Schur positivity and log-concavity related to longest increasing subsequences

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Abstract. Chen proposed a conjecture on the log-concavity of the generating function for the symmetric group with respect to the length of longest increasing subsequences of permutations. Motivated by Chen's log-concavity conjecture, Bóna, Lackner and Sagan further studied similar problems by restricting the whole symmetric group to certain of its subsets. They obtained the log-concavity of the corresponding generating functions for these subsets by using the hook-length formula. In this paper, we generalize and prove their results by establishing the Schur positivity of certain symmetric functions. This also enables us to propose a new approach to Chen's original conjecture.

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1 Introduction

Given positive integers m, n and $\lceil \frac{n}{2} \rceil \leq k \leq n$, let $(k^m, (n-k)^m)$ denote the partition with m parts equal to k and m parts equal to $n-k$. Similarly, for $1 \leq k \leq n$, let $(k^m, 1^{m(n-k)})$ denote the partition with m parts equal to k and $m(n-k)$ parts equal to 1. Given a partition λ, let f^λ denote the number of standard Young tableaux of shape λ. The main objective of this paper is to prove the following result.

Theorem 1.1. *Suppose that m, n are two positive integers.*

(1) For $\lceil \frac{n}{2} \rceil < k < n$ we have

$$(f^{(k^m,(n-k)^m)})^2 \geq f^{((k+1)^m,(n-k-1)^m)} f^{((k-1)^m,(n-k+1)^m)}.$$

(2) For $1 < k < n$ we have

$$(f^{(k^m,1^{m(n-k)})})^2 \geq f^{((k+1)^m,1^{m(n-k-1)})} f^{((k-1)^m,1^{m(n-k+1)})}.$$

The roots of this paper lie in the work of Bóna, Lackner and Sagan [2], who first proved the above theorem for the case of $m = 1, 2$ by using the celebrated hook-length formula. We will present two proofs of Theorem 1.1, one of which is similar to Bóna, Lackner and Sagan's proof for small m, and the other is based on some results on Schur positivity due to Lam, Postnikov, and Pylyavskyy [7].

Let us first review some background. We will adopt the notation and terminology found in Bóna, Lackner and Sagan [2]. Given a positive integer n, let $\mathfrak{S}_n$ be the symmetric group of all permutations of $[n] := \{1, 2, \ldots, n\}$. For a given permutation $\pi \in \mathfrak{S}_n$, let $\mathrm{is}(\pi)$ denote the length of the longest increasing subsequence of π. Define $L_{n,k}$ to be the set of permutations $\pi \in \mathfrak{S}_n$ with $\mathrm{is}(\pi) = k$ for $1 \le k \le n$. Let $\ell_{n,k} = |L_{n,k}|$. Chen proposed the following conjecture.

Conjecture 1.2 ([3, Conjecture 1.1])**.** *For any fixed n, the sequence $\{\ell_{n,k}\}_{k=1}^n$ is log-concave, i.e., $\ell_{n,k}^2 \ge \ell_{n,k+1}\ell_{n,k-1}$ for $1 < k < n$.*

Bóna, Lackner and Sagan [2] further made a companion conjecture for involutions. Define $I_{n,k}$ to be the set of involutions $\pi \in \mathfrak{S}_n$ with $\mathrm{is}(\pi) = k$ for $1 \le k \le n$. Let $i_{n,k} = |I_{n,k}|$.

Conjecture 1.3 ([2, Conjecture 1.2])**.** *For any fixed n, the sequence $\{i_{n,k}\}_{k=1}^n$ is log-concave.*

Bóna, Lackner and Sagan showed that there is a close connection between Conjecture 1.2 and Conjecture 1.3 by using the Robinson-Schensted correspondence. It is well known that, under the Robinson-Schensted correspondence, each permutation $\pi \in \mathfrak{S}_n$ is mapped to a pair of standard Young tableaux of the same partition shape, say $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$. Moreover, we have $\mathrm{is}(\pi) = \lambda_1$. We also say that π is of shape λ, denoted $\mathrm{sh}\,\pi = \lambda$. Bóna, Lackner and Sagan proved that if there is a shape-preserving injection from $I_{n,k-1} \times I_{n,k+1}$ to $I_{n,k} \times I_{n,k}$, then there is a shape-preserving injection from $L_{n,k-1} \times L_{n,k+1}$ to $L_{n,k} \times L_{n,k}$, see [2, Theorem 2.2].

Though they could not prove Conjectures 1.2 and 1.3, Bóna, Lackner and Sagan proposed a new way to look at these problems. Given a set Λ of partitions of n, for $1 \le k \le n$ let

$$L_{n,k}^{\Lambda} = \{\pi \in L_{n,k} \mid \mathrm{sh}\,\pi \in \Lambda\}, \quad \ell_{n,k}^{\Lambda} = |L_{n,k}^{\Lambda}|; \qquad I_{n,k}^{\Lambda} = \{\pi \in I_{n,k} \mid \mathrm{sh}\,\pi \in \Lambda\}, \quad i_{n,k}^{\Lambda} = |I_{n,k}^{\Lambda}|.$$

Thus, the sequence $\{\ell_{n,k}\}_{k=1}^n$ (resp. $\{i_{n,k}\}_{k=1}^n$) is just $\{\ell_{n,k}^{\Lambda}\}_{k=1}^n$ (resp. $\{i_{n,k}^{\Lambda}\}_{k=1}^n$) when taking Λ to be the set of all partitions of n. They noted that the log-concavity of $\{\ell_{n,k}^{\Lambda}\}_{k=1}^n$ is equivalent to that of $\{i_{n,k}^{\Lambda}\}_{k=1}^n$ provided that the set Λ contains at most one partition with first row of length k for each $1 \le k \le n$. They further obtained the following results, see [2, Theorems 3.1, 3.2, 4.4 and 4.5].

Theorem 1.4. *Suppose that n is a positive integer and $m = 1, 2$.*

(1) For $\Lambda = \{(j^m, (n-j)^m) \mid \lceil \frac{n}{2} \rceil \le j \le n\}$, both $\{\ell_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ and $\{i_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ are log-concave.

(2) For $\Lambda = \{(j^m, 1^{m(n-j)}) \mid 1 \le j \le n\}$, both $\{\ell_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ and $\{i_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ are log-concave.

The Robinson-Schensted correspondence says that $f^{\lambda} = |\{\pi \mid \pi^2 = id \text{ and } \mathrm{sh}\,\pi = \lambda\}|$. Thus Theorem 1.1 for $m = 1, 2$ is equivalent to the log-concavity of $\{i_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ parts of Theorem 1.4.

To prove the inequalities on f^{λ}, a natural way is to use the hook-length formula, as Bóna, Lackner and Sagan did in their paper [2]. Here we will propose another way based on the property of the exponential specialization in the theory of symmetric functions. We shall follow Stanley [9] to introduce the exponential specialization. Let $\Lambda_{\mathbb{Q}}$ denote the ring of symmetric functions over the field $\mathbb{Q}$ of rational numbers. The exponential specialization $\mathrm{ex} : \Lambda_{\mathbb{Q}} \longrightarrow \mathbb{Q}[t]$ is defined by acting on the power sums p_n as

$$\mathrm{ex}(p_n) = t\delta_{1n}, \tag{1}$$

and then extended to an algebra homomorphism. For any symmetric function f, let $\mathrm{ex}_1(f) = \mathrm{ex}(f)_{t=1}$. It is well known that

$$\mathrm{ex}_1(s_\lambda) = \frac{f^\lambda}{n!}, \qquad \text{or equivalently,} \quad f^\lambda = \mathrm{ex}_1(n! s_\lambda) \tag{2}$$

for any $\lambda \vdash n$. Since ex is an algebra homomorphism, the inequalities on f^λ considered in Theorem 1.1 can be deduced from the Schur positivity of the differences of products of Schur functions s_λ; see Section 3.

The rest of the paper is organized as follows. In Section 2, we give a proof of Theorem 1.1 by using the hook-length formula, and deduce the extension Theorem 1.4 to any value of m. In Section 3, we present an alternative proof of Theorem 1.1 based on the Schur positivity of certain symmetric functions. In Section 4, we propose some conjectures for further study.

2 Proof by the hook-length formula

The aim of this section is to give a proof of Theorem 1.1 by using the hook-length formula.

Let us first give an overview of related definitions and results. Given a partition λ, let $\ell(\lambda)$ denote the number of its nonzero parts. Each partition λ is associated to a left justified array of cells with λ_i cells in the i-th row, called the Ferrers or Young diagram of λ. Here we use English notation and number the rows from top to bottom and the columns from left to right. The cell in the i-th row and j-th column is denoted by (i, j). The hook-length of (i, j), denoted by $h_{(i,j)}$, is defined to be the number of cells directly to the right or directly below (i, j), counting (i, j) itself once. For example, the hook length of the cell $(1, 2)$ in Young diagram of $(6, 5, 2)$ is 7, as illustrated in Figure 1.

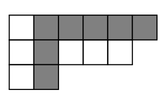

Figure 1: The hook set of $(1, 2)$ composed of gray cells

The classical hook-length formula, which was discovered by Frame, Robinson and Thrall [5], is stated as follows.

Theorem 2.1 ([5]). *For any partition $\lambda \vdash n$, we have*

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{(i,j)}}. \tag{3}$$

For our purpose here, it turns out to be easier to work with the following equivalent form of the hook-length formula, which can be taken as a consequence of the Frobenius character formula, see Fulton and Harris [6].

Theorem 2.2 ([6]). *Given a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{\ell(\lambda)}) \vdash n$, we have*

$$f^\lambda = \frac{n!}{h_{(1,1)}!h_{(2,1)}!\dots h_{(\ell(\lambda),1)}!} \prod_{1\le j_1<j_2\le \ell(\lambda)} (h_{(j_1,1)} - h_{(j_2,1)}). \tag{4}$$

The equivalence between these two formulas is evident by virtue of the equality

$$\prod_{(i,j)\in\lambda} h_{(i,j)} = \frac{h_{(1,1)}!h_{(2,1)}!\dots h_{(\ell(\lambda),1)}!}{\prod_{1\le j_1<j_2\le\ell(\lambda)}(h_{(j_1,1)} - h_{(j_2,1)})}. \tag{5}$$

Now we can give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let us first prove that, for $\lceil \frac{n}{2} \rceil < k < n$,

$$(f^{(k^m,(n-k)^m)})^2 \ge f^{((k+1)^m,(n-k-1)^m)} f^{((k-1)^m,(n-k+1)^m)}. \tag{6}$$

As shown below, we can give a unified proof for any $\lceil \frac{n}{2} \rceil < k < n-1$. The case $k = n-1$ will be treated differently.

We first prove the inequality (6) for $\lceil \frac{n}{2} \rceil < k < n-1$. To this end, we will use the expression for $f^{(k^m,(n-k)^m)}$ ($\lceil \frac{n}{2} \rceil \le k \le n-1$) given by Theorem 2.2. It is easy to see that the hook-lengths of the first column of the partition $(k^m, (n-k)^m)$ are given by

$$h_{(i,1)} = \begin{cases} k + 2m - i, & \text{for } 1 \le i \le m; \\ n - k + 2m - i, & \text{for } m+1 \le i \le 2m. \end{cases} \tag{7}$$

We have

$$\begin{aligned} f^{(k^m,(n-k)^m)} =& \frac{(mn)!}{\prod_{i=1}^m h_{(i,1)}! \times \prod_{j=m+1}^{2m} h_{(j,1)}!} \times \prod_{\substack{1\le i\le m \\ m+1\le j\le 2m}} (h_{(i,1)} - h_{(j,1)}) \\ & \times \prod_{1\le i_1<i_2\le m} (h_{(i_1,1)} - h_{(i_2,1)}) \times \prod_{m+1\le j_1<j_2\le 2m} (h_{(j_1,1)} - h_{(j_2,1)}). \end{aligned}$$

Substituting (7) into the above formula, we obtain

$$\begin{aligned} f^{(k^m,(n-k)^m)} =& \frac{(mn)!}{\prod\limits_{i=1}^{m}(k+2m-i)! \times \prod\limits_{j=m+1}^{2m} (n-k+2m-j)!} \times \prod_{\substack{1\le i\le m \\ m+1\le j\le 2m}} (2k-n+j-i) \\ & \times \prod_{1\le i_1<i_2\le m} (i_2 - i_1) \times \prod_{m+1\le j_1<j_2\le 2m} (j_2 - j_1). \end{aligned}$$

Note that

$$\prod_{i=1}^{m}(k+2m-i)! = \prod_{i=0}^{m-1}(k+m+i)!,$$

$$\prod_{j=m+1}^{2m} (n-k+2m-j)! = \prod_{j=0}^{m-1}(n-k+j)!,$$

and

$$\prod_{\substack{1\le i\le m\\ m+1\le j\le 2m}} (2k-n+j-i) = \prod_{0\le i,j\le m-1} (2k-n+1+j+i).$$

Hence, we have

$$\begin{aligned} f^{(k^m,(n-k)^m)} =& \frac{(mn)!}{\prod_{i=0}^{m-1}(k+m+i)! \times \prod_{j=0}^{m-1}(n-k+j)!} \times \prod_{0\le i,j\le m-1} (2k-n+1+j+i) \\ & \times \prod_{1\le i_1<i_2\le m} (i_2-i_1) \times \prod_{m+1\le j_1<j_2\le 2m} (j_2-j_1). \end{aligned}$$

Note that the last two factors on the right hand side are independent of k. Denote the first factor and the second factor by a_k and b_k respectively, namely,

$$\begin{aligned} a_k &= \frac{(mn)!}{\prod_{i=0}^{m-1}(k+m+i)! \times \prod_{j=0}^{m-1}(n-k+j)!}, \\ b_k &= \prod_{0\le i,j\le m-1} (2k-n+1+j+i). \end{aligned}$$

For $\lceil \frac{n}{2} \rceil < k < n-1$ we have

$$\begin{aligned} \frac{a_k^2}{a_{k-1}a_{k+1}} &= \frac{\prod_{i=0}^{m-1}(k+1+m+i)!(k-1+m+i)!}{\prod_{i=0}^{m-1}(k+m+i)!^2} \times \frac{\prod_{j=0}^{m-1}(n-k-1+j)!(n-k+1+j)!}{\prod_{j=0}^{m-1}(n-k+j)!^2} \\ &= \prod_{i=0}^{m-1} \frac{k+1+m+i}{k+m+i} \prod_{j=0}^{m-1} \frac{n-k+1+j}{n-k+j} \\ &= \frac{2m+k}{m+k} \times \frac{n-k+m}{n-k} > 1. \end{aligned}$$

It is easy to verify that

$$\frac{b_k^2}{b_{k-1}b_{k+1}} = \prod_{0\le i,j\le m-1} \frac{(2k-n+1+j+i)^2}{(2k-n-1+j+i)(2k-n+3+j+i)} > 1,$$

since, for any $\lceil \frac{n}{2} \rceil < k < n-1$ and $0 \le i,j \le m-1$, we have

$$(2k-n+1+j+i)^2 > (2k-n-1+j+i)(2k-n+3+j+i)$$

by the inequality of arithmetic and geometric means.

Thus, for $\lceil \frac{n}{2} \rceil < k < n-1$, we have

$$\frac{(f^{(k^m,(n-k)^m)})^2}{f^{((k+1)^m,(n-k-1)^m)}f^{((k-1)^m,(n-k+1)^m)}} = \frac{a_k^2}{a_{k-1}a_{k+1}} \times \frac{b_k^2}{b_{k-1}b_{k+1}} > 1,$$

as desired.

We now prove (6) for $k = n-1$. Using the hook-length formula given by Theorem 2.1, we get

$$f^{(n^m)} = (mn)! \times \prod_{i=0}^{m-1} \frac{i!}{(n+i)!},$$

$$f^{((n-1)^m,1^m)} = \frac{(mn)!}{m! \times \prod_{i=0}^{m-1}(m+n-1+i)} \times \prod_{i=0}^{m-1} \frac{i!}{(n-2+i)!},$$

$$f^{((n-2)^m,2^m)} = \frac{(mn)!}{m!(m+1)! \times \prod_{i=0}^{m-1}(m+n-2+i)(m+n-3+i)} \times \prod_{i=0}^{m-1} \frac{i!}{(n-4+i)!}.$$

A straightforward computation shows that

$$\frac{(f^{((n-1)^m,1^m)})^2}{f^{(n^m)}f^{((n-2)^m,2^m)}} = (m+1) \times \prod_{i=0}^{m-1} \left(\frac{(m+n-2+i)(n+i)}{(m+n-1+i)(n-2+i)} \cdot \frac{(m+n-3+i)(n-1+i)}{(m+n-1+i)(n-3+i)} \right).$$

Note that for $n, m \geq 1$ and $0 \leq i \leq m-1$, we have

$$(m+n-2+i)(n+i) - (m+n-1+i)(n-2+i) = (n+i) + 2(m-1) > 0$$

and

$$(m+n-3+i)(n-1+i) - (m+n-1+i)(n-3+i) = 2m > 0.$$

Therefore, we have

$$\frac{(f^{((n-1)^m,1^m)})^2}{f^{(n^m)}f^{((n-2)^m,2^m)}} > 1.$$

We proceed to prove the second part of the theorem, namely, for $1 < k < n$,

$$(f^{(k^m,1^{m(n-k)})})^2 \geq f^{((k+1)^m,1^{m(n-k-1)})} f^{((k-1)^m,1^{m(n-k+1)})}.$$

Let us first give an expression of $f^{(k^m,1^{m(n-k)})}$ $(1 \leq k \leq n)$ by using Theorem 2.1. Note that the hook-lengths of the partition $(k^m, 1^{m(n-k)})$ are given by

$$h_{(i,j)} = \begin{cases} m(n-k)+m+k-i, & \text{for } j=1 \text{ and } 1 \leq i \leq m; \\ m(n-k)+m+1-i, & \text{for } j=1 \text{ and } m+1 \leq i \leq m(n-k)+m; \\ h'_{(i,j-1)}, & \text{for } 2 \leq j \leq k \text{ and } 1 \leq i \leq m \end{cases} \tag{8}$$

where $h'_{(i,j)}$ denotes the hook-length of the cell (i,j) in partition $((k-1)^m)$. We have

$$f^{(k^m,1^{m(n-k)})} = \frac{(mn)!}{\prod\limits_{1 \leq i \leq m} h_{(i,1)} \times \prod\limits_{m+1 \leq i \leq m(n-k)+m} h_{(i,1)} \times \prod\limits_{1 \leq i \leq m, 2 \leq j \leq k} h_{(i,j)}}.$$

Substituting (8) into the above formula, we obtain

$$f^{(k^m,1^{m(n-k)})} = \frac{(mn)!}{\left(\prod_{i=0}^{m-1}[m(n-k)+k+i]\right) \times [m(n-k)]! \times \prod_{1\le i\le m, 2\le j\le k} h'_{(i,j-1)}}.$$

Moreover, we see that

$$\prod_{1\le i\le m, 2\le j\le k} h'_{(i,j-1)} = \prod_{(i,j)\in((k-1)^m)} h'_{(i,j)} = \frac{\prod_{i=0}^{m-1}(k+m-2-i)!}{\prod_{0\le i_1<i_2\le m-1}(i_2-i_1)},$$

where the second equality is obtained by applying (5) to the partition $\lambda = ((k-1)^m)$. Note that

$$\prod_{i=0}^{m-1}(k+m-2-i)! = \prod_{i=0}^{m-1}(k-1+i)!.$$

Thus, we have

$$f^{(k^m,1^{m(n-k)})} = \frac{(mn)!}{\left(\prod_{i=0}^{m-1}[m(n-k)+k+i]\right) \times [m(n-k)]!} \times \frac{\prod_{0\le i_1<i_2\le m-1}(i_2-i_1)}{\prod_{i=0}^{m-1}(k-1+i)!}.$$

Let

$$c_k = \left(\prod_{i=0}^{m-1}[m(n-k)+k+i]\right) \times [m(n-k)]! = \frac{(mn-mk)!(mn-mk+k+m-1)!}{(mn-mk+k-1)!}.$$

Then, for $1<k<n$, we have

$$\begin{aligned}
\frac{(f^{(k^m,1^{m(n-k)})})^2}{f^{((k+1)^m,1^{m(n-k-1)})}f^{((k-1)^m,1^{m(n-k+1)})}} &= \frac{\prod_{i=0}^{m-1}(k+i)!\prod_{i=0}^{m-1}(k-2+i)!}{\prod_{i=0}^{m-1}(k-1+i)!\prod_{i=0}^{m-1}(k-1+i)!} \times \frac{c_{k-1}c_{k+1}}{c_k^2} \\
&= \frac{\prod_{i=0}^{m-1}(k+i)}{\prod_{i=0}^{m-1}(k-1+i)} \times \frac{c_{k-1}c_{k+1}}{c_k^2} \\
&= \frac{k+m-1}{k-1} \times \frac{c_{k-1}c_{k+1}}{c_k^2} \\
&> \frac{c_{k-1}c_{k+1}}{c_k^2}.
\end{aligned}$$

Now it suffices to show that $c_k^2 \le c_{k-1}c_{k+1}$ for $1 < k < n$. Let

$$c(z) = \frac{\Gamma(mn - mz + 1)\Gamma(mn - mz + z + m)}{\Gamma(mn - mz + z)}$$

be the continuous function on $[1, n]$, where $\Gamma(z)$ is the Gamma function. Hence, for $1 \le k \le n$, we have $c_k = c(k)$, the value of $c(z)$ evaluated at $z = k$. To prove $c_k^2 \le c_{k-1}c_{k+1}$ for $1 < k < n$, it suffices to show that $(\ln c(z))'' \ge 0$ for $z \in [1, n]$. To this end, we first compute the logarithmic derivative of $c(z)$ as follows:

$$(\ln c(z))' = -m\psi(mn - mz + 1) - (m - 1)\psi(mn - mz + z + m) + (m - 1)\psi(mn - mz + z)$$

where $\psi(z) = (\ln \Gamma(z))'$ is the digamma function. Then we obtain that

$$\begin{aligned}(\ln c(z))'' &= m^2\psi'(mn - mz + 1) + (m - 1)^2\psi'(mn - mz + z + m) - (m - 1)^2\psi'(mn - mz + z)\\ &= m^2\psi'(mn - mz + 1) - (m - 1)^2\psi'(mn - mz + z) + (m - 1)^2\psi'(mn - mz + z + m).\end{aligned}$$

It is known that $\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$ over $z \in (0, +\infty)$, and hence it is positive and decreasing, see [1, p. 260]. Thus,

$$\psi'(mn - mz + 1) \ge \psi'(mn - mz + z)$$

for $z \in [1, n]$, and hence $(\ln c(z))'' \ge 0$ over $[1, n]$. This completes the proof. □

Note that for a fixed integer n and a set Λ of partitions of n, the Robinson-Schensted correspondence shows that

$$\ell_{n,k}^{\Lambda} = \sum_{\lambda \in \Lambda, \lambda_1 = k} \left(f^{\lambda}\right)^2 \quad \text{and} \quad i_{n,k}^{\Lambda} = \sum_{\lambda \in \Lambda, \lambda_1 = k} f^{\lambda} \tag{9}$$

for $1 \le k \le n$. As an immediate consequence of Theorem 1.1, we obtain the following result, which shows that Theorem 1.4 is true for any positive integer m.

Corollary 2.3. *Suppose that m, n are two positive integers.*

(1) For $\Lambda = \{(j^m, (n - j)^m) \mid \lceil \frac{n}{2} \rceil \le j \le n\}$, both $\{\ell_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ and $\{i_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ are log-concave.

(2) For $\Lambda = \{(j^m, 1^{m(n-j)}) \mid 1 \le j \le n\}$, both $\{\ell_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ and $\{i_{mn,k}^{\Lambda}\}_{k=1}^{mn}$ are log-concave.

3 Proof by Schur positivity

The aim of this section is to give another proof of Theorem 1.1 by using Schur positivity. Recall that a symmetric function is said to be Schur positive if it can be written as a non-negative linear combination of Schur functions. Theorem 1.1 is implied by the following result.

Theorem 3.1. *Suppose that m and n are two positive integers.*

(1) For $\lceil \frac{n}{2} \rceil < k < n$, the difference

$$s_{(k^m,(n-k)^m)}^2 - s_{((k+1)^m,(n-k-1)^m)}s_{((k-1)^m,(n-k+1)^m)}$$

is Schur positive.

(2) *For $1 < k < n$, the difference*

$$s^2_{(k^m,1^{m(n-k)})} - s_{((k+1)^m,1^{m(n-k-1)})}s_{((k-1)^m,1^{m(n-k+1)})}$$

is Schur positive.

We shall show why Theorem 3.1 implies Theorem 1.1. For the first case, assume that

$$s^2_{(k^m,(n-k)^m)} - s_{((k+1)^m,(n-k-1)^m)}s_{((k-1)^m,(n-k+1)^m)} = \sum_{\lambda\vdash 2nm} c_\lambda s_\lambda$$

for some nonnegative c_λ. Applying the exponential specialization ex_1 on both sides, we obtain

$$\left(\frac{f^{(k^m,(n-k)^m)}}{(nm)!}\right)^2 - \frac{f^{((k+1)^m,(n-k-1)^m)}}{(nm)!} \times \frac{f^{((k-1)^m,(n-k+1)^m)}}{(nm)!} = \sum_\lambda c_\lambda \frac{f^\lambda}{(2nm)!}$$

by (2). Therefore,

$$\left(f^{(k^m,(n-k)^m)}\right)^2 - f^{((k+1)^m,(n-k-1)^m)} \times f^{((k-1)^m,(n-k+1)^m)} = \sum_\lambda c_\lambda f^\lambda \frac{((nm)!)^2}{(2nm)!} \geq 0.$$

For the second case, we can use similar arguments, which will be omitted here.

Our proof of Theorem 3.1 is based on some Schur positivity results due to Lam, Postnikov and Pylyavskyy [7]. For vectors v, w and a positive integer n, we assume that the operations $v+w$, $\frac{v}{n}$, $\lfloor v \rfloor$ and $\lceil v \rceil$ are performed coordinate-wise. In particular, we have well-defined operations $\lfloor \frac{\lambda+\mu}{2} \rfloor$ and $\lceil \frac{\lambda+\mu}{2} \rceil$ on pairs of any partitions. If λ, μ are partitions with $\lambda_i \geq \mu_i$ for all $i \geq 1$, then the skew diagram λ/μ is the diagram of λ with the diagram of μ removed from its upper left-hand corner. Lam, Postnikov and Pylyavskyy obtained the following result, which answered a conjecture of Okounkov [8].

Theorem 3.2 ([7, Theorem 12])**.** *Given any two skew partitions λ/μ and ν/ρ, the difference*

$$s_{\lfloor \frac{\lambda+\nu}{2} \rfloor / \lfloor \frac{\mu+\rho}{2} \rfloor} s_{\lceil \frac{\lambda+\nu}{2} \rceil / \lceil \frac{\mu+\rho}{2} \rceil} - s_{\lambda/\mu}s_{\nu/\rho}$$

is Schur positive.

Given two partitions λ and μ, let $\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \ldots)$ be the partition obtained by rearranging all parts of λ and μ in the weakly decreasing order. Let $\mathrm{sort}_1(\lambda,\mu) := (\nu_1, \nu_3, \nu_5, \ldots)$ and $\mathrm{sort}_2(\lambda,\mu) := (\nu_2, \nu_4, \nu_6, \ldots)$. Lam, Postnikov and Pylyavskyy also obtained the following result, which was first conjectured by Fomin, Fulton, Li and Poon [4].

Theorem 3.3 ([7, Corollary 14])**.** *For any two partitions λ and μ, the difference*

$$s_{\mathrm{sort}_1(\lambda,\mu)}s_{\mathrm{sort}_2(\lambda,\mu)} - s_\lambda s_\mu$$

is Schur positive.

We are now in the position to give a proof of Theorem 3.1.

Proof of Theorem 3.1. For $\lceil \frac{n}{2} \rceil < k < n$, taking $\lambda = ((k+1)^m, (n-k-1)^m)$, $\nu = ((k-1)^m, (n-k+1)^m)$, and $\mu = \rho = \emptyset$ in Theorem 3.2, we obtain the Schur positivity of

$$s^2_{(k^m,(n-k)^m)} - s_{((k+1)^m,(n-k-1)^m)} s_{((k-1)^m,(n-k+1)^m)}.$$

For $1 < k < n$, taking $\lambda = ((k+1)^m, 1^{m(n-k-1)})$, $\nu = ((k-1)^m, 1^{m(n-k+1)})$ and $\mu = \rho = \emptyset$ in Theorem 3.2, we obtain the Schur positivity of

$$s_{(k^m,1^{m(n-k+1)})} s_{(k^m,1^{m(n-k-1)})} - s_{((k+1)^m,1^{m(n-k-1)})} s_{((k-1)^m,1^{m(n-k+1)})}. \tag{10}$$

Taking $\lambda = (k^m, 1^{m(n-k+1)})$ and $\mu = (k^m, 1^{m(n-k-1)})$ in Theorem 3.3, we obtain the Schur positivity of

$$s^2_{(k^m,1^{m(n-k)})} - s_{(k^m,1^{m(n-k+1)})} s_{(k^m,1^{m(n-k-1)})}. \tag{11}$$

Combining (10) and (11), we obtain the Schur positivity of

$$s^2_{(k^m,1^{m(n-k)})} - s_{((k+1)^m,1^{m(n-k-1)})} s_{((k-1)^m,1^{m(n-k+1)})}.$$

This completes the proof. □

4 Open problems

As we mentioned at the end of Section 2, Theorem 1.1 implies the log-concavity of certain sequences concerning longest increasing subsequences. The approach of Section 3 to Theorem 1.1 inspired us to study Conjecture 1.2 and Conjecture 1.3 from the viewpoint of Schur positivity. We have the following conjectures.

Conjecture 4.1. *For $1 \le k \le n$, let*

$$f_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s^2_\lambda.$$

Then $f^2_{n,k} - f_{n,k+1} f_{n,k-1}$ is Schur positive with the convention that $f_{n,0} = f_{n,n+1} = 0$.

Conjecture 4.2. *For $1 \le k \le n$, let*

$$g_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_\lambda.$$

Then $g^2_{n,k} - g_{n,k+1} g_{n,k-1}$ is Schur positive with the convention that $g_{n,0} = g_{n,n+1} = 0$.

It is easy to see that Conjecture 4.1 implies Conjecture 1.2, and Conjecture 4.2 implies Conjecture 1.3 by (2). We have verified Conjecture 4.1 for $n \le 9$ and Conjecture 4.2 for $n \le 20$.

Chen [3] also put forward a log-concavity conjecture about perfect matchings, which was turned into the form of Conjecture 4.3 below by Bóna, Lackner and Sagan. For any fixed n, let Θ be the set of partitions of n all of whose column lengths are even. Chen's conjecture can be stated as follows.

Conjecture 4.3 ([3, Conjecture 1.5]). *For any fixed n, the sequence $\{i_{n,k}^{\Theta}\}_{k=1}^{n}$ is log-concave.*

Inspired by Conjectures 4.1 and 4.2, we propose the following conjecture, which implies Conjecture 4.3.

Conjecture 4.4. *For $1 \le k \le n$, let*

$$g_{n,k}^{\Theta} = \sum_{\lambda \in \Theta, \lambda_1 = k} s_\lambda.$$

Then $(g_{n,k}^{\Theta})^2 - g_{n,k+1}^{\Theta} g_{n,k-1}^{\Theta}$ is Schur positive with the convention that $g_{n,0}^{\Theta} = g_{n,n+1}^{\Theta} = 0$.

This conjecture has been verified for $n \le 30$. Bóna, Lackner and Sagan further proposed a companion conjecture to Conjecture 4.3.

Conjecture 4.5 ([2, Conjecture 4.3]). *For any fixed n, the sequence $\{\ell_{n,k}^{\Theta}\}_{k=1}^{n}$ is log-concave.*

However, Conjecture 4.5 does not admit a conjecture similar to Conjecture 4.4 as illustrated below. For $1 \le k \le n$, let

$$f_{n,k}^{\Theta} = \sum_{\lambda \in \Theta, \lambda_1 = k} s_\lambda^2.$$

In general, the difference $(f_{n,k}^{\Theta})^2 - f_{n,k+1}^{\Theta} f_{n,k-1}^{\Theta}$ is not Schur positive. For instance, when $n = 10$, we have

$$\begin{aligned}
f_{10,2}^{\Theta} &= s_{(2,2,2,2,1,1)}^2 + s_{(2,2,1,1,1,1,1,1)}^2, \\
f_{10,3}^{\Theta} &= s_{(3,3,2,2)}^2 + s_{(3,3,1,1,1,1)}^2, \\
f_{10,4}^{\Theta} &= s_{(4,4,1,1)}^2.
\end{aligned}$$

However, the symmetric function $(f_{10,3}^{\Theta})^2 - f_{10,2}^{\Theta} f_{10,4}^{\Theta}$ is not Schur positive by computer exploration using the open-source mathematical software `Sage` [10] and its algebraic combinatorics features developed by the `Sage-Combinat` community [11].

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