

# Planar anti-Ramsey numbers of paths and cycles

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## Abstract

Motivated by anti-Ramsey numbers introduced by Erdős, Simonovits and Sós in 1975, we study the anti-Ramsey problem when host graphs are plane triangulations. Given a positive integer  $n$  and a planar graph  $H$ , let  $\mathcal{T}_n(H)$  be the family of all plane triangulations  $T$  on  $n$  vertices such that  $T$  contains a subgraph isomorphic to  $H$ . The *planar anti-Ramsey number of  $H$* , denoted  $ar_{\mathcal{P}}(n, H)$ , is the maximum number of colors in an edge-coloring of a plane triangulation  $T \in \mathcal{T}_n(H)$  such that  $T$  contains no rainbow copy of  $H$ . Analogously to anti-Ramsey numbers and Turán numbers, planar anti-Ramsey numbers are closely related to planar Turán numbers, where the *planar Turán number of  $H$*  is the maximum number of edges of a planar graph on  $n$  vertices without containing  $H$  as a subgraph. The study of  $ar_{\mathcal{P}}(n, H)$  (under the name of rainbow numbers) was initiated by Horňák, Jendrol', Schiermeyer and Soták [J Graph Theory 78 (2015) 248–257]. In this paper we study planar anti-Ramsey numbers for paths and cycles. We first improve existing lower bound for  $ar_{\mathcal{P}}(n, C_k)$  when  $k \geq 5$  and  $n \geq k^2 - k$ . Then, using the main ideas in the above-mentioned paper, we obtain upper bounds for  $ar_{\mathcal{P}}(n, C_6)$  when  $n \geq 8$  and  $ar_{\mathcal{P}}(n, C_7)$  when  $n \geq 13$ , respectively. Finally, we establish lower bounds for  $ar_{\mathcal{P}}(n, P_k)$  when  $n \geq k \geq 8$ .

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**Keywords:** rainbow subgraph; anti-Ramsey number; plane triangulation

## 1 Introduction

All graphs considered in this paper are finite and simple. Motivated by anti-Ramsey numbers introduced by Erdős, Simonovits and Sós [4] in 1975, we study the anti-Ramsey problem when host graphs are plane triangulations. A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. Let  $\mathcal{F}$  be a family of planar graphs. For the purpose of this paper, we call an edge-coloring that contains no rainbow copy of any graph in  $\mathcal{F}$  an  *$\mathcal{F}$ -free edge-coloring*. A graph

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$G$  is  $\mathcal{F}$ -free if no subgraph of  $G$  is isomorphic to any graph in  $\mathcal{F}$ . Let  $n_{\mathcal{F}}$  be the smallest integer such that for any  $n \geq n_{\mathcal{F}}$ , there exists a plane triangulation on  $n$  vertices that is not  $\mathcal{F}$ -free. Such an integer  $n_{\mathcal{F}}$  is well-defined, because for any  $F \in \mathcal{F}$ , we can obtain a plane triangulation from a plane drawing of  $F$  by adding new edges. When  $\mathcal{F} = \{F\}$ , then  $n_{\mathcal{F}} = |F|$ . For any integer  $n \geq n_{\mathcal{F}}$ , let  $\mathcal{T}_n(\mathcal{F})$  be the family of all plane triangulations  $T$  on  $n$  vertices such that  $T$  is not  $\mathcal{F}$ -free. The *planar anti-Ramsey number of  $\mathcal{F}$* , denoted  $ar_{\mathcal{P}}(n, \mathcal{F})$ , is the maximum number of colors in an  $\mathcal{F}$ -free edge-coloring of a plane triangulation in  $\mathcal{T}_n(\mathcal{F})$ . Clearly,  $ar_{\mathcal{P}}(n, \mathcal{F}) < 3n - 6$ . It is worth noting that this problem becomes trivial if the host plane triangulation on  $n$  vertices is  $\mathcal{F}$ -free, because  $3n - 6$  colors can be used.

Analogous to the relation between anti-Ramsey numbers and Turán numbers proved in [4], planar anti-Ramsey numbers are closely related to planar Turán numbers [3]. The *planar Turán number of  $\mathcal{F}$* , denoted  $ex_{\mathcal{P}}(n, \mathcal{F})$ , is the maximum number of edges of an  $\mathcal{F}$ -free planar graph on  $n$  vertices. Given an edge-coloring  $c$  of a host graph  $T$  in  $\mathcal{T}_n(\mathcal{F})$ , we define a *representing graph* of  $c$  to be a spanning subgraph  $R$  of  $T$  obtained by taking one edge of each color under the coloring  $c$  (where  $R$  may contain isolated vertices). Clearly, if  $c$  is an  $\mathcal{F}$ -free edge-coloring of  $T$ , then  $R$  is  $\mathcal{F}$ -free. Thus  $ar_{\mathcal{P}}(n, \mathcal{F}) \leq ex_{\mathcal{P}}(n, \mathcal{F})$  for any  $n \geq n_{\mathcal{F}}$ . When  $\mathcal{F}$  consists of a single graph  $H$ , we write  $ar_{\mathcal{P}}(n, H)$  and  $ex_{\mathcal{P}}(n, H)$  instead of  $ar_{\mathcal{P}}(n, \{H\})$  and  $ex_{\mathcal{P}}(n, \{H\})$ . Given a planar graph  $H$ , let  $\mathcal{H} = \{H - e : e \in E(H)\}$ . Let  $G$  be an  $\mathcal{H}$ -free plane subgraph of a plane triangulation  $T \in \mathcal{T}_n(H)$  with  $e(G) = ex_{\mathcal{P}}(n, \mathcal{H})$ . We then obtain an  $H$ -free edge-coloring of  $T$  by coloring the edges of  $G$  with distinct colors and then coloring the edges in  $E(T) \setminus E(G)$  with a new color. Hence,  $1 + ex_{\mathcal{P}}(n, \mathcal{H}) \leq ar_{\mathcal{P}}(n, H)$  for any  $n \geq |H|$ . We obtain the following analogous result.

**Proposition 1.1** *Given a planar graph  $H$  and a positive integer  $n \geq |H|$ ,*

$$1 + ex_{\mathcal{P}}(n, \mathcal{H}) \leq ar_{\mathcal{P}}(n, H) \leq ex_{\mathcal{P}}(n, H),$$

where  $\mathcal{H} = \{H - e : e \in E(H)\}$ .

Colorings of plane graphs that avoid rainbow faces have also been studied, see, e.g., [5, 7, 15, 16]. Various results on anti-Ramsey numbers can be found in: [1, 2, 8, 9, 10, 11, 13, 14] to name a few. The study of planar anti-Ramsey numbers  $ar_{\mathcal{P}}(n, H)$  was initiated by Horňák, Jendrol', Schiermeyer and Soták [6] (under the name of rainbow numbers). We summarize their results in [6] as follows, where given two positive integers  $a$  and  $b$ , we use  $a \bmod(b)$  to denote the remainder when  $a$  is divided by  $b$ . We use  $P_k$  and  $C_k$  to denote the path and cycle on  $k$  vertices, respectively.

**Theorem 1.2** ([6]) *Let  $n, k$  be positive integers.*

- (a)  $ar_{\mathcal{P}}(n, C_3) = \lfloor (3n - 6)/2 \rfloor$  for  $n \geq 4$ .
- (b)  $ar_{\mathcal{P}}(n, C_4) \leq 2(n - 2)$  for  $n \geq 4$ , and  $ar_{\mathcal{P}}(n, C_4) \geq (9(n - 2) - 4r)/5$  for  $n \geq 42$  and  $r = (n - 2) \bmod(20)$ .

- (c)  $ar_{\mathcal{P}}(n, C_5) \leq 5(n-2)/2$  for  $n \geq 5$ , and  $ar_{\mathcal{P}}(n, C_5) \geq (19(n-2) - 10r)/9$  for  $n \geq 20$  and  $r = (n-2) \bmod(18)$ .
- (d)  $ar_{\mathcal{P}}(n, C_k) \geq (3n-6) \cdot \frac{k-3}{k-2} - \frac{2k-7}{k-2}$  for  $6 \leq k \leq n$ .

Finding exact values of  $ar_{\mathcal{P}}(n, H)$  is far from trivial. As observed in [6], an induction argument in general cannot be applied to compute  $ar_{\mathcal{P}}(n, H)$  because deleting a vertex from a plane triangulation may result in a graph that is no longer a plane triangulation.

Dowden [3] began the study of planar Turán numbers  $ex_{\mathcal{P}}(n, H)$  (under the name of “extremal” planar graphs) and proved Theorem 1.3 below, where each bound is tight.

**Theorem 1.3** ([3]) *Let  $n$  be a positive integer.*

- (a)  $ex_{\mathcal{P}}(n, C_3) = 2n - 4$  for  $n \geq 3$ .
- (b)  $ex_{\mathcal{P}}(n, C_4) \leq 15(n-2)/7$  for  $n \geq 4$ .
- (c)  $ex_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$  for  $n \geq 11$ .

By Proposition 1.1 and Theorem 1.3(c), we see that  $ar_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$  for  $n \geq 11$ . This improves the upper bound for  $ar_{\mathcal{P}}(n, C_5)$  in Theorem 1.2(c) when  $n \geq 11$ . Notice that the upper bound in Proposition 1.1 in general is quite loose, for example,  $ex_{\mathcal{P}}(n, C_3) - ar_{\mathcal{P}}(n, C_3) = \lceil n/2 \rceil - 1$  for all  $n \geq 4$ . We study planar anti-Ramsey numbers for paths and cycles in this paper, and planar Turán numbers for Theta graphs and paths of order at most 9 in [12]. In Section 3, we establish lower bounds for  $ar_{\mathcal{P}}(n, P_k)$  when  $n \geq k \geq 8$ . In Section 2, we first improve the existing lower bounds for  $ar_{\mathcal{P}}(n, C_k)$  when  $k \geq 5$  and  $n \geq k^2 - k$ , which improves Theorem 1.2(c,d). We then use the main ideas in [6] by studying lower and upper bounds for the planar anti-Ramsey numbers when host graphs are wheels to obtain upper bounds for  $ar_{\mathcal{P}}(n, C_6)$  when  $n \geq 8$  and  $ar_{\mathcal{P}}(n, C_7)$  when  $n \geq 13$ , respectively.

We need to introduce more notation. For a graph  $G$  we use  $V(G)$ ,  $|G|$ ,  $E(G)$ ,  $e(G)$ ,  $\delta(G)$  and  $\alpha(G)$  to denote the vertex set, number of vertices, edge set, number of edges, minimum degree, and independence number of  $G$ , respectively. For a vertex  $x \in V(G)$ , we will use  $N_G(x)$  to denote the set of vertices in  $G$  which are adjacent to  $x$ . We define  $N_G[x] = N_G(x) \cup \{x\}$  and  $d_G(x) = |N_G(x)|$ . The subgraph of  $G$  induced by  $A$ , denoted  $G[A]$ , is the graph with vertex set  $A$  and edge set  $\{xy \in E(G) : x, y \in A\}$ . We denote by  $B \setminus A$  the set  $B - A$  and  $G \setminus A$  the subgraph of  $G$  induced on  $V(G) \setminus A$ , respectively. If  $A = \{a\}$ , then we simply write  $B \setminus a$  and  $G \setminus a$ , respectively. Given two graphs  $G$  and  $H$ , the union of  $G$  and  $H$ , denoted  $G \cup H$ , is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . Given two isomorphic graphs  $G$  and  $H$ , we may (with a slight but common abuse of notation) write  $G = H$ . Given a plane graph  $G$  and an integer  $i \geq 3$ , an  $i$ -face in  $G$  is a face of size  $i$ . Let  $f_i(G)$  denote the number of  $i$ -faces in  $G$  and  $n_i(G)$  denote the number of vertices of degree  $i$  in  $G$ . Given an edge-coloring  $c$  of  $G$ , let  $c(G)$  denote the number of colors used under  $c$ . For any positive integer  $k$ , let  $[k] := \{1, 2, \dots, k\}$ .

## 2 Rainbow Cycles

In this section, we study planar anti-Ramsey numbers for cycles.

### 2.1 Improving the existing lower bound for $ar_{\mathcal{P}}(n, C_k)$

We first prove a new lower bound for  $ar_{\mathcal{P}}(n, C_k)$  when  $k \geq 5$ , which improves Theorem 1.2(d) when  $n$  is sufficiently large with respect to  $k$ .

**Theorem 2.1** *For integers  $k \geq 5$ ,  $n \geq k^2 - k$ , and  $r = (n - 2) \bmod (k^2 - k - 2)$ ,*

$$ar_{\mathcal{P}}(n, C_k) \geq \left( \frac{k-3}{k-2} + \frac{2}{3(k+1)(k-2)} \right) (3n-6) - \frac{2k^2-5k-5}{k^2-k-2} r.$$

**Proof.** Let  $n, k, r$  be given as in the statement. Let  $t \geq 3$  be an integer satisfying  $(k^2 - k - 2)(t - 2) + 2 + r = n$ . This is possible because  $r = (n - 2) \bmod (k^2 - k - 2)$  and  $n \geq k^2 - k$ . Let  $T$  be a plane triangulation on  $t$  vertices. Then  $f_3(T) = 2t - 4$ . Let  $k := 3m + q$ , where  $q \in \{0, 1, 2\}$  and  $m \geq 1$  is an integer. Let  $T'$  be obtained from  $T$  as follows. For each face  $F$  in  $T$ : first subdivide each of the  $q + 1$  of the edges of  $F$   $m$  times; next, subdivide each of the remaining  $2 - q$  edges of  $F$   $m - 1$  times; and finally, replace each edge from the subdivision of  $T$  by any plane triangulation on  $k - 1$  vertices. Examples of constructions of  $T'$  when  $k \in \{5, 6, 7\}$  are depicted in Figure 1 and Figure 2.

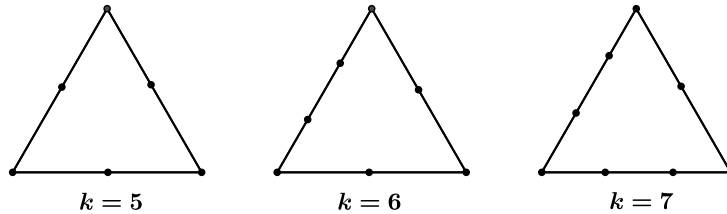


Figure 1: Subdividing one 3-face of  $T$  when  $k \in \{5, 6, 7\}$ .

It is worth noting that different edges of the subdivision of  $T$  may be replaced by different plane triangulations on  $k - 1$  vertices. Such a subdivision of  $T$  is possible when  $q \in \{0, 1, 2\}$  because when  $q = 2$ , every edge of  $T$  is subdivided  $m$  times; and when  $q \in \{0, 1\}$ , the dual of  $T$  has a perfect matching, say  $M$ . Let  $M^*$  be the dual edges of  $M$  in  $T$ . Then every face  $F$  in  $T$  contains exactly one edge in  $M^*$  and  $|M^*| = t - 2$ . When  $q = 0$ , each edge in  $M^*$  is subdivided  $m$  times, and when  $q = 1$ , each edge in  $M^*$  is subdivided  $m - 1$  times. Thus  $(q + 1)(t - 2)$  many edges of  $T$  are each subdivided  $m$  times and  $(2 - q)(t - 2)$  many edges of  $T$  are each subdivided  $m - 1$  times. One can

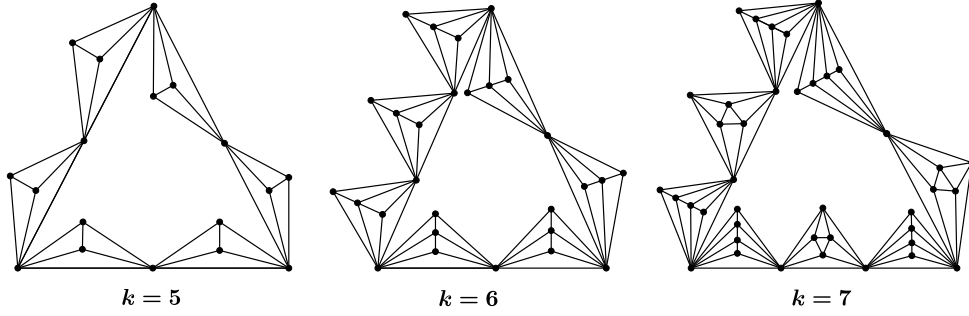


Figure 2: One possible construction of  $T'$  when  $k \in \{5, 6, 7\}$ .

check that

$$\begin{aligned}
|T'| &= t + (q+1)(t-2)[(m+1)(k-3) + m] + (2-q)(t-2)[((m-1)+1)(k-3) + (m-1)] \\
&= t + (t-2)[(q+1)(mk - 2m + k - 3) + (2-q)(mk - 2m - 1)] \\
&= t + (t-2)[(q+1+2-q)(mk - 2m) + (q+1)(k-3) - (2-q)] \\
&= t + (k^2 - k - 5)(t-2)
\end{aligned}$$

and

$$\begin{aligned}
e(T') &= (q+1)(t-2)(m+1)[(3(k-1) - 6] + (2-q)(t-2)m[3(k-1) - 6] \\
&= (t-2)(3k-9)[(q+1+2-q)m + q+1] \\
&= (t-2)(3k-9)(3m+q+1) \\
&= (t-2)(3k-9)(k+1) = 3(k^2 - 2k - 3)(t-2).
\end{aligned}$$

By the construction of  $T'$ , we see that  $T'$  is  $C_k$ -free (but contains  $C_{k+1}$  as a subgraph),  $T'$  has  $f_3(T)$  many  $i$ -faces with  $i > 3$  and at least

$$\begin{aligned}
&(q+1)(t-2)(m+1)[(2(k-1) - 5] + (2-q)(t-2)m[(2(k-1) - 5] \\
&= (t-2)(2k-7)[(q+1+2-q)m + q+1] \\
&= (t-2)(2k^2 - 5k - 7) \\
&\geq k^2 - k - 2
\end{aligned}$$

many 3-faces because  $t \geq 3$  and  $k \geq 5$ . Let  $\mathcal{F}$  be a set which consists of all  $i$ -faces of  $T'$  with  $i > 4$  and  $r$  of the 3-faces of  $T'$ . Let  $T^*$  be the plane triangulation obtained from  $T'$  by adding a new vertex to each face  $F \in \mathcal{F}$  and then joining it to all vertices on the boundary of  $F$ . Then  $|T^*| = |T'| + f_3(T) + r = [t + (k^2 - k - 5)(t-2)] + (2t-4) + r = (k^2 - k - 2)(t-2) + 2 + r = n$  and so  $T^* \in \mathcal{T}_n(C_k)$ . Now let  $c$  be an edge-coloring of  $T^*$  defined as follows: edges in  $E(T')$  are colored with distinct colors under  $c$  (that is,  $T^*$  contains a rainbow copy of  $T'$  under  $c$ ), and for each  $F \in \mathcal{F}$ , all the new edges added inside  $F$  are colored the same, but for distinct  $F, F' \in \mathcal{F}$ , new

edges inside  $F$  are colored differently than the new edges inside  $F'$ . We see that  $T^*$  has no rainbow  $C_k$  (but contains a rainbow copy of  $C_{k+1}$ ) under  $c$  because  $T'$  is  $C_k$ -free (but contains  $C_{k+1}$  as a subgraph) and no rainbow  $C_k$  in  $T^*$  can contain any new edges added to  $T'$ . Hence,

$$\begin{aligned}
c(T^*) &= e(T') + f_3(T) + r = 3(k^2 - 2k - 3)(t - 2) + 2(t - 2) + r \\
&= (3k^2 - 6k - 7)(t - 2) + r \\
&= (3k^2 - 6k - 7) \frac{n - r - 2}{k^2 - k - 2} + r \\
&= \left( \frac{k - 3}{k - 2} + \frac{2}{3(k + 1)(k - 2)} \right) (3n - 6) - \frac{2k^2 - 5k - 5}{k^2 - k - 2} r,
\end{aligned}$$

since  $n = (k^2 - k - 2)(t - 2) + 2 + r$ . Therefore,  $ar_{\mathcal{P}}(n, C_k) \geq c(T^*) \geq \left( \frac{k-3}{k-2} + \frac{2}{3(k+1)(k-2)} \right) (3n - 6) - \frac{2k^2-5k-5}{k^2-k-2} r$ . This completes the proof of Theorem 2.1.  $\blacksquare$

We next obtain a better lower bound for  $ar_{\mathcal{P}}(n, C_5)$ , which improves Theorem 1.2(c) for all  $n \geq 119$ .

**Theorem 2.2** *Let  $n \geq 119$  be an integer and let  $r = (n + 7) \bmod(18)$ . Then  $ar_{\mathcal{P}}(n, C_5) \geq (39n - 123 - 21r)/18$ .*

**Proof.** Let  $r, n$  be given as in the statement. Let  $t \geq 6$  be a positive integer satisfying  $18t + 11 + r = n$ . This is possible because  $n \geq 119$  and  $r = (n + 7) \bmod(18)$ . Let  $H$  be a connected  $C_5$ -free plane graph with  $15t + 9$  vertices and  $(12|H| - 33)/5$  edges such that  $H$  has only 3-faces and 6-faces, and no two 6-faces share an edge in common. The existence of such a graph  $H$  is due to Dowden (see Theorem 4 in [3]). Notice that  $f_6(H) = 3t + 2$  and  $f_3(H) = 18t + 6$ . Let  $\mathcal{F}$  be a set which consists of all 6-faces and  $r$  of the 3-faces of  $H$ . Then  $|\mathcal{F}| = f_6(H) + r$ . Let  $T^*$  be the plane triangulation obtained from  $H$  by adding a new vertex to each face  $F \in \mathcal{F}$  and then joining it to all vertices on the boundary of  $F$ . Then  $|T^*| = |H| + |\mathcal{F}| = |H| + f_6(H) + r = (15t + 9) + (3t + 2) + r = 18t + 11 + r = n$  and so  $T^* \in \mathcal{T}_n(C_5)$ . Finally, let  $c$  be an edge-coloring of  $T^*$  defined as follows: edges in  $E(H)$  are colored with distinct colors under  $c$  (that is,  $T^*$  contains a rainbow copy of  $H$  under  $c$ ), and for each  $F \in \mathcal{F}$ , all the new edges added inside  $F$  are colored the same, but for distinct  $F, F' \in \mathcal{F}$ , new edges inside  $F$  are colored differently than the new edges inside  $F'$ . We see that  $T^*$  has no rainbow  $C_5$  under  $c$  because  $H$  is  $C_5$ -free and no rainbow  $C_5$  in  $T^*$  can contain any new edges added to  $H$ . Then

$$c(T^*) = e(H) + f_6(H) + r = (36t + 15) + (3t + 2) + r = (39n - 123 - 21r)/18,$$

since  $n = 18t + 11 + r$ . Therefore,  $ar_{\mathcal{P}}(n, C_5) \geq c(T^*) \geq (39n - 123 - 21r)/18$ , as desired. This completes the proof of Theorem 2.2.  $\blacksquare$

**Remark.** By Proposition 1.1 and Theorem 1.3(c),  $ar_{\mathcal{P}}(n, C_5) \leq ex_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$  for all  $n \geq 11$ . It then follows from Theorem 2.2 that  $(39n - 123 - 21r)/18 \leq ar_{\mathcal{P}}(n, C_5) \leq ex_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$  for all  $n \geq 119$ , where  $r = (n + 7) \bmod(18)$ .

## 2.2 New upper bounds for $ar_{\mathcal{P}}(n, C_k)$ when $k \in \{6, 7\}$

Finally, we use the main ideas in [6] to establish upper bounds for  $ar_{\mathcal{P}}(n, C_k)$  when  $k \in \{6, 7\}$ . We need to introduce more notation. Let  $C_q$  be a cycle with vertices  $v_1, v_2, \dots, v_q$  in order, where  $q \geq 3$ . Let  $W_q$  be a wheel obtained from  $C_q$  by adding a new vertex  $v$ , the *central vertex* of  $W_q$ , and joining  $v$  to all vertices of  $C_q$ . Vertices  $v_1, v_2, \dots, v_q$  are called *rim vertices* of  $W_q$ . A cycle  $C \subseteq W_q$  is a *central  $k$ -cycle* if it contains the central vertex of  $W_q$  and  $|C| = k$ . For any plane triangulation  $T$  with at least four vertices and any  $v \in V(T)$ , the subgraph of  $T$  induced by  $N_T[v]$  contains the wheel  $W_{d_T(v)}$  with central vertex  $v$  as a subgraph. Let  $c(v)$  be the set of all colors such that each is used to color the edges of  $W_{d_T(v)}$  under any edge-coloring  $c$  of  $T$ . Lemma 2.3 below will be used in our proof.

**Lemma 2.3** ([6]) *Let  $T$  be a plane triangulation and let  $c : E(T) \rightarrow [m]$  be a surjection, where  $m$  is a positive integer. Then*

$$\sum_{v \in V(T)} |c(v)| \geq 4m.$$

To establish an upper bound for  $ar_{\mathcal{P}}(n, C_k)$  when  $k \in \{6, 7\}$ , we use the main ideas in [6] by studying lower and upper bounds for the planar anti-Ramsey numbers when host graphs are wheels. For integers  $k \geq 4$  and  $q \geq k - 1$ , we define  $ar_{\mathcal{P}}(W_q, C_k)$  to be the maximum number of colors in an edge-coloring of  $W_q$  that has no rainbow copy of  $C_k$ .

**Theorem 2.4** *For integers  $k \geq 5$  and  $q \geq k - 1$ ,  $\lfloor \frac{2k-7}{k-3}q \rfloor \leq ar_{\mathcal{P}}(W_q, C_k) \leq \lfloor \frac{2k-5}{k-2}q \rfloor$ .*

**Proof.** Let  $W_q$  be a wheel with rim vertices  $v_1, v_2, \dots, v_q$  and central vertex  $v$ . To obtain the desired lower bound, let  $c : E(W_q) \rightarrow [\lfloor (2k - 7)q / (k - 3) \rfloor]$  be an edge-coloring of  $W_q$  defined as follows: for each  $i \in [q]$ , let  $r := i \bmod(k - 3)$  and  $c(vv_i) := i$ ,

$$c(v_i v_{i+1}) = \begin{cases} (k - 4) \cdot \frac{i-r}{k-3} + q + r - 1, & \text{if } r \in \{3, 4, \dots, k - 4\}, \\ (k - 4) \cdot \frac{i-2}{k-3} + q + 1, & \text{if } r = 2, \\ (k - 4) \cdot \frac{i}{k-3} + q, & \text{if } r = 0, \end{cases}$$

and

$$c(v_i v_{i+1}) = \begin{cases} (k - 4) \cdot \frac{i-1}{k-3} + q + 1, & \text{if } i \neq q \text{ and } r = 1, \\ (k - 4) \cdot \frac{i-1}{k-3} + q, & \text{if } i = q \text{ and } r = 1, \end{cases}$$

where all arithmetic on the index  $i + 1$  here and henceforth is done modulo  $q$ . It can be easily checked that  $c$  is a surjection and  $W_q$  has no rainbow  $C_k$  (but contains a rainbow copy of  $C_{k-1}$ ) under the coloring  $c$ . Hence,  $ar_{\mathcal{P}}(W_q, C_k) \geq \lfloor (2k - 7)q/(k - 3) \rfloor$ .

Next we prove that  $ar_{\mathcal{P}}(W_q, C_k) \leq (2k - 5)q/(k - 2)$ . Let  $c : E(W_q) \rightarrow [m]$  be any surjection such that  $W_q$  contains no rainbow  $C_k$  under the coloring  $c$ . It suffices to show that  $m \leq (2k - 5)q/(k - 2)$ . For any integer  $\ell$ , let  $A_\ell$  be the set of colors used  $\ell$  times under the coloring  $c$ . For integers  $\alpha \in [m]$  and  $j \geq 1$ , let:  $\eta_j(\alpha)$  be the number of central  $k$ -cycles in  $W_q$  containing  $j$  edges colored  $\alpha$  under  $c$ ,  $\eta(\alpha) := \sum_{j=2}^k \eta_j(\alpha)$ ,  $\beta(\alpha) := |\{i \in [q] : c(vv_i) = \alpha\}|$  and  $\beta'(\alpha) := |\{i \in [q] : c(v_i v_{i+1}) = \alpha\}|$ . For any integer  $\ell$ , it is easy to check that  $\beta(\alpha) + \beta'(\alpha) = \ell$  for any  $\alpha \in A_\ell$ . Notice that for any integer  $i \in [q]$ ,  $vv_i$  belongs to exactly two central  $k$ -cycles and  $v_i v_{i+1}$  belongs to exactly  $k - 2$  central  $k$ -cycles in  $W_q$ . For any  $\alpha \in A_\ell$ , we see that

$$2\eta(\alpha) \leq 2\eta(\alpha) + \eta_1(\alpha) \leq \sum_{j \geq 1} j\eta_j(\alpha) = 2\beta(\alpha) + (k - 2)\beta'(\alpha) \leq (k - 2)\ell,$$

which implies that  $\eta(\alpha) \leq (k - 2)\ell/2$ . Since each of the  $q$  central  $k$ -cycles of  $W_q$  contains a color  $\alpha$  with  $\eta(\alpha) \geq 1$ , we have

$$q \leq \sum_{\ell \geq 2} \sum_{\alpha \in A_\ell} \eta(\alpha) \leq \sum_{\ell \geq 2} (k - 2)\ell|A_\ell|/2,$$

which implies  $2q/(k - 2) \leq \sum_{\ell \geq 2} \ell|A_\ell|$ . This, together with  $2q = e(W_q) = \sum_{\ell \geq 1} \ell|A_\ell|$ , implies that  $|A_1| \leq (2k - 6)q/(k - 2)$ . Then

$$m = |A_1| + \sum_{\ell \geq 2} |A_\ell| \leq |A_1| + \sum_{\ell \geq 2} |A_\ell|/2 = |A_1|/2 + \sum_{\ell \geq 1} \ell|A_\ell|/2 \leq (2k - 5)q/(k - 2),$$

as desired. ■

Corollary 2.5 below follows from the fact that  $\lfloor \frac{2k-7}{k-3}q \rfloor = 2q - \lfloor \frac{q}{k-3} \rfloor$ ,  $\lfloor \frac{2k-5}{k-2}q \rfloor = 2q - \lfloor \frac{q}{k-2} \rfloor$  and  $ar_{\mathcal{P}}(W_q, C_k) = 2q - \lfloor \frac{q}{k-3} \rfloor$  if  $\lfloor \frac{q}{k-2} \rfloor = \lfloor \frac{q}{k-3} \rfloor$ . One can see that  $\lfloor \frac{q}{k-2} \rfloor = \lfloor \frac{q}{k-3} \rfloor$  when  $q \in \{t(k - 2), \dots, t(k - 2) + k - 4 - t\}$  for any integer  $t \in [k - 4]$ .

**Corollary 2.5** *Let  $k \geq 5$  and  $q \geq k - 1$  be integers. If  $q \in \{t(k - 2), \dots, t(k - 2) + k - 4 - t\}$  for some integer  $t \in [k - 4]$ , then  $ar_{\mathcal{P}}(W_q, C_k) = 2q - \lfloor \frac{q}{k-3} \rfloor$ .*

We are ready to determine the exact value for  $ar_{\mathcal{P}}(W_q, C_6)$  when  $q \geq 5$ .

**Theorem 2.6** *For integer  $q \geq 5$ ,  $ar_{\mathcal{P}}(W_q, C_6) = \lfloor 5q/3 \rfloor$ .*

**Proof.** By Theorem 2.4,  $ar_{\mathcal{P}}(W_q, C_6) \geq \lfloor 5q/3 \rfloor$ . To prove that  $ar_{\mathcal{P}}(W_q, C_6) \leq \lfloor 5q/3 \rfloor$ , it suffices to show that for any surjection  $c : E(W_q) \rightarrow [m]$  such that  $W_q$  contains no rainbow  $C_6$  under the coloring  $c$ , we must have  $m \leq \lfloor 5q/3 \rfloor$ . We do that next.



Let  $A_\ell$  be the set of colors used  $\ell$  times under the coloring  $c$ . For  $\alpha \in [m]$ , let  $\eta_j(\alpha)$  be the number of central 6-cycles in  $W_q$  containing  $j$  edges colored  $\alpha$  under  $c$ ,  $\eta(\alpha) := \sum_{j=2}^6 \eta_j(\alpha)$ ,  $\beta(\alpha) := |\{i \in [q] : c(vv_i) = \alpha\}|$  and  $\beta'(\alpha) := |\{i \in [q] : c(v_i v_{i+1}) = \alpha\}|$ . Then  $\beta(\alpha) + \beta'(\alpha) = \ell$  for all  $\alpha \in A_\ell$ . Notice that for any integer  $i \in [q]$ ,  $vv_i$  belongs to exactly two central 6-cycles and  $v_i v_{i+1}$  belongs to exactly four central 6-cycles. For any  $\alpha \in A_\ell$ , we see that

$$2\eta(\alpha) \leq 2\eta(\alpha) + \eta_1(\alpha) \leq \sum_{j \geq 1} j\eta_j(\alpha) = 2\beta(\alpha) + 4\beta'(\alpha) \leq 4\ell.$$

This implies that  $\eta(\alpha) \leq 2\ell$ . Notice that for any  $\alpha \in A_2$ , two edges of  $W_q$  colored by  $\alpha$  can prevent at most three central 6-cycles from being rainbow under the coloring  $c$ , and so  $\eta(\alpha) = \eta_2(\alpha) \leq 3$ . Since each of the  $q$  central 6-cycles of  $W_q$  contains a color, say  $\alpha \in [m]$ , with  $\eta(\alpha) \geq 1$ , it follows that

$$q \leq \sum_{\ell \geq 2} \sum_{\alpha \in A_\ell} \eta(\alpha) \leq 3|A_2| + \sum_{\ell \geq 3} 2\ell|A_\ell|.$$

Thus  $q/2 \leq 3|A_2|/2 + \sum_{\ell \geq 3} \ell|A_\ell|$ . This, together with  $2q = e(W_q) = \sum_{\ell \geq 1} \ell|A_\ell|$ , implies that  $2|A_1| + |A_2| \leq 3q$ . Then

$$\begin{aligned} m &= |A_1| + |A_2| + \sum_{\ell \geq 3} |A_\ell| \leq |A_1| + |A_2| + \sum_{\ell \geq 3} \ell|A_\ell|/3 \\ &= (2|A_1| + |A_2|)/3 + \sum_{\ell \geq 1} \ell|A_\ell|/3 = (2|A_1| + |A_2|)/3 + 2q/3 \leq 5q/3, \end{aligned}$$

as desired. ■

Finally, we obtain new upper bounds for  $ar_{\mathcal{P}}(n, C_6)$  when  $n \geq 8$  and  $ar_{\mathcal{P}}(n, C_7)$  when  $n \geq 13$ , respectively.

**Theorem 2.7**  $ar_{\mathcal{P}}(n, C_6) \leq 17(n-2)/6$  for all  $n \geq 8$ , and  $ar_{\mathcal{P}}(n, C_7) \leq (59n-113)/20$  for all  $n \geq 13$ .

**Proof.** We first prove that  $ar_{\mathcal{P}}(n, C_6) \leq 17(n-2)/6$  for all integers  $n \geq 8$ . Let  $n \geq 8$  be given and let  $T$  be any plane triangulation on  $n$  vertices such that  $T$  contains  $C_6$  as a subgraph. Let  $c : E(T) \rightarrow [m]$  be any surjection such that  $T$  contains no rainbow  $C_6$  under the coloring  $c$ . It suffices to show that  $m \leq 17(n-2)/6$ . Since  $e(T) = 3n-6$  and  $n \geq 8$ ,  $T$  must have at least two vertices each with degree at least five. Thus,  $n_4(T) \leq n-2-n_3(T)$  and  $n_3(T) \geq 0$ . For any  $v \in V(T)$ , we see that  $|c(v)| \leq e(W_{d_T(v)}) = 2d_T(v)$ . But for any  $v \in V(T)$  with  $d_T(v) \geq 5$ , by Theorem 2.6,  $|c(v)| \leq ar_{\mathcal{P}}(W_{d_T(v)}, C_6) = \lfloor 5d_T(v)/3 \rfloor$ . By Lemma 2.3,

$$\begin{aligned} 4m &\leq \sum_{v \in V(T)} |c(v)| \leq 6n_3(T) + 8n_4(T) + \sum_{v \in V(T), d_T(v) \geq 5} \lfloor 5d_T(v)/3 \rfloor \\ &\leq n_3(T) + 4n_4(T)/3 + 5/3 \cdot \sum_{v \in V(T)} d_T(v) \\ &\leq 4(n-2)/3 - n_3(T)/3 + 5/3 \cdot 2(3n-6) \leq 34(n-2)/3, \end{aligned}$$

which implies that  $m \leq 17(n-2)/6$ , as desired.

It remains to prove that  $ar_{\mathcal{P}}(n, C_7) \leq (59n-113)/20$  for all  $n \geq 13$ . The proof is similar to the proof of  $ar_{\mathcal{P}}(n, C_6) \leq 17(n-2)/6$ . We include a proof here for completeness. Let  $n \geq 13$  be given and let  $T$  be any plane triangulation on  $n$  vertices such that  $T$  contains  $C_7$  as a subgraph. Let  $c : E(T) \rightarrow [m]$  be any surjection such that  $T$  contains no rainbow  $C_7$  under the coloring  $c$ . It suffices to show that  $m \leq (59n-113)/20$ . Since  $e(T) = 3n-6$  and  $n \geq 13$ ,  $T$  must have at least one vertex of degree six. Thus,  $n_5(T) \leq n-1-n_3(T)-n_4(T)$  and  $n_i(T) \geq 0$  ( $i=3,4$ ). For any  $v \in V(T)$ , we see that  $|c(v)| \leq e(W_{d_T(v)}) = 2d_T(v)$ . But for any  $v \in V(T)$  with  $d_T(v) \geq 6$ , by Theorem 2.4,  $|c(v)| \leq ar_{\mathcal{P}}(W_{d_T(v)}, C_7) \leq \lfloor 9d_T(v)/5 \rfloor$ . By Lemma 2.3,

$$\begin{aligned} 4m &\leq \sum_{v \in V(T)} |c(v)| \leq 6n_3(T) + 8n_4(T) + 10n_5(T) + \sum_{v \in V(T), d_T(v) \geq 6} \lfloor 9d_T(v)/5 \rfloor \\ &\leq 3n_3(T)/5 + 4n_4(T)/5 + n_5(T) + 9/5 \cdot \sum_{v \in V(T)} d_T(v) \\ &\leq n-1-2n_3(T)/5-n_4(T)/5+9/5 \cdot 2(3n-6) \leq 59(n-2)/5+1, \end{aligned}$$

which implies that  $m \leq (59n-113)/20$ , as desired.

This completes the proof of Theorem 2.7. ■

### 3 Rainbow Paths

In this section, we study planar anti-Ramsey numbers for paths. We begin with a construction of a plane triangulation  $T_H$  that will be needed in the proof of Theorem 3.3.

**Lemma 3.1** *For any integers  $p \geq 1$  and  $n = 3p+2$ , there exist plane triangulations  $H$  on  $p+2$  vertices and  $T_H$  on  $n$  vertices such that  $H$  and  $T_H$  satisfy the following.*

- (a)  $H \subseteq T_H$  and  $H$  is hamiltonian;
- (b)  $V(T_H) \setminus V(H)$  is an independent set in  $T_H$ ;
- (c) The longest path in  $T_H$  has  $2p+5-\max\{0, 3-p\}$  vertices; and
- (d) The longest path in  $T_H$  with both endpoints in  $V(H)$  has  $2p+3$  vertices.

**Proof.** Let  $P$  be a path with vertices  $v_1, v_2, \dots, v_p$  in order. Let  $H$  be the plane triangulation obtained from  $P$  by adding two adjacent vertices  $x, y$  and joining each of  $x$  and  $y$  to all vertices on  $P$  with the outer face of  $H$  having vertices  $x, y, v_p$  on its boundary. Then  $|H| = p+2$  and  $H$  is hamiltonian. Let  $T_H$  be the plane triangulation obtained from  $H$  by adding a new vertex to each 3-face  $F$  of  $H$  and then joining it to all vertices on the boundary of  $F$ . For each  $i \in \{1, 2, \dots, p-1\}$ , let  $u_i$  and  $w_i$  be the new vertices added to the faces with vertices  $v_i, v_{i+1}, x$  and  $v_i, v_{i+1}, y$  on the boundary, respectively. Let  $w, z$  be the new vertices added to the outer face of  $H$  and the face of  $H$  with vertices  $x, y, v_1$  on the boundary. The construction of  $T_H$  when  $p=5$  is depicted in Figure

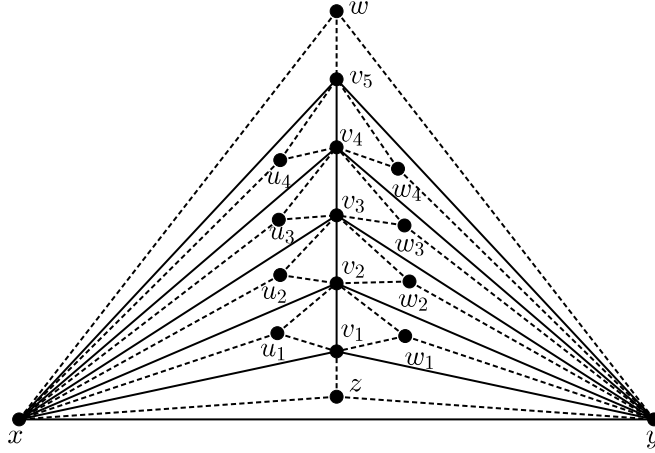


Figure 3: The construction of  $T_H$  when  $p = 5$ .

3. Then  $|T_H| = |H| + f_3(H) = 3p + 2 = n$  and  $V(T_H) \setminus V(H) = \{u_1, \dots, u_{p-1}, w_1, \dots, w_{p-1}, w, z\}$ . Clearly,  $V(T_H) \setminus V(H)$  is a maximal independent set of  $T_H$  with  $|V(T_H) \setminus V(H)| = f_3(H) = 2p$  and  $|V(T_H) \setminus V(H)| \geq |H| + 1 - \max\{0, 3 - p\}$ . It can be easily checked that the longest path in  $T_H$  has  $2p + 5 - \max\{0, 3 - p\}$  vertices, and the longest path with both endpoints in  $V(H)$  has  $2p + 3$  vertices.  $\blacksquare$

**Theorem 3.2** *For any  $k \in \{8, 9\}$ , let  $\varepsilon = k \bmod(2)$  and  $n \geq k$  be an integer. Then  $ar_{\mathcal{P}}(n, P_k) \geq (3n + 3\varepsilon - \varepsilon^* - 3)/2$ , where  $\varepsilon^* = (n + 1 + \varepsilon) \bmod(2)$ .*

**Proof.** Let  $k \in \{8, 9\}$ ,  $n, \varepsilon, \varepsilon^*$  be given as in the statement. Let  $t$  be a positive integer satisfying  $2t - 3 - \varepsilon + \varepsilon^* = n$ . Then  $t \geq k - 3$  because  $n \geq k$ . Let  $H$  be a plane drawing of  $K_2 + (\overline{K}_{t-3-\varepsilon} \cup K_{\varepsilon+1})$ . Clearly,  $H$  has 3-faces and 4-faces only. Notice that  $|H| = t$ ,  $f_3(H) = 2 + 2\varepsilon$ ,  $f_4(H) = t - 3 - \varepsilon$ ,  $e(H) = 2t - 3 + \varepsilon$ , and  $H$  is  $P_{k-2}$ -free but not  $P_{k-3}$ -free. Let  $\mathcal{F}$  be a set which consists of all 4-faces of  $H$  and  $\varepsilon^*$  of the 3-faces of  $H$ . Let  $T^*$  be the plane triangulation obtained from  $H$  by adding a new vertex to each face  $F \in \mathcal{F}$  and then joining it to all vertices on the boundary of  $F$ . Then  $|T^*| = |H| + |\mathcal{F}| = |H| + f_4(H) + \varepsilon^* = 2t - 3 - \varepsilon + \varepsilon^* = n$ . Clearly,  $T^* \in \mathcal{T}_n(P_k)$ . Now let  $c$  be an edge-coloring of  $T^*$  defined as follows: edges in  $E(H)$  are colored with distinct colors under  $c$  (that is,  $T^*$  contains a rainbow copy of  $H$  under  $c$ ), and for each  $F \in \mathcal{F}$ , all the new edges added inside  $F$  are colored the same, but for distinct faces  $F, F' \in \mathcal{F}$ , new edges inside  $F$  are colored differently than the new edges inside  $F'$ . It can be easily checked that  $T^*$  has no rainbow  $P_k$  but contains a rainbow copy of  $P_{k-1}$  under  $c$ . Then  $c(T^*) = e(H) + f_4(H) + \varepsilon^* = 3t - 6 + \varepsilon^* = (3n + 3\varepsilon - \varepsilon^* - 3)/2$ , since  $n = 2t - 3 - \varepsilon + \varepsilon^*$ . Hence,  $ar_{\mathcal{P}}(n, P_k) \geq c(T^*) \geq (3n + 3\varepsilon - \varepsilon^* - 3)/2$ , as desired. This completes the proof of Theorem 3.2.  $\blacksquare$

We next prove a lower bound for  $ar_{\mathcal{P}}(n, P_k)$  when  $k \geq 10$ .

**Theorem 3.3** *Let  $k$  and  $n$  be two integers such that  $n \geq k \geq 10$ . Let  $\varepsilon = k \bmod(2)$ . Then*

$$ar_{\mathcal{P}}(n, P_k) \geq \begin{cases} n + 2k - 12 & \text{if } k \leq n < 3\lfloor k/2 \rfloor + \varepsilon - 5, \\ (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 43)/2 & \text{if } 3\lfloor k/2 \rfloor + \varepsilon - 5 \leq n \leq 5\lfloor k/2 \rfloor + \varepsilon - 15, \\ 2n + k - 14 & \text{if } n > 5\lfloor k/2 \rfloor + \varepsilon - 15. \end{cases}$$

**Proof.** Let  $k, n, \varepsilon$  be given as in the statement. Assume first that  $k \leq n < 3\lfloor k/2 \rfloor + \varepsilon - 5$ . Then  $k \geq 12$ . Let  $p = k - 5$  and let  $P$  and  $H$  be defined in the proof of Lemma 3.1. By Lemma 3.1,  $|H| = k - 3$ ,  $f_3(H) = 2k - 10$ ,  $e(H) = 3k - 15$  and  $H$  is hamiltonian. Since  $n < 3\lfloor k/2 \rfloor + \varepsilon - 5$ , we see that  $n - k + 3 < f_3(H)$ . Let  $\mathcal{F}$  be a set which consists of  $n - k + 3$  many 3-faces of  $H$ . Let  $T^*$  be the plane triangulation obtained from  $H$  by adding a new vertex to each face  $F \in \mathcal{F}$  and then joining it to all vertices on the boundary of  $F$ . Clearly,  $T^* \in \mathcal{T}_n(P_k)$ . Now let  $c$  be an edge-coloring of  $T^*$  defined as follows: edges in  $E(H)$  are colored with distinct colors under  $c$  (that is,  $T^*$  contains a rainbow copy of  $H$  under  $c$ ), and for each  $F \in \mathcal{F}$ , all the new edges added inside  $F$  are colored the same, but for distinct faces  $F, F' \in \mathcal{F}$ , new edges inside  $F$  are colored differently than the new edges inside  $F'$ . It can be easily checked that  $T^*$  has no rainbow  $P_k$  but contains a rainbow copy of  $P_{k-1}$  under  $c$ . Then  $c(T^*) = e(H) + |\mathcal{F}| = 3k - 15 + n - k + 3 = n + 2k - 12$ . Hence,  $ar_{\mathcal{P}}(n, P_k) \geq c(T^*) \geq n + 2k - 12$ .

Next assume that  $3\lfloor k/2 \rfloor + \varepsilon - 5 \leq n \leq 5\lfloor k/2 \rfloor + \varepsilon - 15$ . Let  $\varepsilon^* = (n + \lfloor k/2 \rfloor) \bmod(2)$ . By the choice of  $\varepsilon^*$ , let  $t$  be a positive integer satisfying  $2t + \varepsilon^* + 10 - 3\lfloor k/2 \rfloor - \varepsilon = n$ . Since  $n \geq 3\lfloor k/2 \rfloor + \varepsilon - 5$ , it follows that  $t - 3\lfloor k/2 \rfloor + 10 \geq 2 + \varepsilon$ . Let  $p = \lfloor k/2 \rfloor - 4$  and let  $P, H, T_H, x, y, w, v_{\lfloor k/2 \rfloor - 4}$  be defined in the proof of Lemma 3.1. By Lemma 3.1,  $|H| = \lfloor k/2 \rfloor - 2$ ,  $f_3(H) = 2|H| - 4 = 2\lfloor k/2 \rfloor - 8$  and  $|T_H| = |H| + f_3(H) = 3\lfloor k/2 \rfloor - 10 \geq k - 5 - \varepsilon$ . Let  $F^*$  be the outer face of  $T_H$  and  $F_0$  be the 3-face of  $T_H$  with vertices  $x, w, v_{\lfloor k/2 \rfloor - 4}$  on its boundary. Let  $T$  be the plane graph on  $t$  vertices obtained from  $T_H$  by adding  $t - 3\lfloor k/2 \rfloor + 10 \geq 2 + \varepsilon$  new vertices to the face  $F^*$  and then joining each of the new vertices to both  $x$  and  $y$  (and further adding exactly one edge among the new vertices added inside  $F^*$  when  $\varepsilon = 1$ ). Then all 4-faces of  $T$  are inside the face  $F^*$  of  $T_H$ ,  $e(T) = e(T_H) + 2(t - 3\lfloor k/2 \rfloor + 10) + \varepsilon = 2t + 3\lfloor k/2 \rfloor - 16 + \varepsilon$  and  $f_4(T) = t - 3\lfloor k/2 \rfloor + 10 - \varepsilon$ . Let  $\mathcal{F}$  be a set which consists of all 4-faces of  $T$  (and  $F_0$  when  $\varepsilon^* = 1$ ). Finally, let  $T^*$  be the plane triangulation obtained from  $T$  by adding a new vertex to each face  $F \in \mathcal{F}$  and then joining it to all vertices on the boundary of  $F$ . Then  $|T^*| = |T| + f_4(T) + \varepsilon^* = 2t - 3\lfloor k/2 \rfloor + 10 + \varepsilon^* - \varepsilon = n$ . By Lemma 3.1, the longest  $(x, y)$ -path in  $T_H$  has  $k - 5 - \varepsilon$  vertices. Clearly, the longest  $(x, y)$ -path in  $T^*$  with all its internal vertices inside the face  $F^*$  contains all the new vertices added to  $F^*$ . Thus  $T^*$  contains  $P_k$  as a subgraph and so  $T^* \in \mathcal{T}_n(P_k)$ . Now let  $c$  be an edge-coloring of  $T^*$  defined as follows: edges in  $E(T)$  are colored with distinct colors under  $c$  (that is,  $T^*$  contains a rainbow copy of  $T$  under  $c$ ), and for each  $F \in \mathcal{F}$ , all the new edges added inside  $F$  are colored the same, but for distinct  $F, F' \in \mathcal{F}$ , new edges inside  $F$  are colored differently than the new edges inside  $F'$ . We see

that  $T^*$  has no rainbow  $P_k$  but contains a rainbow  $P_{k-1}$  under  $c$ . Since  $n = 2t + \varepsilon^* + 10 - 3\lfloor k/2 \rfloor - \varepsilon$ , we see that

$$\begin{aligned} c(T^*) &= e(T) + f_4(T) + \varepsilon^* = (2t + 3\lfloor k/2 \rfloor - 16 + \varepsilon) + (t - 3\lfloor k/2 \rfloor + 10 - \varepsilon) + \varepsilon^* \\ &= (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 42 - \varepsilon^*)/2 \\ &\geq (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 43)/2. \end{aligned}$$

Hence,  $ar_{\mathcal{P}}(n, P_k) \geq c(T^*) \geq (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 43)/2$ , as desired.

Finally assume that  $n \geq 5\lfloor k/2 \rfloor + \varepsilon - 14$ . Let  $n - k + 7 = 3m + r$ , where  $m$  is a positive integer and  $r \in \{0, 1, 2\}$ . Since  $k \geq 10$  and  $n \geq 5\lfloor k/2 \rfloor + \varepsilon - 14$ , we have  $m \geq 3$  or  $m = r = 2$ . Let  $t := k + 2m - 7 + \lfloor r/2 \rfloor$ . Then  $t \geq k - 2$  because  $m \geq 3$  or  $m = r = 2$ , and  $t + \lceil (t - k + 7)/2 \rceil = n - \varepsilon'$ , where  $\varepsilon' = 1$  when  $r = 1$  and  $\varepsilon' = 0$  when  $r \in \{0, 2\}$ . Let  $p = k - 9$  and let  $P, H, x, y, v_1, \dots, v_{k-9}$  be defined in the proof of Lemma 3.1. Then  $|H| = k - 7$  and the longest path between  $x$  and  $y$  in  $H$  has  $k - 7$  vertices. Let  $T'$  be the plane triangulation on  $t$  vertices obtained from  $H$  by: adding  $t - k + 7 \geq 5$  new vertices to the outer face of  $H$ , then adding a matching of size  $\lfloor (t - k + 7)/2 \rfloor \geq 2$  among the new vertices, and finally joining each of the new vertices to both  $x$  and  $y$ . We see that  $T'$  is a connected  $P_{k-2}$ -free plane graph with only 3-faces and 4-faces. It can be easily checked that  $f_4(T') = \lfloor (t - k + 7)/2 \rfloor$  and  $e(T') = 2t + k - 13 + \lfloor (t - k + 7)/2 \rfloor$ . Let  $F_0$  be the 3-face of  $T'$  with vertices  $x, y, v_{k-9}$  when  $k = 10$  and  $x, v_{k-10}, v_{k-9}$  when  $k \geq 11$  on its boundary. Let  $\mathcal{F}$  be a set which consists of all 4-faces of  $T'$  (and  $F_0$  when  $\varepsilon' = 1$ ). Let  $T^*$  be the plane triangulation obtained from  $T'$  by adding a new vertex to each  $F \in \mathcal{F}$  and then joining it to all vertices on the boundary of  $F$ . Then  $|T^*| = |T'| + |\mathcal{F}| = |T'| + f_4(T') + \varepsilon' = t + \lceil (t - k + 7)/2 \rceil + \varepsilon' = n$ . Clearly,  $T^*$  contains  $P_k$  as a subgraph and so  $T^* \in \mathcal{T}_n(P_k)$ . Now let  $c$  be an edge-coloring of  $T^*$  defined as follows: edges in  $E(T')$  are colored with distinct colors under  $c$  (that is,  $T^*$  contains a rainbow copy of  $T'$  under  $c$ ), and for each  $F \in \mathcal{F}$ , all the new edges added inside  $F$  are colored the same, but for distinct faces  $F, F' \in \mathcal{F}$ , new edges inside  $F$  are colored differently than the new edges inside  $F'$ . We see that  $T^*$  has no rainbow  $P_k$  under  $c$  but contains a rainbow copy of  $P_{k-1}$ . Then

$$\begin{aligned} c(T^*) &= e(T') + f_4(T') + \varepsilon' = \left(2t + k - 13 + \left\lfloor \frac{t - k + 7}{2} \right\rfloor\right) + \left(\left\lceil \frac{t - k + 7}{2} \right\rceil + \varepsilon'\right) \\ &= 2n + k - 13 - \varepsilon' + \left\lfloor \frac{t - k + 7}{2} \right\rfloor - \left\lceil \frac{t - k + 7}{2} \right\rceil \\ &= 2n + k - 13 - \varepsilon' + \left\lfloor \frac{\lfloor r/2 \rfloor}{2} \right\rfloor - \left\lceil \frac{\lfloor r/2 \rfloor}{2} \right\rceil \\ &\geq 2n + k - 14, \end{aligned}$$

since  $n = t + \lceil (t - k + 7)/2 \rceil + \varepsilon'$  and  $t = k + 2m - 7 + \lfloor r/2 \rfloor$ . Hence,  $ar_{\mathcal{P}}(n, P_k) \geq c(T^*) \geq 2n + k - 14$ , as desired. This completes the proof of Theorem 3.3.  $\blacksquare$

**Remark.** In the proofs of Theorem 3.2 and Theorem 3.3,  $T^* \in \mathcal{T}_n(P_k)$  has no rainbow  $P_k$  but does contain a rainbow copy of  $P_{k-1}$  under the coloring  $c$  we found.

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