Extremal Theta-free planar graphs

Yongxin Lan[1], Yongtang Shi[2,*], Zi-Xia Song[3]

[1] *School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin, 300387, China*
[2] *Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China*
[3] *Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA*

Abstract

Given a family $\mathcal{F}$, a graph is $\mathcal{F}$*-free* if it does not contain any graph in $\mathcal{F}$ as a subgraph. We continue to study the topic of "extremal" planar graphs initiated by Dowden [J. Graph Theory 83 (2016) 213–230], that is, how many edges can an $\mathcal{F}$-free planar graph on n vertices have? We define $ex_{\mathcal{P}}(n,\mathcal{F})$ to be the maximum number of edges in an $\mathcal{F}$-free planar graph on n vertices. Dowden obtained the tight bounds $ex_{\mathcal{P}}(n,C_4) \le 15(n-2)/7$ for all $n \ge 4$ and $ex_{\mathcal{P}}(n,C_5) \le (12n-33)/5$ for all $n \ge 11$. In this paper, we continue to promote the idea of determining $ex_{\mathcal{P}}(n,\mathcal{F})$ for certain classes $\mathcal{F}$. Let Θ_k denote the family of Theta graphs on $k \ge 4$ vertices, that is, graphs obtained from a cycle C_k by adding an additional edge joining two non-consecutive vertices. The study of $ex_{\mathcal{P}}(n,\Theta_4)$ was suggested by Dowden. We show that $ex_{\mathcal{P}}(n,\Theta_4) \le 12(n-2)/5$ for all $n \ge 4$, $ex_{\mathcal{P}}(n,\Theta_5) \le 5(n-2)/2$ for all $n \ge 5$, and then demonstrate that these bounds are tight, in the sense that there are infinitely many values of n for which they are attained exactly. We also prove that $ex_{\mathcal{P}}(n,C_6) \le ex_{\mathcal{P}}(n,\Theta_6) \le 18(n-2)/7$ for all $n \ge 6$.

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1 Introduction

All graphs considered in this paper are finite and simple. We use P_k and C_k to denote the path and cycle on k vertices, respectively. Let $\mathcal{F}$ be a family of graphs. A graph is $\mathcal{F}$-free if it does not contain any graph in $\mathcal{F}$ as a subgraph. When $\mathcal{F} = \{F\}$ we write F-free. One of the best known results in extremal graph theory is Turán's Theorem [12], which gives the maximum number of edges that a K_k-free graph on n vertices can have. The celebrated Erdős-Stone Theorem [4] then extends this to the case when K_k is replaced by an arbitrary graph H, showing that the maximum number of edges possible is $(1+o(1))\left(\frac{\chi(H)-2}{\chi(H)-1}\right)n$, where $\chi(H)$ denotes the chromatic number of H. This latter result has been called the "fundamental theorem of extremal graph theory" [1]. Turán-type problems when host graphs are hypergraphs are notoriously difficult. A large quantity

*E-mail addresses: yxlan0@126.com (Y. Lan); shi@nankai.edu.cn (Y. Shi); Zixia.Song@ucf.edu (Z-X. Song)

of work in this area has been carried out in determining the maximum number of edges in a k-uniform hypergraph on n vertices without containing k-uniform linear paths and cycles (see, for example, [6, 7, 10]). Surveys on Turán-type problems of graphs and hypergraphs can be found in [5] and [9].

Recently, Dowden [3] initiated the study of Turán-type problems when host graphs are planar graphs, i.e., how many edges can an $\mathcal{F}$-free planar graph on n vertices have? The *planar Turán number of* $\mathcal{F}$, denoted $ex_{\mathcal{P}}(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free planar graph on n vertices. When $\mathcal{F} = \{F\}$ we write $ex_{\mathcal{P}}(n, F)$. Dowden [3] observed that it is straightforward to determine the exact values of $ex_{\mathcal{P}}(n, H)$ when H is a complete graph or non-planar graph; he also obtained the tight bounds $ex_{\mathcal{P}}(n, C_4) \le 15(n-2)/7$ for all $n \ge 4$ and $ex_{\mathcal{P}}(n, C_5) \le (12n-33)/5$ for all $n \ge 11$. Recently, Lan, Shi and Song observed in [11] that planar Turán numbers are closely related to planar anti-Ramsey numbers. The *planar anti-Ramsey number of* $\mathcal{F}$, denoted $ar_{\mathcal{P}}(n, \mathcal{F})$, is the maximum number of colors in an edge-coloring of a plane triangulation T (which is not $\mathcal{F}$-free) on n vertices such that T contains no rainbow copy of any $F \in \mathcal{F}$. When $\mathcal{F} = \{F\}$ we write $ar_{\mathcal{P}}(n, F)$. The study of planar anti-Ramsey numbers was initiated by Horňák, Jendrol′, Schiermeyer and Soták [8] (under the name of rainbow numbers). The following result is observed in [11].

Proposition 1.1 ([11]) *Given a planar graph H and a positive integer $n \ge |H|$,*

$$1 + ex_{\mathcal{P}}(n, \mathcal{H}) \le ar_{\mathcal{P}}(n, H) \le ex_{\mathcal{P}}(n, H),$$

where $\mathcal{H} = \{H - e : e \in E(H)\}$.

In this paper, we continue to promote the idea of determining $ex_{\mathcal{P}}(n, \mathcal{F})$ for certain classes $\mathcal{F}$. This paper focuses on the family of Theta graphs, where a graph on at least 4 vertices is a *Theta graph* if it can be obtained from a cycle by adding an additional edge joining two non-consecutive vertices. For integer $k \ge 4$, let Θ_k be the family of non-isomorphic Theta graphs on k vertices. Note that the only graph in Θ_4 is isomorphic to K_4 minus one edge, and Θ_5 has only one graph. By abusing notation, we also use Θ_4 and Θ_5 to denote the only graph in Θ_4 and Θ_5, respectively. Note that the study of $ex_{\mathcal{P}}(n, \Theta_4)$ was suggested by Dowden [3]. We need to introduce more notation. For a graph G, we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree and $\overline{G}$ the complement of G. For a vertex $x \in V(G)$, we will use $N_G(x)$ to denote the set of vertices in G which are adjacent to x. We define $d_G(x) = |N_G(x)|$. Given vertex sets $A, B \subseteq V(G)$, the subgraph of G induced on A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \backslash A$ the set $B - A$ and $G \backslash A$ the subgraph of G induced on $V(G) \backslash A$, respectively. We say that A is *complete to* (resp. *anti-complete to*) B if for every $a \in A$ and every $b \in B$, $ab \in E(G)$ (resp. $ab \notin E(G)$).

If $A = \{a\}$, we simply say a is complete to (resp. anti-complete to) B, and write $B\backslash a$ and $G\backslash a$, respectively. The *join* $G + H$ (resp. *union* $G \cup H$) of two vertex disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \,|\, x \in V(G), y \in V(H)\}$. (resp. $E(G) \cup E(H)$). For a positive integer t, we use tH to denote disjoint union of t copies of a graph H. Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. A graph H is a *spanning subgraph* of a graph G if H is a subgraph of G with $V(H) = V(G)$. For any positive integer k, let $[k] := \{1, 2, \ldots, k\}$.

We state and prove our main results in Section 2.

2 Planar Turán number of Theta graphs

In this section, using the method developed in [3], we study planar Turán numbers of Θ_k when $k \in \{4, 5, 6\}$. The study of $ex_{\mathcal{P}}(n, \Theta_4)$ was suggested by Dowden [3]. Our technique relies heavily on Euler's formula. We need to introduce more notation that shall be used in this section only.

An $\mathcal{F}$-free planar graph G on n vertices with the largest possible number of edges is called *extremal for n and $\mathcal{F}$*. If $\mathcal{F} = \{F\}$, then we simply say G is extremal for n and F. Given a plane graph G and integers $i, j \geq 3$, an *i-face* in G is a face of size i; and let: $E_{i,j}$ denote the set of edges in G that each belong to one i-face and one j-face (and belong to two i-faces when $i = j$); E_i denote the set of edges in G that each belong to at least one i-face; and f_i denote the number of i-faces in G. Let $e_{i,j} := |E_{i,j}|$, $e_i := |E_i|$, and $f := \sum_i f_i$. Given three positive integers a, b and c, we use $a \equiv b (\mathrm{mod\, c})$ to denote a and b have the same remainder when divided by c. We will make use of the following observation.

Observation 2.1 *Let G be a plane graph on $n \geq 3$ vertices with $e(G) \geq 2$. For all $i \geq 3$,*

(a) $e_{i,i} \leq e_i \leq e(G)$,

(b) $if_i = e_i + e_{i,i}$,

(c) $\sum_{i\geq 3} e_i - \sum_{3\leq i<j} e_{i,j} = e(G)$, *and*

(d) every face in G is bounded by a cycle if G is 2-connected.

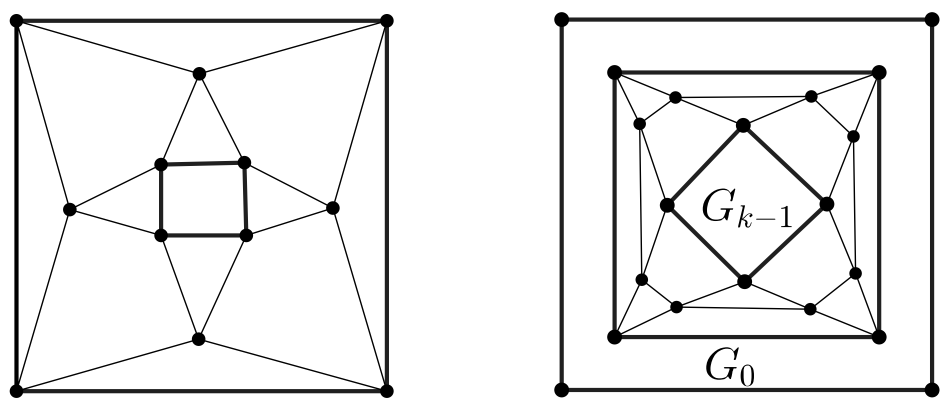
We begin with $\mathcal{F} = \Theta_4$ and prove that $ex_{\mathcal{P}}(n, \Theta_4) \leq 12(n-2)/5$ for all $n \geq 4$ and then demonstrate that this bound is tight, in the sense that there are infinitely many values of n for which it is attained exactly.

Theorem 2.2 $ex_{\mathcal{P}}(n, \Theta_4) \leq 12(n-2)/5$ *for all $n \geq 4$, with equality when $n \equiv 12 (\mathrm{mod}\, 20)$.*

Proof. Let G be a Θ_4-free plane graph on $n \geq 4$ vertices. We shall proceed the proof by induction on n. The statement is trivially true when $n = 4$ because any Θ_4-free plane graph on four vertices has at most four edges. So we may assume that $n \geq 5$. Next assume that there exists a vertex $u \in V(G)$ with $d_G(u) \leq 2$. By the induction hypothesis, $e(G\backslash u) \leq 12(n-3)/5$ and so $e(G) = e(G\backslash u) + d_G(u) \leq 12(n-3)/5 + 2 < 12(n-2)/5$, as desired. So we may assume that $\delta(G) \geq 3$. Then each component of G has at least five vertices because G is Θ_4-free. By the induction hypothesis, we may further assume that G is connected. Then G has no face of size at most two because G is simple. Hence

$$2e(G) = \sum_{i\geq 3} i f_i \geq 3f_3 + 4\sum_{i\geq 4} f_i = 3f_3 + 4(f - f_3) = 4f - f_3, \tag{1}$$

which implies that $f \leq (2e(G) + f_3)/4$. Note that $E_{3,3} = \emptyset$ else G would contain Θ_4 as a subgraph, a contradiction. Thus $e_3 = 3f_3$ by Observation 2.1(b). This, together with $e_3 \leq e(G)$ and $f \leq (2e(G) + f_3)/4$, implies that $f \leq 7e(G)/12$. By Euler's formula, $n - 2 = e(G) - f \geq 5e(G)/12$. Hence $e(G) \leq 12(n-2)/5$, as desired.



(a) (b)

Figure 1: Construction of G_k.

From the proof above, we see that equality in $e(G) \leq 12(n-2)/5$ is achieved for n if and only if equalities hold both in (1) and in $e_3 \leq e(G)$. This implies that $e(G) = 12(n-2)/5$ for n if and only if G is a connected Θ_4-free plane graph on n vertices such that each edge in G belongs to one 3-face and one 4-face. We next construct such an extremal graph for n and Θ_4. Let $n = 20k + 12$ for some integer $k \geq 0$. Let G_0 be the graph depicted in Figure 1(a), we then construct G_k on n vertices recursively for all $k \geq 1$ via the illustration given in Figure 1(b): the entire graph G_{k-1} is placed into the center quadrangle of Figure 1(b), and the entire G_0 is then placed between the two given bold quadrangles of Figure 1(b) (in such a way that these are identified with the bold quadrangles of Figure 1(a)). One can check that G_k is Θ_4-free with $n = 20k + 12$ vertices and $12(n-2)/5$ edges for all $k \geq 0$. ■

We next prove that $ex_{\mathcal{P}}(n,\Theta_5) \le 5(n-2)/2$ and then demonstrate that this bound is tight, in the sense that there are infinitely many values of n for which it is attained exactly.

Theorem 2.3 *$ex_{\mathcal{P}}(n,\Theta_5) \le 5(n-2)/2$ for all $n \ge 5$, with equality when $n \equiv 50 (\mathrm{mod}\, 120)$.*

Proof. Let G be a Θ_5-free plane graph on $n \ge 5$ vertices. We show by induction on n that $e(G) \le 5(n-2)/2$. The statement is trivially true when $n = 5$ because any Θ_5-free plane graph on five vertices has at most seven edges. So we may assume that $n \ge 6$. Next assume that there exists a vertex $u \in V(G)$ with $d_G(u) \le 2$. By the induction hypothesis, $e(G\backslash u) \le 5(n-3)/2$ and so $e(G) = e(G\backslash u) + d_G(u) \le 5(n-3)/2 + 2 < 5(n-2)/2$, as desired. So we may assume that $\delta(G) \ge 3$. Assume next that G is disconnected. Let $G_1, \ldots, G_s, G_{s+1}, \ldots, G_{s+t}$ be all components of G such that $|G_1| = \cdots = |G_s| = 4$ and $5 \le |G_{s+1}| \le \cdots \le |G_{s+t}|$, where $s \ge 0$ and $t \ge 0$ are integers with $s + t \ge 2$ and $4s + |G_{s+1}| + \cdots + |G_{s+t}| = n$. Then $e(G_i) = 6$ for all $i \in [s]$ because $\delta(G) \ge 3$, and $e(G_j) \le 5(|G_j| - 2)/2$ for all $j \in \{s+1, \ldots, s+t\}$ by the induction hypothesis. Therefore,

$$\begin{aligned} e(G) &\le 6s + \frac{5(|G_{s+1}| + \cdots + |G_{s+t}| - 2t)}{2} \\ &= \frac{5(n-2)}{2} - \frac{(8(s+t) + 2t - 10)}{2} < \frac{5(n-2)}{2}, \end{aligned}$$

as desired. So we may further assume that G is connected.

Since G is a connected plane graph on $n \ge 6$ vertices, we see that G has no face of size at most two. Hence

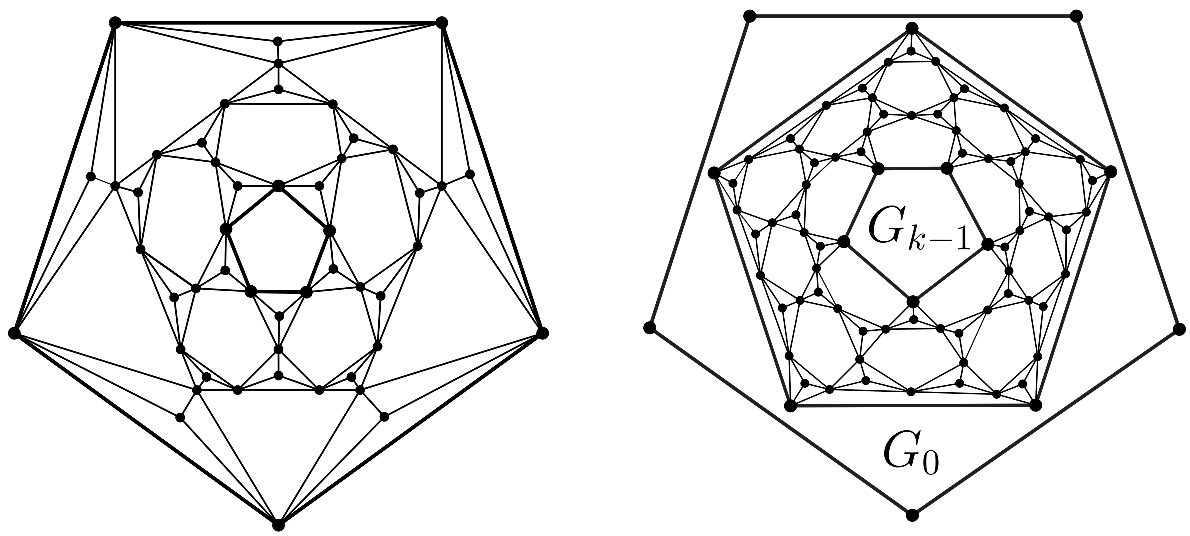
$$2e(G) = 3f_3 + 4f_4 + \sum_{i \ge 5} i f_i \ge 3f_3 + 4f_4 + 5(f - f_3 - f_4) = 5f - 2f_3 - f_4, \tag{2}$$

which implies that $f \le (2e(G) + 2f_3 + f_4)/5$. Note that no 3-face in G has its three edges in $E_{3,3}$ because G is Θ_5-free and $n \ge 6$. It follows that $e_{3,3} \le f_3$. By Observation 2.1(b),

$$3f_3 = e_3 + e_{3,3} \le e_3 + f_3 \text{ and so } f_3 \le e_3/2. \tag{3}$$

It is worth noting that a 4-face and a 3-face in G cannot have exactly one edge in common, else G would contain Θ_5 as a subgraph. Since $\delta(G) \ge 3$, we see that a 4-face and a 3-face in G cannot have exactly two edges in common. Hence, every 4-face and every 3-face in G have no edge in common and so $E_{3,4} = \emptyset$. Thus, $e_3 + e_4 \le e(G)$. By Observation 2.1(a,b), $e_{4,4} \le e_4$ and $4f_4 = e_4 + e_{4,4}$. It follows that

$$4f_4 \le 2e_4 \le 2(e(G) - e_3) \text{ and so } f_4 \le (e(G) - e_3)/2. \tag{4}$$



(a) (b)

Figure 2: Construction of G_k.

Now with the last inequalities in (3) and (4), and the fact that $f \le (2e(G) + 2f_3 + f_4)/5$ and $e_3 \le e(G)$, we obtain $f \le 3e(G)/5$. By Euler's formula, $n - 2 = e(G) - f \ge 2e(G)/5$. Hence $e(G) \le 5(n-2)/2$, as desired.

From the proof above, we see that equality in $e(G) \le 5(n-2)/2$ is achieved for n if and only if equalities hold in (2), (3) and (4) and in $e_3 \le e(G)$. This implies that $e(G) = 5(n-2)/2$ for n if and only if G is a connected Θ_5-free plane graph on n vertices satisfying: each 3-face in G has exactly two edges in $E_{3,3}$; each edge in G belongs to either one 3-face and one 5-face or two 3-faces. We next construct such an extremal plane graph for n and Θ_5. Let $n = 120k + 50$ for some integer $k \ge 0$. Let G_0 be the graph depicted in Figure 2(a), we then construct G_k of order n recursively for all $k \ge 1$ via the illustration given in Figure 2(b): the entire graph G_{k-1} is placed into the center pentagon of Figure 2(b), and the entire G_0 is then placed between the two given bold pentagons of Figure 2(b) (in such a way that these are identified with the bold pentagons of Figure 2(a)). One can check that G_k is Θ_5-free with $n = 120k + 50$ vertices and $5(n-2)/2$ edges for all $k \ge 0$. ■

Finally, we prove an upper bound for $ex_{\mathcal{P}}(n, \Theta_6)$ in Theorem 2.4. Figure 3 illustrates all possible graphs for which equality in Theorem 2.4 is attained when $n = 9$. However, we shall see in Corollary 2.5 that equality is not possible for all $n \ge 10$.

Theorem 2.4 *$ex_{\mathcal{P}}(n, \Theta_6) \le 18(n-2)/7$ for all $n \ge 6$, with equality when $n = 9$.*

Proof. Let G be an extremal plane graph for Θ_6 and $n \ge 6$. We shall prove that $e(G) \le 18(n-2)/7$ by induction on n. When $n = 6$, we show that $e(G) \le 10$. Suppose that $e(G) \ge 11$.

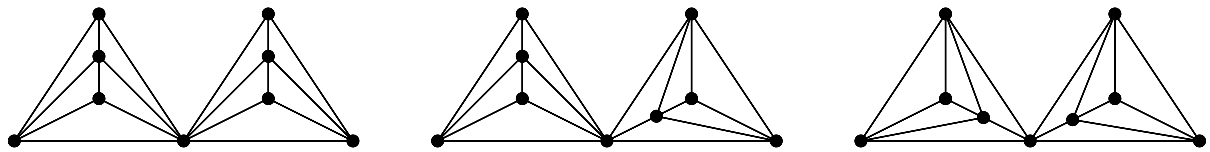


Figure 3: All possible graphs achieving equality in Theorem 2.4 and Corollary 2.5 when $n = 9$.

Then G is isomorphic to either a plane triangulation on six vertices or a plane triangulation on six vertices with one edge removed. Note that all plane triangulations on 6 vertices are depicted in Figure 4. It is easy to check that G has a Hamiltonian cycle and so G contains a graph in Θ_6 as subgraph, a contradiction. Hence, $e(G) \le 10 < 18(n-2)/7$ when $n = 6$. So we may assume that $n \ge 7$. Next assume that there exists a vertex $u \in V(G)$ with $d_G(u) \le 2$. By the induction hypothesis, $e(G\backslash u) \le 18(n-3)/7$ and so $e(G) = e(G\backslash u) + d_G(u) \le 18(n-3)/7 + 2 < 18(n-2)/7$, as desired. So we may assume that $\delta(G) \ge 3$. Assume next that G is disconnected. Then each component of G has exactly four, five or at least six vertices because $\delta(G) \ge 3$. Let $G_1, \ldots, G_r, G_{r+1}, \ldots, G_{r+s}, G_{r+s+1}, \ldots, G_{r+s+t}$ be all components of G such that

$$|G_1| = \cdots = |G_r| = 4, |G_{r+1}| = \cdots = |G_{r+s}| = 5, \text{ and } 6 \le |G_{r+s+1}| \le \cdots \le |G_{r+s+t}|,$$

where $r, s, t \ge 0$ are integers with $r + s + t \ge 2$ and $4r + 5s + |G_{r+s+1}| + \cdots + |G_{r+s+t}| = n$. Since G is an extremal plane graph for Θ_6, we see that $e(G_i) = 6$ for all $i \in [r]$ and $e(G_j) = 9$ for all $j \in \{r+1, \ldots, r+s\}$. By the induction hypothesis, $e(G_k) \le 18(|G_k| - 2)/7$ for all $k \in \{r+s+1, \ldots, r+s+t\}$. Therefore,

$$\begin{aligned} e(G) &\le 6r + 9s + \frac{18(|G_{r+s+1}| + \cdots + |G_{r+s+t}| - 2t)}{7} \\ &= \frac{18(n-2)}{7} - \frac{(27(r+s+t) + 3r + 9t - 36)}{7} \\ &< \frac{18(n-2)}{7}, \end{aligned}$$

as desired. So we may assume that G is connected.

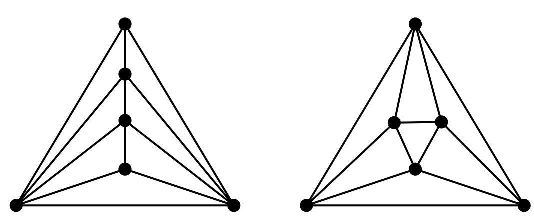


Figure 4: All plane triangulations on 6 vertices.

Next assume that G contains a cut-vertex, say u. Let H be a smallest component of $G\backslash u$, and let $G_1 := G[V(H) \cup \{u\}]$ and $G_2 := G\backslash V(H)$. Then $|G_1| \le |G_2|$ and $|G_1| + |G_2| = n + 1$. Since $\delta(G) \ge 3$, we see that $4 \le |G_1| \le |G_2|$. Assume first that $|G_2| \le 5$. Then $e(G_i) \le 3|G_i| - 6$ for all $i \in \{1, 2\}$. Hence, $e(G) = e(G_1)+e(G_2) \le 3(|G_1|+|G_2|)-12 = 3n-9 \le 18(n-2)/7$ because $n \le 9$, with equality when both G_1 and G_2 are isomorphic to K_5 minus one edge, and so G is isomorphic to the graphs depicted in Figure 3. Assume next that $|G_2| \ge 6$. Then $e(G_2) \le 18(|G_2| - 2)/7$ by the induction hypothesis. Note that $e(G_1) \le 3|G_1| - 6$ when $|G_1| \le 5$ and $e(G_1) \le 18(|G_1| - 2)/7$ when $|G_1| \ge 6$ by the induction hypothesis. Therefore, when $|G_1| \le 5$,

$$\begin{aligned} e(G) = e(G_1) + e(G_2) &\le 3|G_1| - 6 + \frac{18(n + 1 - |G_1| - 2)}{7} \\ &= \frac{18(n-2)}{7} - \frac{(24 - 3|G_1|)}{7} < \frac{18(n-2)}{7}; \end{aligned}$$

when $|G_1| \ge 6$,

$$e(G) = e(G_1) + e(G_2) \le \frac{18(|G_1| + |G_2| - 4)}{7} < \frac{18(n-2)}{7}.$$

So we may assume that G is 2-connected. By Observation 2.1(d) and the fact that $\delta(G) \ge 3$, each face in G is bounded by a cycle.

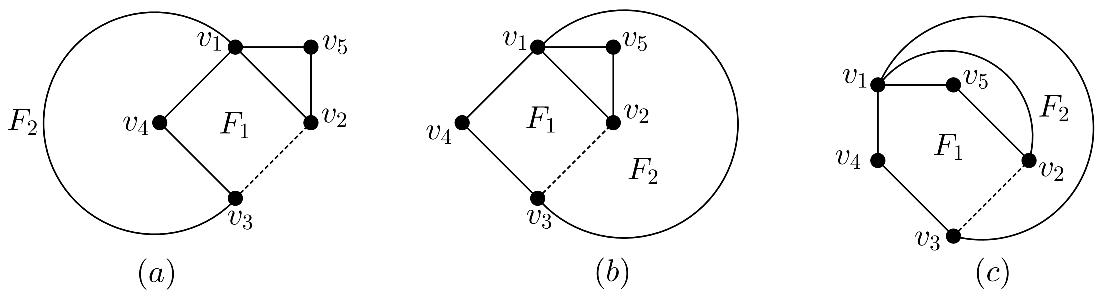


Figure 5: Three possible configurations of H with $v_2v_3 \in E_{4,4} \cup E_{3,5}$.

Assume next that $E_{4,4} \cup E_{3,5} \neq \emptyset$. Let $v_2v_3 \in E_{4,4} \cup E_{3,5}$. Since $\delta(G) \ge 3$, let F_1 and F_2 be the two faces of G having v_2v_3 in common such that the size of F_1 is at least the size of F_2. Then G must contain a plane subgraph H isomorphic to the graphs depicted in Figure 5(a,b) when $v_2v_3 \in E_{4,4}$, and in Figure 5(c) when $v_2v_3 \in E_{3,5}$, because G is Θ_6-free. Let H_1 and H_2 be the induced plane subgraphs of G with boundary v_1, v_2, v_5 and v_1, v_3, v_4, respectively. Then $|H_1| + |H_2| = n + 1$, and $|H_i| \ge 6$ for all $i \in [2]$ because G is Θ_6-free, 2-connected and $\delta(G) \ge 3$. By the induction hypothesis, $e(H_i) \le 18(|H_i| - 2)/7$ for all $i \in [2]$. Thus,

$$e(G) = e(H_1) + e(H_2) + |\{v_2v_3\}| < 18(n-2)/7.$$

We may now further assume that $E_{4,4} \cup E_{3,5} = \emptyset$. Then $e_{4,4} = 0$ and $e_{3,5} = 0$.

It is easy to see that G is not a plane triangulation and so $\sum_{i\geq 4} f_i \geq 0$. We next show that $\sum_{i\geq 5} f_i \neq 0$. Suppose $\sum_{i\geq 5} f_i = 0$. Then $f_3 + f_4 = f$ and $f_4 > 0$. Note that $e_{4,4} = 0$. It follows that every edge of a 4-face of G belongs to $E_{3,4}$, and so G contains a Θ_6 subgraph, a contradiction. Thus $\sum_{i\geq 5} f_i \neq 0$. We may further assume that the outer face of G is neither a 3-face nor a 4-face. Then

$$\begin{aligned} 2e(G) &= 3f_3 + 4f_4 + 5f_5 + \sum_{i\geq 6} i f_i \\ &\geq 3f_3 + 4f_4 + 5f_5 + 6(f - f_3 - f_4 - f_5) \\ &= 6f - 3f_3 - 2f_4 - f_5, \end{aligned} \tag{5}$$

which implies that $6f \leq 2e(G) + 3f_3 + 2f_4 + f_5$.

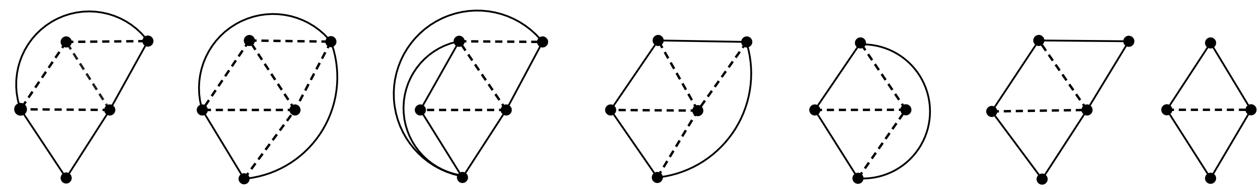


Figure 6: All possible configurations of H_F, where all dashed edges are in $E_{3,3}$, and no solid edges are in $E_{3,3}$.

We next find an upper bound for each of f_3, f_4 and f_5. To get an upper bound for f_3, we first show that $5e_{3,3} \leq 6f_3$. Let F be a 3-face in G with $|E(F) \cap E_{3,3}| \geq 1$. Clearly, $|E(F) \cap E_{3,3}| \leq 3$. Since G is Θ_6-free and the outer face of G is not a 3-face, there exists a plane subgraph H_F of G with $|H_F| \leq 5$ such that F is a face (not the outer face) of H_F; all faces of H_F, except the outer face of H_F and any face of H_F that is not a face in G, are 3-faces; and no edges on the boundary of the outer face of H_F and any face of H_F that is not a face in G are in $E_{3,3}$. The possible configurations of H_F are shown in Figure 6. When H_F is isomorphic to the graph depicted in Figure 6(b), H_F contains six edges in $E_{3,3}$ and five 3-faces of G. From all possible configurations of H_F, we see that $e_{3,3} \leq 6f_3/5$. Hence,

$$3f_3 = e_3 + e_{3,3} \leq e_3 + 6f_3/5, \text{ and so } f_3 \leq 5e_3/9. \tag{6}$$

To get an upper bound for f_4, we next show that $4f_4 \leq 2(e(G) - e_3)$. By Observation 2.1(b,c),

$$4f_4 = e_4 \text{ and } e(G) \geq e_3 + e_4 - e_{3,4}. \tag{7}$$

We next show that $e_{3,4} \leq e(G) - e_3$.

This is trivially true when $e_{3,4} = 0$. Assume that $e_{3,4} \neq 0$. Let F and F' be a 4-face and a 3-face in G, respectively, such that F and F' share an edge in common. We may assume that F

has vertices v_1, v_2, v_3, v_4 in order and F' has vertices v_1, v_4, v_5 in order. Note that F and F' are not outer face in G. Observe that if v_iv_{i+1} belongs to $E_{3,4}$ for any $i \in \{1, 2, 3\}$, then v_iv_{i+1} belongs to the 4-face F and the 3-face with vertices v_i, v_{i+1}, v_5 in order, else G would not be Θ_6-free. Since $n \geq 7$, there exists some $k \in \{1, 2, 3\}$ such that $v_kv_{k+1} \notin E_{3,4}$. Then $v_kv_{k+1} \in E_{4,j}$ for some $j \geq 5$ because $e_{4,4} = 0$. We next show that F has at most two edges in $E_{3,4}$. Suppose $|E(F) \cap E_{3,4}| = 3$. We may assume that $k = 2$. Then $v_1v_2, v_3v_4 \in E_{3,4}$. Thus v_1v_2 belongs to the 4-face F and the 3-face with vertices v_1, v_2, v_5 in order; and v_3v_4 belongs to the 4-face F and the 3-face with vertices v_3, v_4, v_5 in order. Since G is Θ_6-free, we see that $v_5v_2 \in E_{3,j}$ for some $j \geq 6$, $v_5v_3 \in E_{3,j}$ for some $j \geq 6$, and $v_2v_3 \in E_{4,j}$ for some $j \geq 6$. But then $G + v_2v_4$ is Θ_6-free, contrary to the choice of G. Thus F has at most two edges in $E_{3,4}$, and so F has at least two edges not in E_3. This holds for each 4-face in G. Hence, $e_{3,4} \leq e(G) - e_3$.

By (7),

$$4f_4 = e_4 \leq e(G) - e_3 + e_{3,4} \leq 2(e(G) - e_3). \tag{8}$$

Note that $e_{3,5} = 0$. By Observation 2.1(a), $e_{5,5} \leq e_5 \leq e(G) - e_3$. By Observation 2.1(b),

$$5f_5 = e_5 + e_{5,5} \leq 2(e(G) - e_3). \tag{9}$$

Combining $e_3 \leq e(G)$ with the upper bounds on f_3, f_4, f_5 given in (6), (8), (9), we have

$$\begin{aligned}
6f &\leq 2e(G) + 3f_3 + 2f_4 + f_5 \\
&\leq 2e(G) + 5e_3/3 + (e(G) - e_3) + 2(e(G) - e_3)/5 \\
&= 17e(G)/5 + 4e_3/15 \\
&\leq 11e(G)/3.
\end{aligned}$$

It follows that $f \leq 11e(G)/18$. By Euler's formula, $n - 2 \geq e(G) - f \geq 7e(G)/18$. Hence $e(G) \leq 18(n - 2)/7$, as desired. ■

Corollary 2.5 *Let K_5^- be the graph obtained from K_5 by deleting one edge. Then*

(a) $ex_{\mathcal{P}}(n, \Theta_6 \cup \{K_5^-\}) \leq 12(n - 2)/5$ for all $n \geq 7$.

(b) $ex_{\mathcal{P}}(n, \Theta_6) < 18(n - 2)/7$ for all $n \geq 10$.

Proof. To prove (a), let G be an extremal plane graph for $n \geq 7$ and $\Theta_6 \cup \{K_5^-\}$. We prove that $e(G) \leq 12(n-2)/5$ by induction on n. Since G is Θ_6-free, by Theorem 2.4, $e(G) \leq \lfloor 18(n-2)/7 \rfloor = 12(n - 2)/5$ when $n = 7$. So we may assume that $n \geq 8$. Similar to the proof of Theorem 2.4, we see that $e_{3,3} \leq f_3$, because G is K_5^--free and so no H_F is isomorphic to the graphs depicted in

Figure 6(b, c). By Observation 2.1(b), $f_3 \le e_3/2$. This, together with the upper bounds for f_4, f_5 given in (8) and (9), implies that

$$\begin{aligned} 6f &\le 2e(G) + 3f_3 + 2f_4 + f_5 \\ &\le 2e(G) + 3e_3/2 + e(G) - e_3 + 2(e(G) - e_3)/5 \\ &= 17e(G)/5 + e_3/10 \\ &\le 7e(G)/2. \end{aligned}$$

It follows that $f \le 7e(G)/12$. By Euler's formula, $n - 2 \ge e(G) - f \ge 5e(G)/12$. Hence $e(G) \le 12(n-2)/5$.

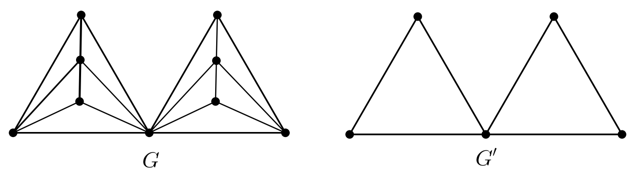


Figure 7: An example of constructing the graph G' from a graph G.

To prove (b), let G be an extremal plane graph for Θ_6 and $n \ge 10$. By Theorem 2.4, $e(G) \le 18(n-2)/7$. Suppose $e(G) = 18(n-2)/7$. By Corollary 2.5(a), G is not K_5^--free. From the proof of Theorem 2.4, we see that equality in $e(G) \le 18(n-2)/7$ is achieved for n if and only if all the equalities hold in (5), (6), (8), (9) and in $e_3 \le e(G)$. This implies that $e(G) = 18(n-2)/7$ for n if and only if G is a 2-connected Θ_6-free plane graph on n vertices satisfying: G consists entirely of K_5^-'s and 6-faces, no two K_5^-'s share an edge, and no two 6-faces have an edge in common. Let G' be the graph obtained from G by deleting the two vertices not on the outer face in each $H_F = K_5^-$, an example is shown in Figure 7. Then G' consists of 3-faces and 6-faces such that each edge of G' belongs to one 3-face and one 6-face. Clearly, G' is 2-connected because G is 2-connected. Let f'_i be the number of i-faces in G'. Let $f' = \sum_{i \ge 1} f'_i$. Then $3f'_3 = e(G') = 6f'_6$ and $f' = f'_3 + f'_6$. Thus $|G'| - 2 = e(G') - f' = e(G')/2$ and so $e(G') = 2|G'| - 4$, which implies that $\delta(G') \le 3$. Since G' is 2-connected, we have $\delta(G') \ge 2$. Note that each vertex of G' must have even degree because the adjacent faces are alternatively of size 3 and size 6. Thus $\delta(G') = 2$. Let $v \in V(G')$ be a vertex of degree two in G'. Let $u_1 v u_2$ and $u_1 v u_2 u_3 u_4 u_5$ be the vertices in order on the boundary of the two adjacent faces containing v, respectively. Then $G'[\{u_1, v, u_2, u_3, u_4, u_5\}]$ contains a graph in Θ_6 as a subgraph. Thus G is not Θ_6-free, a contradiction. ∎

It is worth noting that every C_6-free graph is certainly Θ_6-free. Hence, $ex_{\mathcal{P}}(n, C_6) \le ex_{\mathcal{P}}(n, \Theta_6)$. Corollary 2.6 follows immediately from Theorem 2.4.

Corollary 2.6 $ex_{\mathcal{P}}(n, C_6) \le 18(n-2)/7$ *for all* $n \ge 6$, *with equality when* $n = 9$.

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