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Adjacency Rank and Independence Number of a Signed Graph

Xueliang Li[1,2] · Wen Xia[1]

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Abstract
A signed graph $\Gamma = (G, \sigma)$ is obtained from a simple graph G by assigning to each edge of G a sign $+$ or $-$. Let $A(\Gamma)$ denote the adjacency matrix of Γ and $\alpha(G)$ be the independence number of G. We study the rank of $A(\Gamma)$ and the independence number $\alpha(G)$. We show that $r(\Gamma) + 2\alpha(G) \geq 2n - 2d(G)$, where n is the order of G and $d(G)$ is the dimension of the cycle space of G. Moreover, we obtain sharp lower bounds for $r(\Gamma) + \alpha(G)$, $r(\Gamma) - \alpha(G)$, $r(\Gamma)/\alpha(G)$ and we characterize all corresponding extremal graphs.

Keywords Rank of adjacency matrix · Signed graph · Underlying graph · Independence number

Mathematics Subject Classification 05C50

1 Introduction

The rank of the adjacency matrix of a graph is an important research topic in spectral graph theory and has been a hot issue for scholars. Collatz and Sinogowitz [4] first posed the open problem of characterizing all graphs satisfying that their rank is smaller than their order. The problem has not been fully solved until now. Let

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✉ Xueliang Li
lxl@nankai.edu.cn

Wen Xia
honeyxiaw@163.com

1 Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China

2 School of Mathematics and Statistics, Qinghai Normal University, Xining 810008, Qinghai, China

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$G = (V(G), E(G))$ be a simple graph. We can get some new kinds of graphs if we add some properties to the edges $E(G)$ (e.g., oriented graph, signed graph). A lot of scholars pay attention to these new kinds of graph and have made some progresses in these graphs.

For an oriented graph G^τ, Ma et al. [15] presented a relation between skew-rank of an oriented graph and the rank of its underlying graph. Huang et al. [9] established sharp lower bounds on $sr(G^\tau)+2\alpha(G)$, $sr(G^\tau)+\alpha(G)$, $sr(G^\tau)-\alpha(G)$ and $sr(G^\tau)/\alpha(G)$ of an oriented graph and characterized the corresponding extremal oriented graphs, where $sr(G^\tau)$ is the skew-rank of G^τ, $\alpha(G)$ is the independence number of G, whereas we recently established sharp upper bounds for them in [11]. For more results and comprehensive study of the skew-adjacency matrices of oriented graphs, we refer to [1,3] a survey paper by Li and Lian [10].

Fan et al. [6] introduced the rank of signed graphs, and they characterized the rank of signed graphs with pendant trees and the unicyclic signed graphs of order n with rank 2, 3, 4 and 5, respectively. Fan et al. [5] characterized the signed graph of order n with rank 2 or 3, and introduced a graph transformation which preserves the rank. They also determined the unbalanced bicyclic signed graphs of order n with rank 3 or 4 and signed bicyclic graphs (including simple bicyclic graphs) of order n with rank 5. For more details, we refer to papers [8,13].

For a signed graph $\Gamma = (G, \alpha)$, Lu et al. [14] proved that $r(G)-2d(G) \leq r(\Gamma) \leq r(G) + 2d(G)$ for an unbalanced signed graph and characterized all corresponding extremal graphs where $r(G)$ is the rank of G, $d(G)$ is the dimension of the cycle space of G. In this paper, we first establish sharp lower bound on $r(\Gamma) + 2\alpha(G)$ for a signed graph. We then apply the same fundamental idea to determine a lower bound on $r(\Gamma)+\alpha(G)$, $r(\Gamma)-\alpha(G)$ and $r(\Gamma)/\alpha(G)$ and we characterize the corresponding extremal signed graphs.

2 Notation and Definition

All graphs considered in this paper are finite, simple (i.e., have no multiple edges and loops) and connected. For terminology and notation not defined here, we refer to Bondy and Murty [2]. Let G be a simple graph of order n with vertex-set $V(G)$ and edge-set $E(G)$. The *adjacency matrix* $A(G)$ of G is an $n \times n$ symmetric matrix $(a_{ij})_{n\times n}$ such that $a_{ij} = 1$ if the vertices i and j are adjacent in G, and $a_{ij} = 0$, otherwise. The *rank* $r(G)$ of G means the rank of $A(G)$.

Given a graph G, a signed graph $\Gamma = (G, \sigma)$ is obtained from G by assigning to each edge of G a sign. Formally, a signed graph $\Gamma = (G, \sigma)$ consists of the underlying graph G of Γ, and a sign function $\sigma : E \to \{+, -\}$. The adjacency matrix associated with Γ, written as $A(\Gamma)$, is defined to be an $n \times n$ matrix (a^σ_{ij}) such that $a^\sigma_{ij} = \sigma(v_i v_j)a_{ij}$, where a_{ij} is an element of the adjacency matrix $A(G)$ of its underlying graph G. An unsigned graph is considered an all-positive signed graph, in this case, replacing matrix $A(\Gamma)$ by matrix $A(G)$. The rank $r(\Gamma)$ of a signed graph Γ is defined as the rank of $A(\Gamma)$.

An induced subgraph $H = (G', \sigma)$ of Γ is a signed graph such that G' is an induced subgraph of G and each edge of H has the same sign as that in Γ. For an induced subgraph H of Γ, let $\Gamma - H$ be the subgraph obtained from Γ by removing all vertices of H and their incident edges. For $W \subseteq V(\Gamma)$, $\Gamma - W$ is the subgraph obtained from Γ by removing all vertices in W and all incident edges.

Let C be a cycle of Γ. The *sign* $\sigma(C)$ of C is the product of the signs of all edges. The signed cycle C is said to be *positive* (or *negative*) if $\sigma(C) = +$ (or $\sigma(C) = -$). As the *edge space* $\mathcal{E} = \mathcal{E}(G)$, we take the vector space $\{0, 1\}^E$ over $\mathbb{F}_2$, which we view as the power set of E with symmetric differences as addition. We treat a cycle $C \subseteq G$ as an element of the edge space. The *cycle space* $\mathcal{C} = \mathcal{C}(G)$ of G is the subspace of $\mathcal{E}$ generated by the cycles in G. Denote by $d(G)$ the dimension of the cycle space of G, that is $d(G) = |E(G)| - |V(G)| + c(G)$, where $c(G)$ is the number of components of G.

Denote by P_n, C_n, S_n and K_n a path, a cycle, a star and a complete graph of order n, respectively. The set of neighbors of a vertex v in G is denoted by $N_G(v)$ or simply $N(v)$. A signed graph is called *acyclic* (resp. *connected, bipartite*) if its underlying graph is acyclic (resp. connected, bipartite). A graph is called *an empty graph* if it has no edges. We call v *a cut-vertex* of a connected graph Γ if $\Gamma - v$ is disconnected.

A vertex of Γ is called a *pendant vertex* if it is adjacent to a unique vertex, and the unique neighbor of a pendant vertex is called a *quasi-pendant vertex*. An induced subgraph C_q of a graph Γ is called a *pendant cycle* if C_q is a cycle and is connected to the rest of the graph by a single edge.

Two distinct edges in a graph G are independent if they do not share a common end-vertex. A *matching* is a set of pairwise independent edges of G, while a maximum matching of G is a matching with the maximum cardinality. *The matching number* of G, denoted by $\alpha'(G)$, is the cardinality of a maximum matching of G. Two vertices of a graph G are said to be independent if they are not adjacent. A subset I of $V(G)$ is called *an independent set* if any two vertices of I are independent in G. An independent set I is maximum if G has no independent set I' with $|I'| > |I|$. The number of vertices in a maximum independent set of G is called *the independence number* of G, denoted by $\alpha(G)$.

3 Main Results

A *block* in G is a maximal subgraph with no cut-vertex. By contracting each 2-connected block into a vertex we obtain an acyclic graph T_G. Let $W_\mathcal{C}$ denote the set of vertices of T_G that correspond to the cycles in G. Moreover, L_G is the graph obtained by deleting every 2-connected block. It is the subgraph induced by the vertices of G that are not in any 2-connected block.

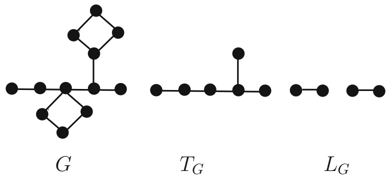
Now, we state our main results as follows.

Theorem 3.1 *Let $\Gamma = (G, \sigma)$ be a signed simple, connected graph on n vertices. Then,*

$$r(\Gamma) + 2\alpha(G) \geq 2n - 2d(G). \tag{1}$$

The equality in (1) holds if and only if the following conditions hold for Γ:

Fig. 1 Graphs G, T_G and Γ_G



(i) *the cycles (if any) of* Γ *are pairwise vertex-disjoint;*
(ii) Γ *is bipartite and a cycle* C *of* Γ *is positive if and only if its length* $|C|$ *is a multiple of 4;*
(iii) $\alpha(T_G) = \alpha(L_G) + d(G)$.

For example, if all cycles of G in Fig. 1 are positive, since they all have orders that are multiples 4, then Γ satisfies the three conditions of Theorem 3.1 and $r(\Gamma) + 2\alpha(G) = 2n - 2d(G)$ holds with $r(\Gamma) = 8, \alpha(G) = 6, n = 12$ and $d(G) = 2$.

Next, we will establish sharp lower bounds on $r(\Gamma) + \alpha(G), r(\Gamma) - \alpha(G)$ and $r(\Gamma)/\alpha(G)$.

Theorem 3.2 *Let* $\Gamma = (G, \sigma)$ *be a signed simple, connected graph with n vertices and m edges. Then,*

$$r(\Gamma) + \alpha(G) \geq 4n - 2m - \sqrt{n(n-1) - 2m + \frac{1}{4}} - \frac{5}{2}, \tag{2}$$

with equality if and only if $G \cong S_n$.

Theorem 3.3 *Let* $\Gamma = (G, \sigma)$ *be a signed simple, connected graph with n vertices and m edges. Then,*

$$r(\Gamma) - \alpha(G) \geq 4n - 2m - 3\sqrt{n(n-1) - 2m + \frac{1}{4}} - \frac{7}{2}, \tag{3}$$

with equality if and only if $G \cong S_n$.

Theorem 3.4 *Let* $\Gamma = (G, \sigma)$ *be a signed simple, connected graph with n vertices and m edges. Then,*

$$\frac{r(\Gamma)}{\alpha(G)} \geq \frac{4(2n - m - 1)}{\sqrt{4n(n-1) - 8m + 1} + 1} - 2, \tag{4}$$

with equality if and only if $G \cong S_n$.

In order to give proofs for our main results, we need to do some preparations in the next section.

4 Preliminary Results

Some known results are listed in this section which will be used in the sequel.

Lemma 4.1 [17, Lemma 2.6] *Let Γ be a signed graph.*

(i) *If H is an induced subgraph of Γ, then $r(H) \leq r(\Gamma)$;*
(ii) *If $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ are all the components of Γ, then $r(\Gamma) = \sum_{i=1}^{t} r(\Gamma_i)$;*
(iii) *$r(\Gamma) \geq 0$ with equality if and only if Γ is an empty graph.*

Lemma 4.2 [16, Lemma 3.1] *Let G be a graph and $x \in V(G)$.*

(i) *$d(G) = d(G - x)$ if x is not on any cycle of G;*
(ii) *$d(G - x) \leq d(G) - 1$ if x lies on a cycle;*
(iii) *$d(G - x) \leq d(G) - 2$ if x is a common vertex of distinct cycles;*
(iv) *If the cycles of G are pairwise vertex-disjoint, then $d(G)$ is exactly the number of cycles in G.*

Lemma 4.3 [9, Lemma 1.8] *Let G be a simple connected graph. Then,*

(i) *$\alpha(G) - 1 \leq \alpha(G - v) \leq \alpha(G)$ for any $v \in V(G)$;*
(ii) *$\alpha(G - e) \geq \alpha(G)$ for any $e \in E(G)$.*

Lemma 4.4 [2] *Let G be a bipartite graph with n vertices. Then,*

$$\alpha(G) + \alpha'(G) = n.$$

The *join* of two disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 by an edge.

Lemma 4.5 [7, Theorem 1] *Let G be a simple connected graph with n vertices and m edges. Then,*

$$\frac{1}{2}\left[(2m+n+1) - \sqrt{(2m+n+1)^2 - 4n^2}\right] \leq \alpha(G) \leq \sqrt{n(n-1) - 2m + \frac{1}{4}} + \frac{1}{2}.$$

The equality on the right-hand side holds if and only if $G \cong K_{n-\alpha(G)} \vee \alpha(G)K_1$.

Note that any acyclic signed graph Γ switches to an all-positive, i.e., unsigned, graph. So it is easy to obtain the following two lemmas.

Lemma 4.6 [6, Lemma 2.1] *Let P_n be a signed path of order n. Then, $r(P_n) = n$ if n is even, and $r(P_n) = n - 1$ if n is odd.*

Lemma 4.7 [12, Lemma 2.2] *Let $\Gamma = (G, \sigma)$ be a signed acyclic graph with matching number $\alpha'(G)$. Then,*

$$r(\Gamma) = r(G) = 2\alpha'(G).$$

Lemma 4.8 [6, Lemma 2.2] *Let C_n be a positive signed cycle of order n. Then, $r(C_n) = n-2$ if $n \equiv 0 \pmod 4$, and $r(C_n) = n$ otherwise. Let C_n be a negative signed cycle of order n. Then, $r(C_n) = n-2$ if $n \equiv 2 \pmod 4$, and $r(C_n) = n$ otherwise.*

Lemma 4.9 [14, Lemma 2.3] *Let x be a vertex of Γ. Then, $r(\Gamma)-2 \le r(\Gamma-x) \le r(\Gamma)$.*

Lemma 4.10 [6, Lemma 2.4] *Let y be a pendant vertex of Γ, and x be the neighbor of y. Then,*

$$r(\Gamma) = r(\Gamma - x - y) + 2.$$

Lemma 4.11 [9, Lemma 2.3] *Let y be a pendant vertex of G with neighbor x. Then,*

$$\alpha(G) = \alpha(G - x) = \alpha(G - x - y) + 1.$$

Let T be a tree with at least one edge, and denote by $\tilde{T}$ the subtree obtained from T by removing all the pendant vertices of T.

Lemma 4.12 [15, Lemma 4.2] *Let T be a tree with at least one edge. Then,*

(i) $r(\tilde{T}) < r(T)$;
(ii) *If $r(T - D) = r(T)$ for a subset D of $V(T)$, then there is a pendant vertex v such that $v \notin D$.*

Denote by $p(G)$ the number of pendant vertices of G.

Lemma 4.13 [9, Corollary 1.17] *Let T be a tree with at least one edge. Then,*

(i) $\alpha(T) < \alpha(\tilde{T}) + p(T)$;
(ii) *If $\alpha(T) = \alpha(T - D) + |D|$ for a subset D of $V(T)$, then there is a pendant vertex v such that $v \notin D$.*

Lemma 4.14

$$r(\Gamma) + 2\alpha(G) \ge 2n - 2d(G). \tag{5}$$

Proof By induction on $d(G)$. If $d(G) = 0$, then Γ is a signed tree, and the result follows immediately from Lemmas 4.4 and 4.7.

Now suppose $d(G) \ge 1$, and let x be a vertex on some cycle of G. By Lemma 4.2 (ii), we have

$$d(G - x) \le d(G) - 1. \tag{6}$$

By the induction hypothesis, one has

$$r(\Gamma - x) + 2\alpha(G - x) \ge 2(n-1) - 2d(G - x). \tag{7}$$

By Lemmas 4.1 (i) and 4.3 (i), we have

$$r(\Gamma - x) \le r(\Gamma), \quad \alpha(G - x) \le \alpha(G). \tag{8}$$

From inequalities (6)–(8), we obtain the inequality (5). □

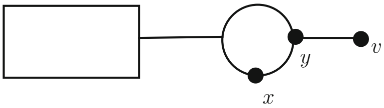


Fig. 2 Graph in the proof of Lemma 4.15

For convenience, we call a graph Γ *lower-optimal* if it achieves the equality in inequality (5). In the rest of this section, we aim to provide some fundamental characterizations of lower-optimal signed graphs.

Lemma 4.15 *Let* $\Gamma = (G, \sigma)$ *be a signed graph with* $d(G) \geq 1$ *and* x *be a vertex lying on a cycle of* Γ. *If* Γ *is lower-optimal, then*

(i) $r(\Gamma) = r(\Gamma - x)$;
(ii) $\alpha(G) = \alpha(G - x)$;
(iii) $d(G) = d(G - x) + 1$;
(iv) $\Gamma - x$ *is lower-optimal;*
(v) x *lies on just one cycle of* G *and* x *is not a quasi-pendant vertex of* G.

Proof The lower-optimality condition for Γ together with the proof of Lemma 4.14 forces equalities in (6)–(8). So we have (i)–(iv).

By (iii) and Lemma 4.2 (iii), we obtain that x lies on just one cycle of G, as shown in Fig. 2. If x is a quasi-pendant vertex adjacent to a pendant vertex v, then by Lemma 4.10, we have $r(\Gamma) = r(\Gamma - x) + 2$, a contradiction to (i). This completes the proof of (v). $\square$

Lemma 4.16 [14, Theorem 4.1] *Let* $\Gamma = (G, \sigma)$ *be a signed graph and* C_q *be a pendant cycle of* Γ *with* x *being the unique vertex of* C_q *of degree* 3, *and let* $H = \Gamma - C_q$, $M = \Gamma - (C_q - x)$.

(i) *If* C_q *is positive with order* $q \equiv 0 \pmod 4$, *or* C_q *is negative with order* $q \equiv 2 \pmod 4$, *then* $r(\Gamma) = q - 2 + r(M)$;
(ii) *If* C_q *is positive with order* $q \equiv 2 \pmod 4$, *or* C_q *is negative with order* $q \equiv 0 \pmod 4$, *then* $r(\Gamma) = q + r(H)$;
(iii) *If* q *is odd, then* $q - 1 + r(M) \leq r(\Gamma) \leq q + r(M)$.

Lemma 4.17 *Let* $\Gamma = (G, \sigma)$ *be a signed graph and* C_q *be a pendant cycle of* Γ *with* x *being the unique vertex of* C_q *of degree* 3, *and let* $H = \Gamma - C_q$, $M = \Gamma - (C_q - x)$.

(i) *If* C_q *is positive with order* $q \equiv 0 \pmod 4$, *or* C_q *is negative with order* $q \equiv 2 \pmod 4$, *then* $r(\Gamma) = q - 2 + r(M)$;
(ii) *If* C_q *is positive with order* $q \equiv 2 \pmod 4$, *or* C_q *is negative with order* $q \equiv 0 \pmod 4$, *or* q *is odd, then* $r(\Gamma) = q + r(H)$.

Proof When q is even, the results follow from Lemma 4.16. Now, we only need to consider the case of q is odd.

When q is odd, we assume that $q = 2k - 1$ ($k \geq 2$) and $x = v_{2k-1}$. The adjacency matrix $A(\Gamma)$ of Γ can be expressed as

$$A(\Gamma) = \begin{pmatrix} 0 & a^{\sigma}_{12} & & & \mathbf{0} & a^{\sigma}_{1,2k-1} & & & \\ a^{\sigma}_{12} & 0 & a^{\sigma}_{23} & & & 0 & & & \\ \mathbf{0} & \ddots & \ddots & & a^{\sigma}_{(2k-3),(2k-2)} & 0 & \mathbf{0} & & \\ & & a^{\sigma}_{(2k-3),(2k-2)} & & 0 & a^{\sigma}_{(2k-2),(2k-1)} & & & \\ a^{\sigma}_{1,2k-1} & 0 & \cdots & 0 & a^{\sigma}_{(2k-2),(2k-1)} & 0 & \alpha & 0 \cdots & 0 \\ & & & & & \alpha & & & \\ & & & & & 0 & & & \\ & & \mathbf{0} & & & \vdots & & \mathrm{F} & \\ & & & & & 0 & & & \end{pmatrix},$$

where $F = A(H), \alpha \in \{1, -1\}$. By some elementary row and column transformations, we can show that $A(\Gamma)$ is congruent to

$$A(\Gamma_1) = \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_{k-1} & \\ & & & C \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 0 & a_{(2i-1),2i} \\ a_{(2i-1),2i} & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 2b & \alpha & 0 \cdots 0 \\ \alpha & & \\ 0 & \mathrm{F} & \\ \vdots & & \\ 0 & & \end{pmatrix},$$

$i = 1, 2, \ldots k - 1$ and

$$b = (-1)^{k-1} \frac{a_{1,(2k-1)} a_{23} \ldots a_{(2k-2),(2k-1)}}{a_{12} a_{34} \ldots a_{(2k-3),(2k-2)}}.$$

With the fact that A_i has rank 2 and $r(C) = 1 + r(F) = 1 + r(H)$, we then have $r(\Gamma) = 2k - 2 + r(C) = q + r(H)$. □

Lemma 4.18 *Let $\Gamma = (G, \sigma)$ be a signed graph and C_q be a pendant signed cycle of Γ with x being the unique vertex of C_q of degree 3, and let $H = \Gamma - C_q$, $M = \Gamma - (C_q - x)$. If Γ is lower-optimal, then*

- (i) *C_q is positive with order $q \equiv 0 \pmod 4$, or C_q is negative with order $q \equiv 2 \pmod 4$;*
- (ii) *$r(\Gamma) = q - 2 + r(\Gamma_1)$, $\alpha(G) = \alpha(H) + \frac{q}{2}$;*

(iii) *both* H *and* M *are lower-optimal;*
(iv) $r(M) = r(H)$ *and* $\alpha(M) = \alpha(H) + 1$.

Proof (i) By contradiction, supposing that C_q is positive with order $q \equiv 2 \pmod 4$ or C_q is negative with order $q \equiv 0 \pmod 4$, or q is odd, then by Lemma 4.17 we have

$$r(\Gamma) = q + r(H). \tag{9}$$

Let $\delta = 0$ for even q and $\delta = 1$ for odd q. Note that x lies on the cycle C_q. So by Lemma 4.15 (ii), we have

$$\alpha(G) = \alpha(G - x) = \alpha(P_{q-1}) + \alpha(H) = \frac{q - \delta}{2} + \alpha(H). \tag{10}$$

Since C_q is a pendant cycle of G, we have

$$d(G) = d(M) + 1 = d(H) + 1. \tag{11}$$

Note that $|V(G)| = n$ and Γ is lower-optimal, we have

$$r(\Gamma) + 2\alpha(G) = 2n - 2d(G). \tag{12}$$

From (9)–(12), we have $r(H) + 2\alpha(H) = 2(n - q) - 2d(H) - 2 + \delta$, which is a contradiction to (1).

(ii) Since x lies on a cycle of Γ, by Lemma 4.15 (i)–(ii) we have

$$r(\Gamma) = r(\Gamma - x) = r(P_{q-1}) + r(H) = q - 2 + r(H), \tag{13}$$

$$\alpha(G) = \alpha(G - x) = \alpha(P_{q-1}) + \alpha(H) = \frac{q}{2} + \alpha(H). \tag{14}$$

(iii) Let x_1 be on C_q such that it is adjacent to x. By applying Lemma 4.15 to Γ (resp. G) and Lemma 4.10 (resp. Lemma 4.11) to $\Gamma - x_1$ (resp. $G - x_1$), we have

$$r(\Gamma) = r(\Gamma - x_1) = q - 2 + r(M), \tag{15}$$

$$\alpha(G) = \alpha(G - x_1) = \frac{q - 2}{2} + \alpha(M). \tag{16}$$

From (11)–(12) and (13)–(14), we have $r(H) + 2\alpha(H) = 2(n - q) - 2d(H)$, implying that H is lower-optimal.

Combining (11)–(12) and (15)–(16), one has $r(M) + 2\alpha(M) = 2(n - q + 1) - 2d(M)$, which implies that M is also lower-optimal.

(iv) Combining (13) and (15) yields $r(M) = r(H)$, and equalities (14) and (16) lead to $\alpha(M) = \alpha(H) + 1$. □

Lemma 4.19 *Let* y *be a pendant vertex of* Γ *with neighbor* x*, and let* $H = \Gamma - x - y$*. If* Γ *is lower-optimal, then*

(i) x *does not lie on any cycle of* G*;*

(ii) *H is also lower-optimal.*

Proof (i) Since x is a quasi-pendant vertex of Γ, Lemma 4.15 (v) states that x does not lie on any cycle of Γ.

(ii) By Lemmas 4.10 and 4.11, we have

$$r(\Gamma) = r(H) + 2, \quad \alpha(G) = \alpha(H) + 1. \tag{17}$$

Since x does not lie on any cycle of G, by Lemma 4.2 (i) we have

$$d(G) = d(H). \tag{18}$$

Equalities (17)–(18) together with the lower-optimality condition of Γ imply that $r(H) + 2\alpha(H) = 2(n-2) - 2d(H)$, i.e., H is lower-optimal. □

Lemma 4.20 *If Γ is lower-optimal, then*

(i) *the cycles (if any) of Γ are pairwise vertex-disjoint;*
(ii) *Γ is bipartite and each cycle C has sign $(-1)^{|C|/2}$;*
(iii) $\alpha(G) = \alpha(T_G) + \sum_{C\in\mathcal{C}(G)} \frac{|V(C)|}{2} - d(G)$.

Proof (i) If G contains cycles, then let x be a vertex on some cycle. By Lemma 4.15 (iii), we have $d(G) = d(G-x)+1$. By Lemma 4.2 (iii), x cannot be a common vertex of distinct cycles. Hence, the cycles of Γ are pairwise vertex-disjoint. This completes the proof of (i).

We will prove (ii)–(iii) by induction on the order n of G. The initial case $n = 1$ is trivial.

Suppose that (ii) and (iii) hold for any lower-optimal signed graph of order smaller than n, and suppose that Γ is a lower-optimal signed graph of order $n \geq 2$.

If T_G is empty graph, then Γ is a simple signed cycle C_q. By Lemma 4.8, (ii) follows, and (iii) holds from the fact that $\alpha(C_q) = \frac{q}{2}$ because q is even.

If T_G has at least one edge, then T_G contains at least one pendant vertex, say y. Then, y is either a pendant vertex of G or $y \in W_{\mathcal{C}}$, in which case G contains a pendant cycle. Now we consider both cases.

Case 1 G contains a pendant vertex y. In this case, let x be the neighbor of y in G and let $H = \Gamma - x - y$. By Lemma 4.19, x is not a vertex on any cycle of G and H is also lower-optimal. By the induction hypothesis, we have property (ii) for H since all cycles of Γ also in H. Similarly, we have

$$d(G) = d(H). \tag{19}$$

Since $T_H = T_G - x - y$, by Lemma 4.11 and (19) we have

$$\begin{aligned}\alpha(G) = \alpha(H) + 1 &= \alpha(T_H) + \sum_{C\in\mathcal{C}(H)} \frac{|V(C)|}{2} - d(H) + 1 \\ &= \alpha(T_G) + \sum_{C\in\mathcal{C}(G)} \frac{|V(C)|}{2} - d(G).\end{aligned}$$

Thus, (iii) holds.

Case 2 Γ has a pendant cycle C_q.

In this case, let x be the unique vertex of C_q of degree 3, $H = \Gamma - C_q$ and $M = \Gamma - (C_q - x)$. It follows from Lemma 4.18 (iii) that M is lower-optimal. Applying the induction hypothesis to M yields property (ii) for M. Applying Lemma 4.18 (i), we have property (ii) for Γ. Thus, (ii) holds.

Combining Lemma 4.18 (ii), (iv) and assertion (d), we have

$$\alpha(G) = \alpha(M) + \frac{q}{2} - 1 = \alpha(T_M) + \sum_{C \in \mathcal{C}(M)} \frac{|V(C)|}{2} + \frac{q}{2} - d(M) - 1. \tag{20}$$

Since C_q is a pendant cycle of Γ, we have

$$d(G) = d(M) + 1. \tag{21}$$

Note that $T_M \cong T_G$ and $\frac{q}{2} + \sum_{C \in \mathcal{C}(M)} \frac{|V(C)|}{2} = \sum_{C \in \mathcal{C}(G)} \frac{|V(C)|}{2}$. Together with (20)–(21), we have $\alpha(G) = \alpha(T_G) + \sum_{C \in \mathcal{C}(G)} \frac{|V(C)|}{2} - d(G)$ as desired. □

5 Proofs of Our Main Results

With the above preparations, we are now ready to give the proofs of our main results stated in Sect. 3.

5.1 Proof of Theorem 3.1

Proof Lemma 4.14 already established (1). We now characterize all the simple and connected signed graphs Γ which attain the lower bound by considering the sufficient and necessary conditions for the equality in (1).

Sufficiency: We proceed by induction on the order n of G to show that Γ is lower-optimal if Γ satisfies the conditions (i)–(iii).

The $n = 1$ case is trivial. Suppose that any graph with order smaller than n which satisfies (i)–(iii) is lower-optimal, and suppose that Γ is a signed graph with order $n \geq 2$ that satisfies (i)-(iii). Since the cycles (if any) of Γ are pairwise vertex-disjoint, Lemma 4.2 states that G has exactly $d(G)$ cycles, implying that $|W_{\mathcal{C}}| = d(G)$.

If T_G is an empty graph, it follows from (ii) that Γ is a positive cycle with order $q \equiv 0 \pmod 4$, or a negative cycle with order $q \equiv 2 \pmod 4$, leading to the fact that Γ is lower-optimal. So in what follows, we assume that T_G has at least one edge. Note that $\alpha(T_G) = \alpha(L_G) + d(G) = \alpha(T_G - W_{\mathcal{C}}) + d(G)$. Then, by Lemma 4.13 (ii) there exists a pendant vertex of T_G not in $W_{\mathcal{C}}$. Thus, G has at least one pendant vertex, say y. Let x be the unique neighbor of y in G and $H = \Gamma - x - y$. Then, y is also a pendant vertex of T_G adjacent to x. By Lemma 4.11, we have

$$\alpha(T_G) = \alpha(T_G - x) = \alpha(T_G - x - y) + 1. \tag{22}$$

If $x \in W_{\mathcal{C}}$, then the graph $L_G \cup (d(G)K_1)$ can be obtained from $(T_G - x) \cup K_1$ by removing some edges. By Lemma 4.3 (ii), we get

$$\alpha(L_G) + d(G) \geq \alpha(T_G - x) + 1. \tag{23}$$

Now from (22)–(23), we have $\alpha(L_G) \geq \alpha(T_G - x) - d(G) + 1 = \alpha(T_G) - d(G) + 1$, a contradiction to (iii). Thus, x does not lie on any cycle of G. Then, y is also a pendant vertex of L_G adjacent to x and $L_H = L_G - x - y$. By Lemma 4.11, we have

$$\alpha(L_G) = \alpha(L_H) + 1. \tag{24}$$

Since x does not lie on any cycle of G, Lemma 4.2 (i) implies that

$$d(G) = d(H). \tag{25}$$

Now from condition (iii) and (22), as well as (24)–(25), we have $\alpha(T_H) = \alpha(L_H) + d(H)$. Also note that all cycles of G are cycles of H. We conclude that H satisfied conditions (i)–(iii). By the induction hypothesis, we have

$$r(H) + 2\alpha(H) = 2(n-2) - 2d(H). \tag{26}$$

Furthermore, it follows from Lemmas 4.10 and 4.11 that

$$r(\Gamma) = r(H) + 2, \quad \alpha(G) = \alpha(H) + 1. \tag{27}$$

By (25)–(27), we have $r(\Gamma) + 2\alpha(G) = 2n - 2d(G)$, implying that Γ is lower-optimal.

Necessity: Let Γ be lower-optimal. By Lemma 4.20, Γ satisfies (i) and (ii).

We proceed by induction on the order n of G to prove (iii). The $n = 1$ case is trivial. Suppose that (iii) holds for all lower-optimal signed graph of order smaller than n, and suppose that Γ is lower-optimal signed graph of order $n \geq 2$.

If T_G is an empty graph, then Γ is a cycle of even order and (iii) follows immediately.

Now suppose T_G has at least one edge. Then, T_G has at least one pendant vertex, say y. As in the proof of Lemma 4.20, either G contains y as a pendant vertex, or G contains a pendant cycle.

Case 1 G has a pendant vertex y.

Let x be the neighbor of y in G and $H = \Gamma - x - y$. By Lemma 4.19, x is not on any cycle of G and H is also lower-optimal. Applying the induction hypothesis to H yields

$$\alpha(T_H) = \alpha(L_H) + d(H). \tag{28}$$

Since x dose not lie on any cycle of G, Lemma 4.2 (i) states that

$$d(G) = d(H). \tag{29}$$

Note that y is also a pendant vertex of T_G (resp. L_G) adjacent to x and $T_H = T_G - x - y$ (resp. $L_H = L_G - x - y$). Then, by Lemma 4.11 we have

$$\alpha(T_G) = \alpha(T_H) + 1, \quad \alpha(L_G) = \alpha(L_H) + 1. \tag{30}$$

From (28)–(30), we have $\alpha(T_G) = \alpha(L_G) + d(G)$, as desired.

Case 2 G has a pendant cycle C_q.

Let x be the unique vertex of C_q of degree 3, and $H = \Gamma - C_q$. By Lemma 4.18 (iii), H is lower-optimal.

Applying the induction hypothesis to H yields

$$\alpha(T_H) = \alpha(L_H) + d(H). \tag{31}$$

From Lemma 4.18 (ii), we have

$$\alpha(G) = \alpha(H) + \frac{q}{2}. \tag{32}$$

Note that $\mathcal{C}(G) = \mathcal{C}(H) \cup C_q$. Together with (32) and Lemma 4.20 (iii), we have

$$\alpha(T_G) = \alpha(H) + \frac{q}{2} - \sum_{C \in \mathcal{C}(G)} \frac{|V(C)|}{2} + d(G) = \alpha(H) - \sum_{C \in \mathcal{C}(H)} \frac{|V(C)|}{2} + d(G). \tag{33}$$

Since H is lower-optimal, Lemma 4.20 (iii) states that

$$\alpha(T_H) = \alpha(H) - \sum_{C \in \mathcal{C}(H)} \frac{|V(C)|}{2} + d(H). \tag{34}$$

Since C_q is a pendant cycle of G, we have

$$d(G) = d(H) + 1. \tag{35}$$

Combining (33)–(35) yields

$$\alpha(T_G) = \alpha(T_H) + 1. \tag{36}$$

Note that $L_G \cong L_H$. Then, the required equality $\alpha(T_G) = \alpha(L_G) + d(G)$ follows from (31) and (35)–(36). This completes the proof. □

5.2 Proofs of Theorems 3.2, 3.3 and 3.4

By Theorem 3.1 and Lemma 4.5, we can obtain the proofs of Theorems 3.2, 3.3 and 3.4, easily. So we only give the proof of Theorem 3.2 and omit the other proofs.

The proof of Theorem 3.2 Note that for a given simple connected graph G with $|V(G)| = n$ and $|E(G)| = m$, by (1) and Lemma 4.5 we have

$$r(\Gamma) + \alpha(G) = r(\Gamma) + 2\alpha(G) - \alpha(G) \geq 4n - 2m - \sqrt{n(n-1) - 2m + \frac{1}{4}} - \frac{5}{2},$$

as stated in (2).

Now, we prove the sufficient and necessary conditions for equality in (2).

Sufficiency: If $n = 1$, we have $G \cong K_1$, and then (2) holds, trivially. If $n \geq 2$, we have $G \cong S_n$, and we obtain $r(\Gamma) = 2$, $\alpha(G) = n - 1$. Together with the fact that $m = n - 1$, we have the equality in (2).

Necessity: Combining Theorem 3.1 and Lemma 4.5, the equality in (2) holds if and only if Γ is lower-optimal and $G \cong K_{n-\alpha(G)} \vee \alpha(G)K_1$. Note that the cycles (if any) of Γ are pairwise vertex-disjoint, and each cycle C_q of Γ is positive with order $q \equiv 0 \pmod 4$, or negative with order $q \equiv 2 \pmod 4$. So $n - \alpha(G) = 1$ and $\alpha(G) = n - 1$, which implies $G \cong S_n$. This completes the proof. □

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