jq^{-2j^2-j}, \tag{2.14}$$

$$B'_n(q^2,-q,0,q^2)=\frac{1}{(-q^3;q^2)_n}; \tag{2.15}$$

$$A'_n(1,-1,0,q^2)=q^{3n^2+n}\sum_{j=-n}^{n}(-1)^jq^{-2j^2}-q^{3n^2-n}\sum_{j=-n+1}^{n-1}(-1)^jq^{-2j^2}, \tag{2.16}$$

$$B'_n(1,-1,0,q^2)=\frac{1}{(-q^2;q^2)_n}; \tag{2.17}$$

$$A'_n(1,-q^{-1},0,q^2)=q^{3n^2+2n}\sum_{j=-n}^{n}(-1)^jq^{-2j^2-j}-q^{3n^2-2n}\sum_{j=-n+1}^{n-1}(-1)^jq^{-2j^2-j}, \tag{2.18}$$

$$B'_n(1,-q^{-1},0,q^2)=\frac{1}{(-q;q^2)_n}. \tag{2.19}$$

Proof. First, by means of Lemma 2.2, we obtain (2.6)-(2.15). Next, we prove the last two Bailey pairs. With the aid of (2.3)-(2.5) with replacing b, c, and q by -1, 0, and q^2, respectively, we have

$$\begin{aligned}&A'_n(1,-1,0,q^2)\\&=2(-1)^nq^{n(n+1)}-2q^{3n^2-n}(1-q^{2n})\left(\frac{1}{2}+\sum_{j=1}^{n-1}(-1)^jq^{-2j^2}\right)\end{aligned}$$

$$
\begin{aligned}
&= 2(-1)^n q^{n(n+1)} - q^{3n^2-n}(1-q^{2n}) - 2q^{3n^2-n}(1-q^{2n})\sum_{j=1}^{n-1}(-1)^j q^{-2j^2} \\
&= 2(-1)^n q^{n(n+1)} - q^{3n^2-n}(1-q^{2n}) - 2q^{3n^2-n}\sum_{j=1}^{n-1}(-1)^j q^{-2j^2} \\
&\qquad + 2q^{3n^2+n}\sum_{j=1}^{n-1}(-1)^j q^{-2j^2} \\
&= 2q^{3n^2+n}\sum_{j=1}^{n}(-1)^j q^{-2j^2} - q^{3n^2-n}(1-q^{2n}) - 2q^{3n^2-n}\sum_{j=1}^{n-1}(-1)^j q^{-2j^2} \\
&= q^{3n^2+n}\sum_{\substack{j=-n \\ j\neq 0}}^{n}(-1)^j q^{-2j^2} - q^{3n^2-n} + q^{3n^2+n} - q^{3n^2-n}\sum_{\substack{j=-n+1 \\ j\neq 0}}^{n-1}(-1)^j q^{-2j^2} \\
&= q^{3n^2+n}\sum_{j=-n}^{n}(-1)^j q^{-2j^2} - q^{3n^2-n}\sum_{j=-n+1}^{n-1}(-1)^j q^{-2j^2}
\end{aligned}
$$

and

$$B'_n(1,-1,0,q^2) = \frac{1}{(-q^2;q^2)_n}.$$

Similarly, in light of (2.3)-(2.5) with replacing b, c, and q by $-q^{-1}$, 0, and q^2, respectively, we derive that

$$
\begin{aligned}
A'_n(1,-q^{-1},0,q^2) = \frac{(-1)^n q^{n^2+n}(1+q^{2n})(1+q^{-1})}{1+q^{2n-1}} - q^{3n^2-2n}(1-q^{4n}) \\
\times\left(\frac{q}{1+q} + \sum_{j=1}^{n-1}\frac{(-1)^j q^{-2j^2+j}(1+q)(1+q^{2j})}{(1+q^{2j-1})(1+q^{2j+1})}\right)
\end{aligned}
$$

and

$$B'_n(1,-q^{-1},0,q^2) = \frac{1}{(-q;q^2)_n}.$$

Let

$$A := \frac{q}{1+q} + \sum_{j=1}^{n-1}\frac{(-1)^j q^{-2j^2+j}(1+q)(1+q^{2j})}{(1+q^{2j-1})(1+q^{2j+1})}.$$

Since

$$\frac{(1+q)(1+q^{2j})}{(1+q^{2j-1})(1+q^{2j+1})} = \frac{1}{1+q^{2j-1}} + \frac{q}{1+q^{2j+1}},$$

we have

$$A = \frac{q}{1+q} + \sum_{j=1}^{n-1}\frac{(-1)^j q^{-2j^2+j}}{1+q^{2j-1}} + \sum_{j=1}^{n-1}\frac{(-1)^j q^{-2j^2+j+1}}{1+q^{2j+1}}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n-1} \frac{(-1)^j q^{-2j^2+j}}{1+q^{2j-1}} + \sum_{j=0}^{n-1} \frac{(-1)^j q^{-2j^2+j+1}}{1+q^{2j+1}} \\
&= \sum_{j=1}^{n-1} \frac{(-1)^j q^{-2j^2+j}}{1+q^{2j-1}} - \sum_{j=1}^{n} \frac{(-1)^j q^{-2j^2+5j-2}}{1+q^{2j-1}} \\
&= \sum_{j=1}^{n-1} \frac{(-1)^j q^{-2j^2+j}(1-q^{4j-2})}{1+q^{2j-1}} - \frac{(-1)^n q^{-2n^2+5n-2}}{1+q^{2n-1}} \\
&= \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+j}(1-q^{2j-1}) - \frac{(-1)^n q^{-2n^2+5n-2}}{1+q^{2n-1}}.
\end{aligned}
$$

Then we derive that

$$
\begin{aligned}
&A_n'(1,-q^{-1},0,q^2) \\
&= (-1)^n q^{n^2+n} + \frac{(-1)^n q^{n^2+n}(q^{-1}+q^{2n})}{1+q^{2n-1}} + \frac{(-1)^n q^{n^2+3n-2}(1-q^{4n})}{1+q^{2n-1}} \\
&\qquad - q^{3n^2-2n}(1-q^{4n}) \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+j}(1-q^{2j-1}) \\
&= (-1)^n q^{n^2+n} + \frac{(-1)^n q^{n^2+n-1} + (-1)^n q^{n^2+3n} + (-1)^n q^{n^2+3n-2} - (-1)^n q^{n^2+7n-2}}{1+q^{2n-1}} \\
&\qquad - \left(q^{3n^2-2n} - q^{3n^2+2n}\right) \left(\sum_{j=1}^{n-1} (-1)^j q^{-2j^2+j} - \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+3j-1} \right) \\
&= (-1)^n q^{n^2+n} + (-1)^n q^{n^2+n-1} + (-1)^n q^{n^2+3n}(1-q^{2n-1}) \\
&\qquad - q^{3n^2-2n} \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+j} + q^{3n^2+2n} \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+j} \\
&\qquad + q^{3n^2-2n} \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+3j-1} - q^{3n^2+2n} \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+3j-1} \\
&= (-1)^n q^{n^2+n} - q^{3n^2-2n} \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+j} + q^{3n^2+2n} \sum_{j=1}^{n} (-1)^j q^{-2j^2+j} \\
&\qquad + q^{3n^2-2n} \sum_{j=1}^{n} (-1)^j q^{-2j^2+3j-1} - q^{3n^2+2n} \sum_{j=1}^{n} (-1)^j q^{-2j^2+3j-1} \\
&= (-1)^n q^{n^2+n} - q^{3n^2-2n} \sum_{j=1}^{n-1} (-1)^j q^{-2j^2+j} + q^{3n^2+2n} \sum_{j=1}^{n} (-1)^j q^{-2j^2+j} \\
&\qquad - q^{3n^2-2n} \sum_{j=0}^{n-1} (-1)^j q^{-2j^2-j} + q^{3n^2+2n} \sum_{j=0}^{n-1} (-1)^j q^{-2j^2-j}
\end{aligned}
$$

$$= -q^{3n^2-2n}\sum_{j=1}^{n-1}(-1)^j q^{-2j^2+j} + q^{3n^2+2n}\sum_{j=1}^{n}(-1)^j q^{-2j^2+j}$$
$$- q^{3n^2-2n}\sum_{j=0}^{n-1}(-1)^j q^{-2j^2-j} + q^{3n^2+2n}\sum_{j=0}^{n}(-1)^j q^{-2j^2-j},$$

which implies (2.18). We compete the proof. □

Next, we review some properties for Hecke-type double sums which were given in Propositions 6.1, 6.2, and Corollary 6.4 of [14].

Lemma 2.4. *[14] For $x, y \in \mathbb{C}^*$,*

$$\begin{aligned} f_{a,b,c}(x,y,q) &= f_{a,b,c}(-x^2q^a, -y^2q^c, q^4) - xf_{a,b,c}(-x^2q^{3a}, -y^2q^{c+2b}, q^4) \\ &\quad - yf_{a,b,c}(-x^2q^{a+2b}, -y^2q^{3c}, q^4) \\ &\quad + xyq^b f_{a,b,c}(-x^2q^{3a+2b}, -y^2q^{3c+2b}, q^4), \end{aligned} \tag{2.20}$$

$$f_{a,b,c}(x,y,q) = -\frac{q^{a+b+c}}{xy} f_{a,b,c}(q^{2a+b}/x, q^{2c+b}/y, q), \tag{2.21}$$

$$f_{a,b,c}(x,y,q) = -yf_{a,b,c}(q^b x, q^c y, q) + j(x;q^a), \tag{2.22}$$

$$f_{a,b,c}(x,y,q) = -xf_{a,b,c}(q^a x, q^b y, q) + j(y;q^c). \tag{2.23}$$

Proof of Theorem 1.3. Substituting (2.6) and (2.7) into (2.1) with replacing q, a, ρ_1, and ρ_2 by q^2, q^4, $-q^2$, and $-q^3$, respectively, we get

$$\sum_{n=0}^{\infty} q^n(-q^2;q^2)_n(-q^3;q^2)_n B'_n(q^4,-q,q,q^2)$$
$$= \frac{(-q^3;q^2)_\infty(-q^4;q^2)_\infty}{(q^6;q^2)_\infty(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n(-q^2;q^2)_n}{(-q^4;q^2)_n} A'_n(q^4,-q,q,q^2).$$

In light of (1.4), we have

$$\begin{aligned} A(q) &= \frac{q}{j(q;q^2)}\sum_{n=0}^{\infty}(-1)^n q^{2n^2+3n}(1-q^{2n+2})\sum_{j=0}^{n} q^{-j^2-j} \\ &= \frac{q}{2j(q;q^2)}\sum_{n=0}^{\infty}(-1)^n q^{2n^2+3n}(1-q^{2n+2})\sum_{j=-n-1}^{n} q^{-j^2-j} \\ &= \frac{q}{2j(q;q^2)}\left(\sum_{n=0}^{\infty}(-1)^n q^{2n^2+3n}\sum_{j=-n-1}^{n} q^{-j^2-j} - \sum_{n=0}^{\infty}(-1)^n q^{2n^2+5n+2}\sum_{j=-n-1}^{n} q^{-j^2-j}\right) \\ &= \frac{q}{2j(q;q^2)}\left(\sum_{n=0}^{\infty}\sum_{j=-n-1}^{n}(-1)^n q^{2n^2+3n-j^2-j} - \sum_{n=2}^{\infty}\sum_{j=-n+1}^{n-2}(-1)^n q^{2n^2-3n-j^2-j}\right) \\ &= \frac{q}{2j(q;q^2)}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^n q^{2n^2+3n-j^2-j} + \sum_{n=0}^{\infty}(-1)^n q^{n^2+2n}\right. \end{aligned}$$

$$-\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n-1}(-1)^n q^{2n^2-3n-j^2-j}+\sum_{n=2}^{\infty}(-1)^n q^{n^2-2n}-q^{-1}\Bigg)$$

$$=\frac{q}{2j(q;q^2)}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^n q^{2n^2+3n-j^2-j}-\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n-1}(-1)^n q^{2n^2-3n-j^2-j}+j(q^3;q^2)\right)$$

$$=\frac{q}{2j(q;q^2)}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^n q^{2n^2+3n-j^2-j}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}(-1)^n q^{2n^2+3n-j^2-j}+j(q^3;q^2)\right)$$

$$=\frac{q}{2j(q;q^2)}\left(\left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}}-\sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^n q^{2n^2+3n-j^2-j}+j(q^3;q^2)\right). \tag{2.24}$$

Then setting $n=\frac{r+s}{2}$ and $j=\frac{r-s}{2}$ in (2.24) yields

$$A(q)=\frac{q}{2j(q;q^2)}\left(\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}}(-1)^{\frac{r+s}{2}}sg(r)q^{\frac{r^2+s^2}{4}+\frac{3rs}{2}+r+2s}+j(q^3;q^2)\right)$$
$$=\frac{q}{2j(q;q^2)}\left(f_{1,3,1}(q^5,q^3,q^2)-q^5 f_{1,3,1}(q^7,q^9,q^2)+j(q^3;q^2)\right)$$
$$=\frac{q}{j(q;q^2)}f_{1,3,1}(q^5,q^3,q^2) \quad \text{(by (2.23))},$$

which implies (1.6).

Next, substituting (2.8) and (2.9) into (2.1) with replacing q, a, ρ_1, and ρ_2 by q^2, q^2, q, and $-q^2$, respectively, we arrive at

$$\sum_{n=0}^{\infty}(-1)^n q^n (q;q^2)_n(-q^2;q^2)_n B_n'(q^2,-1,-q,q^2)$$
$$=\frac{(q;q^2)_\infty(-q^2;q^2)_\infty}{(q^4;q^2)_\infty(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{1-q^{2n+1}}A_n'(q^2,-1,-q,q^2).$$

Then based on (1.5), we have

$$B(-q)=\frac{1}{j(-q;q^2)}\sum_{n=0}^{\infty}(-1)^n q^{2n^2+2n}\sum_{j=-n}^{n}(-1)^j q^{-j^2}. \tag{2.25}$$

Multiplying 2 on both sides in (2.25), we derive

$$2B(-q)=\frac{1}{j(-q;q^2)}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^{n+j}q^{2n^2+2n-j^2}$$
$$=\frac{1}{j(-q;q^2)}\left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}}-\sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^{n+j}q^{2n^2+2n-j^2}$$

$$= \frac{1}{j(-q;q^2)} \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} (-1)^r sg(r) q^{\frac{r^2+s^2}{4}+\frac{3rs}{2}+r+s}$$

$$= \frac{1}{j(-q;q^2)} \left(f_{1,3,1}(-q^3,-q^3,q^2) - q^4 f_{1,3,1}(-q^7,-q^7,q^2)\right)$$

$$= \frac{2}{j(-q;q^2)} f_{1,3,1}(-q^3,-q^3,q^2) \quad \text{(by (2.21))}, \tag{2.26}$$

which implies (1.7) by replacing q by $-q$.

Similarly, substituting (2.8) and (2.9) into (2.1) with replacing q, a, ρ_1, and ρ_2 by q^2, q^2, $-q^3$, and q, respectively, we deduce

$$\sum_{n=0}^{\infty}(-1)^n(-q^3;q^2)_n(q;q^2)_n B'_n(q^2,-1,-q,q^2)$$
$$= \frac{(-q;q^2)_\infty(q^3;q^2)_\infty}{(q^4;q^2)_\infty(-1;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(1-q)(1+q^{2n+1})}{(1+q)(1-q^{2n+1})} A'_n(q^2,-1,-q,q^2). \tag{2.27}$$

Notice that McIntosh [15, p.287] gave that

$$U_0(-q) - 2U_1(-q) = 2\sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q^2;q^2)_n}, \tag{2.28}$$

$$U_0(-q) - 2U_1(-q) + 4A(q) = (-q;q^2)_\infty^5 (q^2;q^2)_\infty, \tag{2.29}$$

where the right-hand side of (2.28) is the average of the even and odd partial sums. In addition, Andrews [1, Eq. (3.28)] showed

$$4A(q) = (-q;q^2)_\infty^5(q^2;q^2)_\infty - \mu(-q). \tag{2.30}$$

Therefore, it can be seen from (2.28)-(2.30) that

$$\mu(-q) = 2\sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q^2;q^2)_n}. \tag{2.31}$$

Combining (2.27) and (2.31), we have

$$\mu(-q) = \frac{1}{j(-q^2;q^8)} \sum_{n=0}^{\infty}(-1)^n q^{2n^2+n}(1+q^{2n+1}) \sum_{j=-n}^{n}(-1)^j q^{-j^2}$$

$$= \frac{1}{j(-q^2;q^8)} \left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{n+j}q^{2n^2+n-j^2} + \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{n+j}q^{2n^2+3n+1-j^2} \right)$$

$$= \frac{1}{j(-q^2;q^8)} \left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{n+j}q^{2n^2+n-j^2} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}(-1)^{n+j}q^{2n^2+n-j^2} \right)$$

$$= \frac{1}{j(-q^2;q^8)} \left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}} \right)(-1)^{n+j}q^{2n^2+n-j^2}$$

$$= \frac{1}{j(-q^2;q^8)} \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} (-1)^r sg(r) q^{\frac{r^2+s^2}{4}+\frac{3rs}{2}+\frac{r+s}{2}}$$
$$= \frac{1}{j(-q^2;q^8)} \left(f_{1,3,1}(-q^2,-q^2,q^2) - q^3 f_{1,3,1}(-q^6,-q^6,q^2)\right)$$
$$= \frac{1}{j(-q^2;q^8)} \left(f_{1,3,1}(-q^2,-q^2,q^2) + q f_{1,3,1}(-q^4,-q^4,q^2)\right) \quad \text{(by (2.21))},$$

which implies (1.8) by replacing q by $-q$. Therefore, we finish the proof. □

Proof of Theorem 1.4. First, replacing q, a, ρ_1 by q^2, 1, $-q$ and letting $\rho_2 \to \infty$ in (2.1), and then applying (2.16) and (2.17), we obtain

$$S_0(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{n^2} B_n'(1,-1,0,q^2)$$
$$= \frac{2}{j(q;-q)} \sum_{n=0}^{\infty} q^{n^2} A_n'(1,-1,0,q^2)$$
$$= \frac{2}{j(q;-q)} \left(\sum_{n=0}^{\infty} q^{4n^2+n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2} - \sum_{n=1}^{\infty} q^{4n^2-n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-2j^2} \right)$$
$$= \frac{2}{j(q;-q)} \left(\sum_{n=0}^{\infty} \sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1} \sum_{j=n+1}^{-n-1} \right) (-1)^j q^{4n^2+n-2j^2}$$
$$= \frac{2}{j(q;-q)} \left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}} \right) (-1)^j q^{4n^2+n-2j^2}$$
$$= \frac{2}{j(q;-q)} \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r) (-1)^{\frac{r-s}{2}} q^{\frac{r^2+s^2}{2}+3rs+\frac{r+s}{2}}$$
$$= \frac{2}{j(q;-q)} \left(f_{1,3,1}(q^3,q^3,q^4) + q^5 f_{1,3,1}(q^{11},q^{11},q^4)\right)$$
$$= \frac{2}{j(q;-q)} f_{1,3,1}(q,-q,-q),$$

where in the last equation, we apply (2.20) with replacing q, x, and y by $-q$, q and $-q$, respectively. So we prove (1.9).

Replacing q, a, ρ_1 by q^2, q^2, $-q$ and letting $\rho_2 \to \infty$ in (2.1), and then using (2.10) and (2.11), we find that

$$S_1(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{n^2+2n} B_n'(q^2,-1,0,q^2)$$
$$= \frac{2}{j(q;-q)} \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(1+q)}{1+q^{2n+1}} A_n'(q^2,-1,0,q^2)$$

$$\begin{aligned}
&= \frac{2}{j(q;-q)}\left(\sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1})\sum_{j=-n}^{n}(-1)^j q^{-2j^2}\right)\\
&= \frac{2}{j(q;-q)}\left(\sum_{n=0}^{\infty} q^{4n^2+3n}\sum_{j=-n}^{n}(-1)^j q^{-2j^2} - \sum_{n=-\infty}^{-1} q^{4n^2+3n}\sum_{j=n+1}^{-n-1}(-1)^j q^{-2j^2}\right)\\
&= \frac{2}{j(q;-q)}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^j q^{4n^2+3n-2j^2}\\
&= \frac{2}{j(q;-q)}\left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^j q^{4n^2+3n-2j^2}\\
&= \frac{2}{j(q;-q)}\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r-s}{2}} q^{\frac{r^2+s^2}{2}+3rs+\frac{3r+3s}{2}}\\
&= \frac{2}{j(q;-q)}\left(f_{1,3,1}(q^5,q^5,q^4)+q^7 f_{1,3,1}(q^{13},q^{13},q^4)\right)\\
&= \frac{2}{j(q;-q)} f_{1,3,1}(q^2,-q^2,-q),
\end{aligned}$$

where in the last equation, we apply (2.20) with replacing q, x, and y by $-q$, q^2 and $-q^2$, respectively. Therefore, we derive (1.10).

Replacing q, a, ρ_1 by q^2, q^4, $-q^2$ and letting $\rho_2 \to \infty$ in (2.1), and then employing (2.12) and (2.13), we get

$$\sum_{n=0}^{\infty}(-q^2;q^2)_n q^{n^2+3n} B_n'(q^4,-q,0,q^2) = \frac{(-q^4;q^2)_\infty}{(q^6;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+3n}(1+q^2)}{1+q^{2n+2}} A_n'(q^4,-q,0,q^2).$$

Rewriting the above identity, we obtain

$$\begin{aligned}
T_0(q) &= \frac{1}{j(q^2;q^4)}\left(\sum_{n=0}^{\infty} q^{4n^2+7n+2}(1-q^{2n+2})\sum_{j=-n-1}^{n}(-1)^j q^{-2j^2-3j}\right)\\
&= \frac{1}{j(q^2;q^4)}\left(\sum_{n=0}^{\infty}\sum_{j=-n-1}^{n} - \sum_{n=-\infty}^{-2}\sum_{j=n+1}^{-n-2}\right)(-1)^j q^{4n^2+7n+2-2j^2-3j}\\
&= \frac{1}{j(q^2;q^4)}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^j q^{4n^2+7n+2-2j^2-3j}\\
&\quad + \frac{1}{j(q^2;q^4)}\left(\sum_{n=0}^{\infty}(-1)^{n+1}q^{2n^2+6n+3} + \sum_{n=-\infty}^{-2}(-1)^{n+1}q^{2n^2+6n+3} + q^{-1}\right)\\
&= \frac{1}{j(q^2;q^4)}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n} - \sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^j q^{4n^2+7n+2-2j^2-3j}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{j(q^2;q^4)}\left(\sum_{n=-\infty}^{\infty}(-1)^{n+1}q^{2n^2+6n+3}\right)\\
&=\frac{1}{j(q^2;q^4)}\left(\sum_{\substack{n+j\ge 0\\ n-j\ge 0}}-\sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^j q^{4n^2+7n+2-2j^2-3j}\\
&=\frac{1}{j(q^2;q^4)}\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r-s}{2}}q^{\frac{r^2+s^2}{2}+3rs+2r+5s+2}\\
&=\frac{1}{j(q^2;q^4)}\left(q^2 f_{1,3,1}(q^6,q^{12},q^4)+q^{13}f_{1,3,1}(q^{14},q^{20},q^4)\right)\\
&=\frac{1}{j(q^2;q^4)}\left(q^5 f_{1,3,1}(q^{10},q^{12},q^4)-q^4 f_{1,3,1}(q^8,q^{14},q^4)\right)\quad \text{(by (2.21) and (2.23))},
\end{aligned}
$$

where using (1.1) in the fifth step, we have

$$\sum_{n=-\infty}^{\infty}(-1)^{n+1}q^{2n^2+6n+3}=-q^3 j(q^8;q^4)=0.$$

Hence, we arrive at (1.11).

Replacing q, a, ρ_1 by q^2, q^2, $-q^2$ and letting $\rho_2\to\infty$ in (2.1), and then invoking (2.14) and (2.15), we derive

$$\sum_{n=0}^{\infty}(-q^2;q^2)_n q^{n^2+n}B_n'(q^2,-q,0,q^2)=\frac{(-q^2;q^2)_\infty}{(q^4;q^2)_\infty}\sum_{n=0}^{\infty}q^{n^2+n}A_n'(q^2,-q,0,q^2).$$

Rearranging the above identity yields that

$$
\begin{aligned}
T_1(q)&=\frac{1}{j(q^2;q^4)}\left(\sum_{n=0}^{\infty}q^{4n^2+3n}(1-q^{2n+1})\sum_{j=-n}^{n}(-1)^j q^{-2j^2-j}\right)\\
&=\frac{1}{j(q^2;q^4)}\left(\sum_{n=0}^{\infty}q^{4n^2+3n}\sum_{j=-n}^{n}(-1)^j q^{-2j^2-j}-\sum_{n=-\infty}^{-1}q^{4n^2+3n}\sum_{j=n+1}^{-n-1}(-1)^j q^{-2j^2-j}\right)\\
&=\frac{1}{j(q^2;q^4)}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}\right)(-1)^j q^{4n^2+3n-2j^2-j}\\
&=\frac{1}{j(q^2;q^4)}\left(\sum_{\substack{n+j\ge 0\\ n-j\ge 0}}-\sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^j q^{4n^2+3n-2j^2-j}\\
&=\frac{1}{j(q^2;q^4)}\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r-s}{2}}q^{\frac{r^2+s^2}{2}+3rs+r+2s}\\
&=\frac{1}{j(q^2;q^4)}\left(f_{1,3,1}(q^4,q^6,q^4)+q^7 f_{1,3,1}(q^{12},q^{14},q^4)\right),
\end{aligned}
$$

which is (1.12).

Note that Gordon and McIntosh [12, p. 323] gave the following identities:

$$U_0(q) = S_0(q^2) + qS_1(q^2), \tag{2.32}$$

$$U_1(q) = T_0(q^2) + qT_1(q^2). \tag{2.33}$$

Combining (1.9), (1.10) and (2.32) yields (1.13). Then in light of (1.11), (1.12) and (2.33), we derive

$$\begin{aligned} j(q^4;q^8)U_1(q) = q^{10} f_{1,3,1}(q^{20}, q^{24}, q^8) - q^8 f_{1,3,1}(q^{16}, q^{28}, q^8) \\ + q f_{1,3,1}(q^8, q^{12}, q^8) + q^{15} f_{1,3,1}(q^{24}, q^{28}, q^8). \end{aligned} \tag{2.34}$$

On the other hand, employing (2.20) with replacing q, x, and y by $-q^2$, $-q^3$, and q^5, respectively, we have

$$\begin{aligned} f_{1,3,1}(-q^3, q^5, -q^2) &= f_{1,3,1}(q^8, q^{12}, q^8) + q^3 f_{1,3,1}(q^{12}, q^{24}, q^8) \\ &\quad - q^5 f_{1,3,1}(q^{20}, q^{16}, q^8) + q^{14} f_{1,3,1}(q^{24}, q^{28}, q^8) \\ &= f_{1,3,1}(q^8, q^{12}, q^8) - q^7 f_{1,3,1}(q^{28}, q^{16}, q^8) \\ &\quad + q^9 f_{1,3,1}(q^{20}, q^{24}, q^8) + q^{14} f_{1,3,1}(q^{24}, q^{28}, q^8) \quad \text{(by (2.21))}. \end{aligned}$$

Examining (2.34) and the above identity yields (1.14).

Finally, inserting (2.18) and (2.19) into (2.1) with replacing q, a, ρ_2 by q^2, 1, q and then setting $\rho_1 \to \infty$, we have

$$\begin{aligned} &1 + V_0(-q) \\ &= \frac{2}{j(-q;q^4)} \left(\sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^{n+j} q^{4n^2+2n-2j^2-j} - \sum_{n=1}^{\infty} \sum_{j=-n+1}^{n-1} (-1)^{n+j} q^{4n^2-2n-2j^2-j} \right) \\ &= \frac{2}{j(-q;q^4)} \left(\sum_{\substack{n+j\geq 0 \\ n-j\geq 0}} - \sum_{\substack{n+j<0 \\ n-j<0}} \right) (-1)^{n+j} q^{4n^2+2n-2j^2-j} \\ &= \frac{2}{j(-q;q^4)} \sum_{\substack{sg(r)=sg(s) \\ r\equiv s \pmod 2}} (-1)^r sg(r) q^{\frac{r^2+s^2}{2}+3rs+\frac{r+3s}{2}} \\ &= \frac{2}{j(-q;q^4)} \left(f_{1,3,1}(-q^3, -q^5, q^4) - q^6 f_{1,3,1}(-q^{11}, -q^{13}, q^4) \right). \end{aligned}$$

That is,

$$j(q;q^4) + j(q;q^4)V_0(q) = 2\left(f_{1,3,1}(q^3, q^5, q^4) - q^6 f_{1,3,1}(q^{11}, q^{13}, q^4) \right). \tag{2.35}$$

On the other hand,

$$\begin{aligned} &f_{1,3,1}(-q, -q^2, -q) \\ &\quad = f_{1,3,1}(q^3, q^5, q^4) - q^6 f_{1,3,1}(q^{11}, q^{13}, q^4) \\ &\qquad + q f_{1,3,1}(q^5, q^{11}, q^4) + q^2 f_{1,3,1}(q^9, q^7, q^4) \quad \text{(by (2.20))} \\ &\quad = f_{1,3,1}(q^3, q^5, q^4) - q^6 f_{1,3,1}(q^{11}, q^{13}, q^4) \\ &\qquad - q^{12} f_{1,3,1}(q^{17}, q^{15}, q^4) + qj(q^5;q^4) - q^6 f_{1,3,1}(q^{11}, q^{13}, q^4) \quad \text{(by (2.22))} \end{aligned}$$

$$
\begin{aligned}
&= f_{1,3,1}(q^3,q^5,q^4) - q^6 f_{1,3,1}(q^{11},q^{13},q^4) \\
&\quad + f_{1,3,1}(q^3,q^5,q^4) - j(q;q^4) - q^6 f_{1,3,1}(q^{11},q^{13},q^4) \quad \text{(by (2.21))} \\
&= 2\left(f_{1,3,1}(q^3,q^5,q^4) - q^6 f_{1,3,1}(q^{11},q^{13},q^4)\right) - j(q;q^4). \qquad (2.36)
\end{aligned}
$$

Based on (2.35) and the above identity, we prove (1.15).

Substituting (2.14) and (2.15) into (2.1) with replacing q, a, ρ_1 by q^2, q^2, q and then letting $\rho_2 \to \infty$, we obtain

$$
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}(q;q^2)_n}{(-q^3;q^2)_n} = \frac{(1+q)(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{4n^2+4n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2-j}.
$$

That is,

$$
\begin{aligned}
&-2V_1(-q) \\
&= \frac{2q}{j(-q;q^4)} \sum_{n=0}^{\infty} (-1)^n q^{4n^2+4n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2-j} \\
&= \frac{q}{j(-q;q^4)} \left(\sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^{n+j} q^{4n^2+4n-2j^2-j} + \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^{n+j} q^{4n^2+4n-2j^2-j} \right) \\
&= \frac{q}{j(-q;q^4)} \left(\sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^{n+j} q^{4n^2+4n-2j^2-j} - \sum_{n=-\infty}^{-1} \sum_{j=n+1}^{-n-1} (-1)^{n+j} q^{4n^2+4n-2j^2-j} \right) \\
&= \frac{q}{j(-q;q^4)} \left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}} \right) (-1)^{n+j} q^{4n^2+4n-2j^2-j} \\
&= \frac{q}{j(-q;q^4)} \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} (-1)^r sg(r) q^{\frac{r^2+s^2}{2}+3rs+\frac{3r+5s}{2}} \\
&= \frac{q}{j(-q;q^4)} \left(f_{1,3,1}(-q^5,-q^7,q^4) - q^8 f_{1,3,1}(-q^{13},-q^{15},q^4) \right).
\end{aligned}
$$

Namely,

$$
2j(q;q^4)V_1(q) = q f_{1,3,1}(q^5,q^7,q^4) - q^9 f_{1,3,1}(q^{13},q^{15},q^4). \qquad (2.37)
$$

By (2.20),

$$
\begin{aligned}
f_{1,3,1}(-q^2,-q^3,-q) = {} & f_{1,3,1}(q^5,q^7,q^4) + q^2 f_{1,3,1}(q^7,q^{13},q^4) \\
& + q^3 f_{1,3,1}(q^{11},q^9,q^4) - q^8 f_{1,3,1}(q^{13},q^{15},q^4). \qquad (2.38)
\end{aligned}
$$

Since

$$
f_{1,3,1}(q^7,q^{13},q^4) = -f_{1,3,1}(q^{13},q^7,q^4) = f_{1,3,1}(q^{13},q^7,q^4),
$$

we have

$$
f_{1,3,1}(q^7,q^{13},q^4) = 0.
$$

Similarly,

$$f_{1,3,1}(q^{11},q^9,q^4)=0.$$

Hence, we have

$$f_{1,3,1}(-q^2,-q^3,-q)=f_{1,3,1}(q^5,q^7,q^4)-q^8f_{1,3,1}(q^{13},q^{15},q^4). \tag{2.39}$$

Combining (2.37), (2.38), and (2.39) indicates (1.16). Therefore, we complete the proof. □

Next, using Bailey's lemma, we state another proof of the following generalized Lambert series for $A(q)$ given by McIntosh [16]:

$$A(q)=\frac{1}{2j(q;q^4)}\sum_{j=-\infty}^{\infty}\frac{(-1)^jq^{2j^2+3j+1}}{1-q^{2j+1}}. \tag{2.40}$$

***Proof of* (2.40).** Based on Lemma 2.2, we have

$$A'_n(q^2,q,q,q^2)=\frac{q^{2n^2+2n}(1+q)(1+q^{2n+1})}{(1-q)(1-q^{2n+1})}\sum_{j=-n}^{n}(-1)^jq^{-j^2-j}$$

and

$$B'_n(q^2,q,q,q^2)=\frac{1}{(q^3;q^2)_n^2}.$$

Substituting the above Bailey pair into (2.1) with replacing q, a, ρ_1 by q^2, q^2, $-q$ and then $\rho_2\to\infty$, we deduce

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q^3;q^2)_n^2}=\frac{(-q;q^2)_\infty(1-q)}{(q^4;q^2)_\infty(1+q)}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\frac{q^{3n^2+4n}(-1)^jq^{-j^2-j}}{1-q^{2n+1}}.$$

Combining (1.4) and the above identity, we arrive at

$$\begin{aligned}
A(q)&=\frac{1}{j(q;q^4)}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\frac{q^{3n^2+4n+1}(-1)^jq^{-j^2-j}}{1-q^{2n+1}}\\
&=\frac{1}{j(q;q^4)}\left(\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{q^{3n^2+4n+1}(-1)^jq^{-j^2-j}}{1-q^{2n+1}}+\sum_{j=-\infty}^{-1}\sum_{n=-j}^{\infty}\frac{q^{3n^2+4n+1}(-1)^jq^{-j^2-j}}{1-q^{2n+1}}\right)\\
&=\frac{1}{j(q;q^4)}\left(\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{q^{3n^2+4n+1}(-1)^jq^{-j^2-j}}{1-q^{2n+1}}-\sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty}\frac{q^{3n^2+4n+1}(-1)^jq^{-j^2-j}}{1-q^{2n+1}}\right)\\
&=\frac{1}{j(q;q^4)}\sum_{j=0}^{\infty}\frac{(-1)^jq^{2j^2+3j+1}}{1-q^{2j+1}}.
\end{aligned}$$

Thus,

$$A(q)=\frac{1}{2j(q;q^4)}\left(\sum_{j=0}^{\infty}\frac{(-1)^jq^{2j^2+3j+1}}{1-q^{2j+1}}+\sum_{j=0}^{\infty}\frac{(-1)^jq^{2j^2+3j+1}}{1-q^{2j+1}}\right),$$

which implies (2.40). We complete the proof. □

Notice that (2.40) is equivalent to

$$A(q) = -m(q, q^4, q^2)$$

which was provided in [14, Eq. (5.1)]. Observe that

$$\begin{aligned}
&\frac{1}{2j(q;q^4)}\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{2j^2+3j+1}}{1-q^{2j+1}}\\
&=\frac{1}{2j(q;q^4)}\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{2j^2+3j+1}}{1-q^{2j+1}}\cdot\frac{1+q^{2j+1}}{1+q^{2j+1}}\\
&=\frac{q}{2j(q;q^4)}\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{2j^2+3j+1}}{1-q^{4j+2}}+\frac{q^2}{2j(q;q^4)}\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{2j^2+5j+1}}{1-q^{4j+2}}\\
&=\frac{qj(q^5;q^4)}{2j(q;q^4)}m(q,q^4,q^5)+\frac{q^2j(q^7;q^4)}{2j(q;q^4)}m(q^{-1},q^4,q^7)\\
&=-\frac{1}{2}m(q,q^4,q^5)-\frac{q^{-1}}{2}m(q^{-1},q^4,q^7)\\
&=-m(q,q^4,q^2),
\end{aligned}$$

where we apply the following properties of Appell-Lerch functions given by Hickerson and Mortenson [14]:

$$\begin{aligned}
m(x,q,z)&=m(x,q,x^{-1}z^{-1}),\\
m(x,q,z)&=m(x,q,qz),\\
m(x,q,z)&=x^{-1}m(x^{-1},q,z^{-1}).
\end{aligned}$$

Gordon and McIntosh [12] established the following generalized Lambert series for $U_0(q)$ and $U_1(q)$ by the Watson-Whipple transformation formula [11, Eq. (III.17)].

$$U_0(q)=\frac{2(-q;q)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{4n}}, \tag{2.41}$$

$$U_1(q)=\frac{(-q;q)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+3n+1}}{1+q^{4n+2}}. \tag{2.42}$$

Next, in light of the Bailey pair given by Andrews and Hickerson [5], we give different proofs of (2.41) and (2.42).

Proof of (2.41)**.** Based on (2.3) and (2.4), we have

$$A'_n(1,i,-i,q^2)=\frac{2(-1)^n q^{n^2+n}(1+q^{2n})}{1+q^{4n}}$$

and

$$B'_n(1,i,-i,q^2)=\frac{1}{(-q^4;q^4)_n}.$$

Substituting the above Bailey pair into (2.1) with replacing q, a, ρ_1 by q^2, 1, $-q$ and setting $\rho_2 \to \infty$, we deduce

$$\begin{aligned}
U_0(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{2(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{4n}}+1\right)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{4n}}+\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{4n}}+1\right)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{4n}}+\sum_{n=-\infty}^{-1}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{4n}}+1\right)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{4n}}\right)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{4n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+3n}}{1+q^{4n}}\right)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{4n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2-3n}}{1+q^{-4n}}\right),
\end{aligned}$$

which implies (2.41). □

Proof of (2.42)**.** Invoking Lemma 2.2, we get

$$A_n'(q^2,-iq,iq,q^2) = \frac{q^{2n^2+2n}(1+q^2)(1-q^{4n+2})}{(1-q^2)(1+q^{4n+2})}\left(\sum_{j=-n}^{n}(-1)^j q^{-j^2-j}\right) \tag{2.43}$$

and

$$B_n'(q^2,-iq,iq,q^2) = \frac{1}{(-q^6;q^4)_n}. \tag{2.44}$$

Substituting (2.43) and (2.44) into (2.1) with replacing q, a, ρ_1 by q^2, q^2, $-q$ and then setting $\rho_2 \to \infty$, we have

$$\sum_{n=0}^{\infty}(-q;q^2)_n q^{n^2+2n} B_n'(q^2,-iq,iq,q^2) = \frac{(-q^3;q^2)_\infty}{(q^4;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(1+q)}{1+q^{2n+1}}A_n'(q^2,-iq,iq,q^2).$$

That is,

$$\begin{aligned}
U_1(q) &= \frac{q(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{3n^2+4n}(1-q^{2n+1})}{1+q^{4n+2}}\sum_{j=-n}^{n}(-1)^j q^{-j^2-j}\\
&= \frac{q(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\frac{q^{3n^2+4n}(-1)^j q^{-j^2-j}}{1+q^{4n+2}}-\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\frac{q^{3n^2+6n+1}(-1)^j q^{-j^2-j}}{1+q^{4n+2}}\right).
\end{aligned} \tag{2.45}$$

Notice that

$$\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\frac{q^{3n^2+4n}(-1)^j q^{-j^2-j}}{1+q^{4n+2}}$$

$$= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} \frac{q^{3n^2+4n}(-1)^j q^{-j^2-j}}{1+q^{4n+2}} + \sum_{j=-\infty}^{-1}\sum_{n=-j}^{\infty} \frac{q^{3n^2+4n}(-1)^j q^{-j^2-j}}{1+q^{4n+2}}$$
$$= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} \frac{q^{3n^2+4n}(-1)^j q^{-j^2-j}}{1+q^{4n+2}} - \sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty} \frac{q^{3n^2+4n}(-1)^j q^{-j^2-j}}{1+q^{4n+2}}$$
$$= \sum_{j=0}^{\infty} \frac{(-1)^j q^{2j^2+3j}}{1+q^{4j+2}}$$

and

$$\sum_{n=0}^{\infty}\sum_{j=-n}^{n} \frac{q^{3n^2+6n+1}(-1)^j q^{-j^2-j}}{1+q^{4n+2}}$$
$$= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} \frac{q^{3n^2+6n+1}(-1)^j q^{-j^2-j}}{1+q^{4n+2}} + \sum_{j=-\infty}^{-1}\sum_{n=-j}^{\infty} \frac{q^{3n^2+6n+1}(-1)^j q^{-j^2-j}}{1+q^{4n+2}}$$
$$= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} \frac{q^{3n^2+6n+1}(-1)^j q^{-j^2-j}}{1+q^{4n+2}} - \sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty} \frac{q^{3n^2+6n+1}(-1)^j q^{-j^2-j}}{1+q^{4n+2}}$$
$$= \sum_{j=0}^{\infty} \frac{(-1)^j q^{2j^2+5j+1}}{1+q^{4j+2}}.$$

Thus, the identity (2.45) is equivalent to the following equation:

$$U_1(q) = \frac{q(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{j=0}^{\infty} \frac{(-1)^j q^{2j^2+3j}}{1+q^{4j+2}} - \sum_{j=0}^{\infty} \frac{(-1)^j q^{2j^2+5j+1}}{1+q^{4j+2}}\right)$$
$$= \frac{q(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{j=0}^{\infty} \frac{(-1)^j q^{2j^2+3j}}{1+q^{4j+2}} + \sum_{j=-\infty}^{-1} \frac{(-1)^j q^{2j^2+3j}}{1+q^{4j+2}}\right),$$

which implies (2.42). Therefore, we complete the proof. □

Notice that (2.41) and (2.42) can also be proved by (1.13), (1.14), Theorem 1.3 in [14], and Corollary 3.10 in [17].

3. The proof of Theorem 1.5

In this section, employing Ramanujan's ${}_1\psi_1$ summation formula and a ${}_2\psi_2$ transformation formula due to Bailey [6], we prove some identities for the second and eighth order mock theta functions.

The general bilateral basic hypergeometric series in base q with r numerator and s denominator parameters is defined by

$${}_r\psi_s\left(\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}; q, z\right) := \sum_{n=-\infty}^{\infty} \frac{(a_1,a_2,\cdots,a_r;q)_n}{(b_1,b_2,\cdots,b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{s-r} z^n.$$

Ramanujan's ${}_1\psi_1$ summation formula [11, Eq. (5.2.1)] says that

$$ {}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix}; q, z\right) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \tag{3.1}$$

where $|b/a| < |z| < 1$. We also require the ${}_2\psi_2$ transformation formula from [6].

$$ {}_2\psi_2\left(\begin{matrix} a, & b \\ c, & d \end{matrix}; q, z\right) = \frac{(az, d/a, c/b, dq/abz; q)_\infty}{(z, d, q/b, cd/abz; q)_\infty} {}_2\psi_2\left(\begin{matrix} a, & abz/d \\ az, & c \end{matrix}; q, \frac{d}{a}\right). \tag{3.2}$$

Proof of Theorem 1.5. Based on (1.5), we have

$$\begin{aligned} B(q) + B(-q) &= \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q^2)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}} + \sum_{n=-\infty}^{-1} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}} \\ &= \sum_{n=-\infty}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}. \end{aligned}$$

Replacing q, b, c, d, and z by q^2, $-q$, a, q^3, and q, respectively, in (3.2), we derive

$$\begin{aligned} &\frac{1}{1-q} \sum_{n=-\infty}^{\infty} \frac{q^n(-q;q^2)_n}{(q^3;q^2)_n} \\ &= \frac{(-a^{-1}q^3;q^2)_\infty(-aq^{-1};q^2)_\infty(aq;q^2)_\infty(a^{-1}q^3;q^2)_\infty}{(-q;q^2)_\infty^2(q;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{a^{-n}q^{3n}(-aq^{-1};q^2)_n}{(aq;q^2)_n}. \end{aligned} \tag{3.3}$$

In addition, it can be seen from (3.1) with replacing q, a, b, and z by q^2, $-aq^{-1}$, aq, and $a^{-1}q^3$, respectively, that

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \frac{a^{-n}q^{3n}(-aq^{-1};q^2)_n}{(aq;q^2)_n} \\ &= \frac{2(-q^2;q^2)_\infty^3(q^2;q^2)_\infty}{(-a^{-1}q^3;q^2)_\infty(-aq^{-1};q^2)_\infty(aq;q^2)_\infty(a^{-1}q^3;q^2)_\infty}. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) yields that

$$\sum_{n=-\infty}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}} = \frac{2(-q^2;q^2)_\infty^3(q^2;q^2)_\infty}{(-q;q^2)_\infty^2(q;q^2)_\infty^2}.$$

Simplifying the above identity yields (1.17).

Next, it follows from (1.2) that

$$\begin{aligned} V_0(q) + V_0(-q) &= -2 + 2\left(\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q^2)_n}\right) \\ &= 2\left(\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q^2)_n}\right) \end{aligned}$$

$$= 2\left(\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} + \sum_{n=-\infty}^{-1}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}\right)$$
$$= 2\sum_{n=-\infty}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}. \tag{3.5}$$

Then replacing q, a, b, c, d by q^2, $-q/z$, $-q$, q, 0 and then letting z tend to 0 in (3.2), we see that

$$\sum_{n=-\infty}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} = 2(-q^2;q^2)_\infty\sum_{n=-\infty}^{\infty}\frac{q^{2n^2}}{(q^2;q^4)_n}. \tag{3.6}$$

Furthermore, applying (3.1) with replacing q, a, b by q^4, $-q^2/z$, q^2 and then letting z tend to 0 yields that

$$\sum_{n=-\infty}^{\infty}\frac{q^{2n^2}}{(q^2;q^4)_n} = \frac{(-q^2,-q^2,q^4;q^4)_\infty}{(-1,q^2;q^4)_\infty}. \tag{3.7}$$

Examining (3.6) and (3.7), we derive

$$\sum_{n=-\infty}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} = (-q^2;q^4)_\infty^4(q^8;q^8)_\infty. \tag{3.8}$$

Thus, we establish (1.18) from (3.5) and (3.8).

Similarly, with the aid of (1.3), we have

$$V_1(q) - V_1(-q) = \sum_{n=0}^{\infty}\frac{q^{n^2+2n+1}(-q;q^2)_n}{(q;q^2)_{n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2n+1}(q;q^2)_n}{(-q;q^2)_{n+1}}$$
$$= \sum_{n=0}^{\infty}\frac{q^{n^2+2n+1}(-q;q^2)_n}{(q;q^2)_{n+1}} + \sum_{n=-\infty}^{-1}\frac{q^{n^2+2n+1}(-q;q^2)_n}{(q;q^2)_{n+1}}$$
$$= \sum_{n=-\infty}^{\infty}\frac{q^{n^2+2n+1}(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{3.9}$$

Additionally, it follows from (3.2) by setting $b = b/z$ that

$${}_2\psi_2\left(\begin{matrix} a, & b/z \\ c, & d\end{matrix}; q, z\right) = \frac{(az, d/a, cz/b, dq/ab;q)_\infty}{(z,d,qz/b,cd/ab;q)_\infty}\,{}_2\psi_2\left(\begin{matrix} a, & ab/d \\ az, & c\end{matrix}; q, \frac{d}{a}\right). \tag{3.10}$$

Then replacing q, a, b, d by q^2, $-q$, $-q^3$, q^3 and then letting c, z tend to 0 in (3.10), we derive

$$\sum_{n=-\infty}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q;q^2)_{n+1}} = (-q^2;q^2)_\infty\sum_{n=-\infty}^{\infty}(-1)^n q^{2n}(q^2;q^4)_n. \tag{3.11}$$

Furthermore, it can be seen from (3.1) with replacing q, a, b, and z by q^4, q^2, 0, and $-q^2$, respectively, that

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{2n}(q^2;q^4)_n = 2(-q^4;q^4)_\infty^2(q^8;q^8)_\infty. \tag{3.12}$$

Examining (3.9), (3.11), and (3.12), we deduce (1.19).

Notice that Gordon and McIntosh [12, Eq. (1.6)] established the generalized Lambert series for $V_1(q)$.

$$V_1(q) = \frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n^2+4n+1}}{1-q^{4n+1}}.$$

Using the above identity, we give another proof of (1.19).

$$\begin{aligned} V_1(q) - V_1(-q) &= \frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty} \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n^2+4n+1}}{1-q^{4n+1}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n^2+4n+1}}{1+q^{4n+1}} \right) \\ &= \frac{2(-q^4;q^4)_\infty}{(q^4;q^4)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n^2+4n+1}}{1-q^{8n+2}} \\ &= \frac{2q(-q^4;q^4)_\infty (q^8;q^8)_\infty^2}{(q^4;q^4)_\infty (q^2;q^4)_\infty}, \end{aligned}$$

where in the last line, we use the following identity given by Andrews and Berndt [4, Entry 12.2.2]:

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{1-q^n z} = \frac{j(q;q^3)^3}{j(z;q)}.$$

Hence, we obtain (1.19).

Finally, we prove (1.20) and (1.21). Replacing q, a, b, and z by q^2, $-q^2$, 0, and q, respectively, in (3.1) yields that

$$\sum_{n=-\infty}^{\infty} q^n (-q^2;q^2)_n = \frac{(-q;q^2)_\infty^3 (q^2;q^2)_\infty}{2q}. \tag{3.13}$$

Combining (3.13) and the following identity [12, Eq. (1.9)]:

$$U_0(q) + 2U_1(q) = (-q;q^2)_\infty^3 (q^2;q^2)_\infty (q^2;q^4)_\infty,$$

we have

$$U_0(q) + 2U_1(q) = 2q(q^2;q^4)_\infty \sum_{n=-\infty}^{\infty} q^n (-q^2;q^2)_n. \tag{3.14}$$

Replacing q, a, b, d by q^2, $-q$, $-q$, $-q^2$ and then letting c, z tend to 0 in (3.10), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2} (-q;q^2)_n}{(-q^2;q^2)_n} = 2(q;q^2)_\infty \sum_{n=-\infty}^{\infty} q^n (-q;q)_{2n-1}. \tag{3.15}$$

Moreover, replacing q, a, b, d by q^2, $-q$, $-q^3$, $-q^2$ and then letting c, z tend to 0 in (3.10), we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2+2n} (-q;q^2)_n}{(-q^2;q^2)_n} = 2(q;q^2)_\infty \sum_{n=-\infty}^{\infty} q^n (-q;q)_{2n}. \tag{3.16}$$

In addition, Gordon and McIntosh [12] gave that

$$S_0(q)+2T_0(q)=\sum_{n=-\infty}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{3.17}$$

$$S_1(q)+2T_1(q)=\sum_{n=-\infty}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(-q^2;q^2)_n}. \tag{3.18}$$

Combining (3.15) and (3.17) implies

$$\begin{aligned}S_0(q^2)+2T_0(q^2)&=2(q^2;q^4)_\infty\sum_{n=-\infty}^{\infty}q^{2n}(-q^2;q^2)_{2n-1}\\&=2q(q^2;q^4)_\infty\sum_{n=-\infty}^{\infty}q^{2n+1}(-q^2;q^2)_{2n+1}\\&=q(q^2;q^4)_\infty\sum_{n=-\infty}^{\infty}\left(1-(-1)^n\right)q^n(-q^2;q^2)_n. \end{aligned}\tag{3.19}$$

Examining (3.14) and (3.19), we arrive at (1.20). Similarly, applying (3.16) and (3.18) yields that

$$\begin{aligned}S_1(q^2)+2T_1(q^2)&=2(q^2;q^4)_\infty\sum_{n=-\infty}^{\infty}q^{2n}(-q^2;q^2)_{2n}\\&=(q^2;q^4)_\infty\sum_{n=-\infty}^{\infty}\left(1+(-1)^n\right)q^n(-q^2;q^2)_n. \end{aligned}\tag{3.20}$$

Using (3.14) and (3.20), we deduce (1.21). Therefore, we complete the proof. □

Notice that based on the following results established by Hickerson and Mortenson [14]:

$$\begin{aligned}B(q)&=-q^{-1}m(1,q^4,q^3),\\V_1(q)&=-m(q^2,q^8,q),\end{aligned}$$

we can also prove (1.17) and (1.19) by using the following identity [14]:

$$m(x,q,z_1)=m(x,q,z_0)+\frac{z_0j^3(q;q^3)j(z_1/z_0;q)j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}.$$

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