CONGRUENCES MODULO POWERS OF 2 FOR GENERALIZED FROBENIUS PARTITIONS WITH SIX COLORS

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Abstract. We establish some congruences modulo powers of 2 for the function $c\phi_6(n)$ which denotes the number of generalized Frobenius partitions of n with six colors.

1. Introduction

The concept of generalized Frobenius partitions with k colors was introduced by Andrews [1]. A generalized Frobenius partition of n with k colors is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{pmatrix},$$

where $\sum_{i=1}^{m}(a_i + b_i + 1) = n$, and where the integer entries are taken from k distinct copies of the non-negative integers distinguished by color, and the rows are ordered first by size and then by color with no two consecutive like entries in any row. The number of this kind of partitions of n is denoted by $c\phi_k(n)$. Andrews [1] showed that

$$c\phi_2(2n+1) \equiv 0 \pmod{2} \quad \text{and} \quad c\phi_2(5n+3) \equiv 0 \pmod{5}.$$

Baruah and Sarmah [3] represented the generating function of $c\phi_6(n)$ in terms of Ramanujan's theta functions and established 2- and 3-dissections of it which imply that for $n \geq 0$,

$$\begin{aligned} c\phi_6(2n+1) &\equiv 0 \pmod{4}, \\ c\phi_6(3n+1) &\equiv 0 \pmod{9}, \\ c\phi_6(3n+2) &\equiv 0 \pmod{9}. \end{aligned}$$

Xia [22] proved the following conjecture posed in [3]:

$$c\phi_6(3n+2) \equiv 0 \pmod{3^3}.$$

Later, Gu et al. [9] and Hirschhorn [10] arrived at more congruences for $c\phi_6(n)$ modulo powers of 3. For more properties of $c\phi_k(n)$, one could see [2, 5, 7, 11–21, 23, 24].

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Here and in what follows, we have made use of the standard q-series notation [8]

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),\quad |q|<1.$$

For convenience, define f_k as

$$f_k:=(q^k;q^k)_\infty.$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}},\quad |ab|<1.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows:

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Thus,

$$\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\frac{f_2^5}{f_1^2f_4^2},$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}=\frac{f_2^2}{f_1},$$

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{n(3n-1)}{2}}=f_1.$$

In this paper, we prove the following results.

Theorem 1.1. *For any prime p with $\left(\frac{-2}{p}\right)=-1$ and $n\geq 0$,*

$$c\phi_6\left(6p^2n+6pj+\frac{11p^2+1}{4}\right)\equiv 0 \pmod{2^3}, \tag{1.1}$$

$$c\phi_6\left(6p^2n+6pj+\frac{19p^2+1}{4}\right)\equiv 0 \pmod{2^3}, \tag{1.2}$$

where $j=1,2,\cdots,p-1$.

Theorem 1.2. *For $\alpha\geq 1$ and $n\geq 0$,*

$$c\phi_6\left(2\cdot 3^{4\alpha}n+\frac{3^{4\alpha-1}+1}{4}\right)\equiv 0 \pmod{2^4}.$$

2. MAIN RESULTS

In this section, we prove Theorems 1.1 and 1.2.

Lemma 2.1. *[6] For any prime $p\geq 5$,*

$$f(-q)=\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^kq^{\frac{3k^2+k}{2}}f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{24}}f(-q^{p^2}).$$

Further, we claim that for $-(p-1)/2 \le k \le (p-1)/2$ *and* $k \ne (\pm p-1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

Lemma 2.2. *[6] For any odd prime p,*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}).$$

Furthermore, we claim that for $0 \le k \le (p-3)/2$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

Setting $p = 3$ in Lemma 2.2, we arrive at

$$\psi(q) = q\psi(q^9) + f(q^3, q^6) = q\psi(q^9) + \varphi(-q^9)\frac{f_6}{f_3}. \tag{2.1}$$

Then

$$\psi(-q) = -q\psi(-q^9) + f(-q^3, q^6). \tag{2.2}$$

Lemma 2.3. *We have*

$$f_1^2 \equiv f_2 \pmod{2},$$
$$f_1^4 \equiv f_2^2 \pmod{4}.$$

Proof of Theorem 1.1. Baruah and Sarmah [3] showed that

$$\sum_{n=0}^{\infty} c\phi_6(2n+1)q^n = 36\frac{f_2^{29}}{f_1^{25}f_4^6}\varphi(q^3) + 448q\frac{f_2^5 f_4^{10}}{f_1^{17}}\varphi(q^3) + 240q\frac{f_2^{23}f_{12}^2}{f_1^{23}f_4^2 f_6} + 256q^2\frac{f_4^{14}f_{12}^2}{f_1^{15}f_2 f_6}.$$

Then

$$\sum_{n=0}^{\infty} c\phi_6(2n+1)q^n \equiv 4\frac{f_2^{29}}{f_1^{25}f_4^6}\varphi(q^3) \pmod{2^4}$$
$$= 4\frac{f_2^{28}}{f_1^{24}f_4^8}\varphi(q^3)\psi(q)\psi(q^2).$$

Employing Lemma 2.3 yields that

$$\frac{f_2^{28}}{f_1^{24}f_4^8} \equiv \frac{f_2^{28}}{f_2^{12}f_2^{16}} = 1 \pmod{4}.$$

So we have

$$4\frac{f_2^{28}}{f_1^{24}f_4^8} \equiv 4 \pmod{2^4}.$$

Therefore,

$$\sum_{n=0}^{\infty} c\phi_6(2n+1)q^n \equiv 4\varphi(q^3)\psi(q)\psi(q^2) \pmod{2^4} \tag{2.3}$$

$$\begin{aligned} &= 4\varphi(q^3)\left(\varphi(-q^9)\frac{f_6}{f_3}+q\psi(q^9)\right)\left(\varphi(-q^{18})\frac{f_{12}}{f_6}+q^2\psi(q^{18})\right) \\ &= 4\varphi(q^3)\left(\varphi(-q^9)\varphi(-q^{18})\frac{f_{12}}{f_3}+q\varphi(-q^{18})\psi(q^9)\frac{f_{12}}{f_6}\right. \\ &\quad \left.+q^2\varphi(-q^9)\psi(q^{18})\frac{f_6}{f_3}+q^3\psi(q^9)\psi(q^{18})\right), \end{aligned} \tag{2.4}$$

where in the penultimate line we employ (2.1). Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} c\phi_6\left(2\left(3n+1\right)+1\right)q^n &= \sum_{n=0}^{\infty} c\phi_6\left(6n+3\right)q^n \\ &\equiv 4\varphi(q)\varphi(-q^6)\psi(q^3)\frac{f_4}{f_2} \pmod{2^4} \\ &= 4\frac{f_2^5}{f_1^2f_4^2}\frac{f_6^2}{f_{12}}\psi(q^3)\frac{f_4}{f_2}. \end{aligned}$$

Using Lemma 2.3, we arrive at

$$\frac{f_2^5}{f_1^2f_4^2}\frac{f_6^2}{f_{12}}\frac{f_4}{f_2} \equiv \frac{f_2^5}{f_2f_2^4}\frac{f_{12}}{f_{12}}\frac{f_2^2}{f_2} = f(-q^2) \pmod{2}.$$

So

$$4\frac{f_2^5}{f_1^2f_4^2}\frac{f_6^2}{f_{12}}\frac{f_4}{f_2} \equiv 4f(-q^2) \pmod{2^3}.$$

Thus,

$$\sum_{n=0}^{\infty} c\phi_6\left(6n+3\right)q^n \equiv 4\psi(q^3)f(-q^2) \pmod{2^3}.$$

Then in view of Lemma 2.1 and Lemma 2.2, we arrive at that for a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$3\cdot\frac{k^2+k}{2}+2\cdot\frac{3m^2+m}{2} \equiv \frac{11p^2-11}{24} \pmod{p}.$$

Namely,

$$(6k+3)^2+2(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right)=-1$, we get $k=(p-1)/2$ and $m=(\pm p-1)/6$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} c\phi_6\left(6\left(pn+\frac{11p^2-11}{24}\right)+3\right)q^n &= \sum_{n=0}^{\infty} c\phi_6\left(6pn+\frac{11p^2+1}{4}\right)q^n \\ &\equiv 4\psi(q^{3p})f(-q^{2p}) \pmod{2^3}. \end{aligned}$$

So we obtain

$$c\phi_6\left(6p\left(pn+j\right)+\frac{11p^2+1}{4}\right)\equiv 0 \pmod{2^3},$$

where $j=1,2,\cdots,p-1$. Moreover, applying (2.4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_6\left(2\left(3n+2\right)+1\right)q^n &= \sum_{n=0}^{\infty} c\phi_6\left(6n+5\right)q^n\\ &\equiv 4\varphi(q)\varphi(-q^3)\psi(q^6)\frac{f_2}{f_1} \pmod{2^4}\\ &= 4\frac{f_2^5}{f_1^2f_4^2}\frac{f_3^2}{f_6}\psi(q^6)\frac{f_2}{f_1}.\end{aligned}$$

Applying Lemma 2.3 yields that

$$\frac{f_2^5}{f_1^2f_4^2}\frac{f_3^2}{f_6}\frac{f_2}{f_1}\equiv\frac{f_2^5}{f_2f_2^4}\frac{f_6}{f_6}\frac{f_1^2}{f_1}=f(-q) \pmod{2}.$$

Hence,

$$\sum_{n=0}^{\infty} c\phi_6\left(6n+5\right)q^n\equiv 4\psi(q^6)f(-q) \pmod{2^3}.$$

Then the proof of (1.2) is similar to that of (1.1). Here we omit it. □

Lemma 2.4. *[4, Corollary (i), p. 49]*

$$\varphi(q)=\varphi(q^9)+2qf(q^3,q^{15}). \tag{2.5}$$

Lemma 2.5. *Let $a(n)$ be defined by*

$$\sum_{n=0}^{\infty} a(n)q^n=\varphi(-q)\varphi(-q^2)\psi(-q^3).$$

Then we have

$$\sum_{n=0}^{\infty} a\left(3n\right)q^n\equiv\varphi(-q^3)\varphi(-q^6)\psi(-q) \pmod{4}.$$

Proof. First, from (2.5), it can be seen that

$$\begin{aligned}&\sum_{n=0}^{\infty} a(n)q^n\\ &=\left(\varphi(-q^9)-2qf(-q^3,-q^{15})\right)\left(\varphi(-q^{18})-2q^2f(-q^6,-q^{30})\right)\psi(-q^3)\\ &=\varphi(-q^9)\varphi(-q^{18})\psi(-q^3)-2q\varphi(-q^{18})\psi(-q^3)f(-q^3,-q^{15})\\ &\quad -2q^2\varphi(-q^9)\psi(-q^3)f(-q^6,-q^{30})+4q^3\psi(-q^3)f(-q^3,-q^{15})f(-q^6,-q^{30})\\ &\equiv\varphi(-q^9)\varphi(-q^{18})\psi(-q^3)-2q\varphi(-q^{18})\psi(-q^3)f(-q^3,-q^{15})\\ &\quad -2q^2\varphi(-q^9)\psi(-q^3)f(-q^6,-q^{30}) \pmod{4}.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} a\left(3n\right)q^n \equiv \varphi(-q^3)\varphi(-q^6)\psi(-q) \pmod 4.$$

Here we complete the proof. □

Lemma 2.6. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\sum_{n=0}^{\infty} c\phi_6\left(2\cdot 3^{4\alpha}n+\frac{3^{4\alpha+1}+1}{4}\right)q^n \equiv 4\varphi(q^3)\psi(q)\psi(q^2) \pmod{2^4}.$$

Proof. First, from (2.4), it can be seen that

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_6\left(2\left(3n\right)+1\right)q^n &= \sum_{n=0}^{\infty} c\phi_6\left(6n+1\right)q^n\\ &= 4\varphi(q)\varphi(-q^3)\varphi(-q^6)\frac{f_4}{f_1}+4q\varphi(q)\psi(q^3)\psi(q^6)\\ &= 4\varphi(-q^3)\varphi(-q^6)\frac{f_2^5}{f_1^3f_4}+4q\varphi(q)\psi(q^3)\psi(q^6).\end{aligned}$$

Applying Lemma 2.3, we derive

$$\frac{f_2^5}{f_1^3f_4}=\frac{f_1f_2f_2^4}{f_1^4f_4}\equiv\frac{f_1f_2f_4^2}{f_2^2f_4}=\psi(-q) \pmod 4.$$

So we have

$$4\varphi(-q^3)\varphi(-q^6)\frac{f_2^5}{f_1^3f_4}\equiv 4\varphi(-q^3)\varphi(-q^6)\psi(-q) \pmod{2^4}.$$

Therefore,

$$\sum_{n=0}^{\infty} c\phi_6\left(6n+1\right)q^n \equiv 4\varphi(-q^3)\varphi(-q^6)\psi(-q)+4q\varphi(q)\psi(q^3)\psi(q^6) \pmod{2^4}. \tag{2.6}$$

Then by means of (2.2) and (2.5), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_6\left(6\left(3n+1\right)+1\right)q^n &= \sum_{n=0}^{\infty} c\phi_6\left(18n+7\right)q^n\\ &\equiv -4\varphi(-q)\varphi(-q^2)\psi(-q^3)+4\varphi(q^3)\psi(q)\psi(q^2) \pmod{2^4}.\end{aligned}$$

Applying (2.3), Lemma 2.5, and (2.6) yields

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_6\left(18\left(3n\right)+7\right)q^n &= \sum_{n=0}^{\infty} c\phi_6\left(54n+7\right)q^n\\ &\equiv -4\varphi(-q^3)\varphi(-q^6)\psi(-q)+4\varphi(-q^3)\varphi(-q^6)\psi(-q)\\ &\quad +4q\varphi(q)\psi(q^3)\psi(q^6) \pmod{2^4}\\ &= 4q\varphi(q)\psi(q^3)\psi(q^6).\end{aligned} \tag{2.7}$$

From (2.5), it can be shown that

$$\sum_{n=0}^{\infty} c\phi_6\left(54\left(3n+1\right)+7\right)q^n = \sum_{n=0}^{\infty} c\phi_6\left(162n+61\right)q^n$$
$$\equiv 4\varphi(q^3)\psi(q)\psi(q^2) \pmod{2^4}.$$

Therefore, combining (2.3) with the above relation, we derive

$$c\phi_6\left(162n+61\right) \equiv c\phi_6\left(2n+1\right) \pmod{2^4}.$$

By induction on α, we conclude the lemma based on the above relation. □

Proof of Theorem 1.2. For $\alpha \geq 0$ and $n \geq 0$, from (2.3) and Lemma 2.6, it can be seen that

$$\sum_{n=0}^{\infty} c\phi_6\left(2n+1\right)q^n \equiv \sum_{n=0}^{\infty} c\phi_6\left(2\cdot 3^{4\alpha}n+\frac{3^{4\alpha+1}+1}{4}\right)q^n \pmod{2^4}. \tag{2.8}$$

In addition, rewrite (2.7) as

$$\sum_{n=0}^{\infty} c\phi_6\left(2(27n+3)+1\right)q^n \equiv 4q\varphi(q)\psi(q^3)\psi(q^6) \pmod{2^4}.$$

Then combining (2.8) with the above congruence yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} c\phi_6\left(2\cdot 3^{4\alpha}\left(27n+3\right)+\frac{3^{4\alpha+1}+1}{4}\right)q^n \\
&= \sum_{n=0}^{\infty} c\phi_6\left(2\cdot 3^{4\alpha+3}n+\frac{3^{4\alpha+3}+1}{4}\right)q^n \\
&\equiv 4q\varphi(q)\psi(q^3)\psi(q^6) \pmod{2^4} \\
&= 4q\left(\varphi(q^9)+2qf(q^3,q^{15})\right)\psi(q^3)\psi(q^6),
\end{aligned} \tag{2.9}$$

where we use (2.5) in the last line. Since there are no terms with powers of q congruent to 0 modulo 3 on the right-hand side of (2.9), we have

$$c\phi_6\left(2\cdot 3^{4\alpha+3}\left(3n\right)+\frac{3^{4\alpha+3}+1}{4}\right) \equiv 0 \pmod{2^4}.$$

Here we complete the proof. □

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