ON VERTEX-TRANSITIVE SELF-COMPLEMENTARY DIGRAPHS

HU YE CHEN AND ZAI PING LU

Abstract. In this paper, we consider vertex-transitive self-complementary digraphs which satisfy certain conditions, and present several analogous results of that on vertex-transitive self-complementary graphs. We first give an elementary proof of the fact that there exists a vertex-transitive self-complementary digraph of order n if and only if n is odd. In Section 3, we show that an arc-transitive self-complementary digraph is either a graph or isomorphic a normal Cayley digraph of $\mathbb{Z}_p^f$, where p is an odd prime. At last, we characterize all vertex-transitive self-complementary digraphs of order a product of two primes.

Keywords: Digraph, self-complementary, complementary map, vertex-transitive, arc-transitive.

1. introduction

Let V be a finite nonempty set, and $V^{(2)} = \{(u, v) \mid u, v \in V, u \neq v\}$. A (simple) digraph on V is a pair (V, A) with $A \subseteq V^{(2)}$, while $|V|$ is the *order* of Γ, and the elements in V and A are called *vertices* and *arcs*, respectively. A subset $A \subseteq V^{(2)}$ is *self-paired* if $A = A^* := \{(v, u) \mid (u, v) \in A\}$. In this paper, we always consider (simple) graphs as digraphs which have self-paired arc set. For a digraph $\Gamma = (V, A)$, we set $\overline{\Gamma} = (V, V^{(2)} \setminus A)$ and call it the *complement* of Γ. Then a digraph $\Gamma = (V, A)$ is *self-complementary* if there is an isomorphism of digraphs between Γ and $\overline{\Gamma}$.

Let $\Gamma = (V, A)$ be a digraph. Denote by $\mathsf{Aut}\Gamma$ the automorphism group of Γ, which is the subgroup of the symmetric group $\mathrm{Sym}(V)$ preserving the adjacency of Γ. The digraph Γ is called *vertex-transitive* if $\mathsf{Aut}\Gamma$ is a transitive permutation group on V, i.e., a transitive subgroup of $\mathrm{Sym}(V)$. (We follows [4] for notation and concepts relative to permutation groups.) Note that each $g \in \mathrm{Sym}(V)$ has a natural action on $V^{(2)}$ by $(u, v)^g =$

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E-mail address: 1120140003@mail.nankai.edu.cn(H.Y. Chen), lu@nankai.edu.cn(Z.P. Lu).

(u^g, v^g). Then the digraph Γ is called *arc-transitive* if $\mathsf{Aut}\Gamma$ is transitive on both V and A. In the literature, an arc-transitive graph (i.e., digraph with self-paired arc set) is also called a *symmetric* graph.

In 1962, Sachs [21] started the study of vertex-transitive self-complementary graphs and constructed some self-complementary circulant graphs. Since then, much work has been done on this topic (see [6, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22], for example), and many intriguing results have been presented. For example, Rao [20] gave a characterization of the orders of vertex-transitive self-complementary graphs, and then Muzychuk [18] presented a Sylow property for such graphs.

Note that a graph can be identified with a digraph with self-paired arc set. It is reasonable to consider various analogous problems about vertex-transitive self-complementary digraphs. To our knowledge, Chia and Lim [2] first studied the class of vertex-transitive self-complementary digraphs, and they give a complete classification for vertex-transitive self-complementary digraphs of prime order. However, there are few progresses on this topic. One can deduce an analogue of Rao's result from [13], which says that there exists a vertex-transitive self-complementary digraph of order n if and only if n is odd. A remarkable result on this topic is the characterization of primitive self-complementary digraphs, which is given by Guralnick, Li, Praeger and Saxl [5]. Therefore we shall make an attempt in this paper to generalize some results on vertex-transitive self-complementary graphs to digraphs.

This paper is organized as follows. In Section 2, we collect some basic properties about vertex-transitive self-complementary digraphs. In Sections 3 and 4, we give a classification of arc-transitive self-complementary digraphs, and a classification of vertex-transitive self-complementary digraphs of order a product of two primes.

2. The orders of vertex-transitive self-complementary digraphs

Let $\Gamma = (V, A)$ be a self-complementary digraph. Let ι be an isomorphism from Γ to $\overline{\Gamma}$, which is called a complementary map. Then $\iota^2 \in \mathsf{Aut}\Gamma$ and $\mathsf{Aut}\overline{\Gamma} = \mathsf{Aut}\Gamma^\iota$, and $\overline{\Gamma}$ has arc set $A^\iota := \{(u^\iota, v^\iota) \mid (u, v) \in A\}$. Moreover, for each odd integer k, ι^k is also a complementary map. Clearly, as a permutation on V, ι has even order and fixes at most one point in V. Consider a factorization of ι into disjoint cycles on V. Suppose that there is a cycle of odd length k in this factorization. Then ι^k fixes at least k points in V. Since ι^k is a complementary map, we have $k = 1$. Then the following lemma holds.

Lemma 2.1. *Let $\mathit{\Gamma} = (V, A)$ be a self-complementary digraph with a complementary map ι. Then, in each factorization of ι into disjoint cycles on V, there is at most one 1-cycle, and the remaining cycles have even length. If further $\mathit{\Gamma}$ is vertex-transitive, then ι has exactly one fixed point; in particular, $|V|$ is odd.*

Proof. The first part of this lemma follows from the argument in the beginning paragraph of this section. Thus we assume that $\mathit{\Gamma}$ is a vertex-transitive self-complementary digraph of order n. For $u \in V$, set $\Gamma^+(u) = \{v \in V \mid (u, v) \in A\}$. Then $\overline{\Gamma}^+(u) = V \setminus (\{u\} \cup \Gamma^+(u))$; in particular, $|\Gamma^+(u)| + |\overline{\Gamma}^+(u)| = |V| - 1$. Since Γ is vertex-transitive, both $|\Gamma^+(u)|$ and $|\overline{\Gamma}^+(u)|$ are independent of the choice of u. Thus if further $\mathit{\Gamma} \cong \overline{\Gamma}$ then $|V| - 1 = |\Gamma^+(u)| + |\overline{\Gamma}^+(u)| = 2|\Gamma^+(u)|$, and so $|V|$ is odd. Then the second part of this lemma follows from the first part. □

Let $\mathit{\Gamma} = (V, A)$ be a vertex-transitive self-complementary digraph with a complementary map ι. Then ι lies in the normalizes $\mathbf{N}_{\mathrm{Sym}(V)}(\mathsf{Aut}\mathit{\Gamma})$ of $\mathsf{Aut}\mathit{\Gamma}$ in the symmetric group $\mathrm{Sym}(V)$, and ι has even order (in $\mathrm{Sym}(V)$) and a unique fixed-point (in V). (Note such an ι may be chosen having order a power of 2). This observation leads to a criterion for a transitive permutation group acting on some vertex-transitive self-complementary digraphs.

Lemma 2.2. *Let G be a transitive permutation group on a finite set V with $|V| > 1$. Suppose that $\iota \in \mathbf{N}_{\mathrm{Sym}(V)}(G)$ fixes a unique point, say u. Suppose that for each odd integer k, ι^k has no fixed-point other than u. Then ι fixes the fixed-point set of G_u, and there is a vertex-transitive self-complementary digraph $\mathit{\Gamma}$ such that $G \leq \mathsf{Aut}\mathit{\Gamma}$.*

Proof. Since $u^\iota = u$ and $G^\iota = G$, we have $G_u^\iota = G_u$. Thus G_u fixes some $v \in V$ if and only if G_u fixes v^ι. Then the first part of this lemma follows.

Now write ι as a product of disjoint cycles of length no less than 2, say $\iota = \iota_1\iota_2 \cdots \iota_t$ For each i, fix a point u_i involved in the cycle ι_i, and let U_i be the $\langle \iota_i^2 \rangle$-orbit containing u_i. Set $A = \cup_{g \in G, 1 \leq i \leq t}(\{u\} \times U_i)^g$. Then it is easily shown that $\mathit{\Gamma} := (V, A)$ is a self-complementary digraph. Clearly, $G \leq \mathsf{Aut}\mathit{\Gamma}$. Thus the lemma holds. □

In the following, we shall see that there exists a vertex-transitive self-complementary digraph of order n if and only if n is odd. Note that this is trivial if $n = 1$.

Let R be a finite group and S be a subset of R with $1 \notin S$, where 1 is the identity of R. Then the Cayley digraph $\mathrm{Cay}(R, S)$ is defined on R such that (x, y) is an arc if and only if $yx^{-1} \in S$. It is well-know that the group R induces by right multiplication a subgroup of $\mathsf{Aut}(\mathrm{Cay}(R, S))$, denoted by $\widehat{R}$, which acts regularly on (the vertex set) R. Thus every Cayley digraph is vertex-transitive. Denote by $\mathsf{Aut}(R)$ the automorphism group of R. Then each $\sigma \in \mathsf{Aut}(R)$, in its natural action, gives an isomorphism from $\mathrm{Cay}(R, S)$ to $\mathrm{Cay}(R, S^\sigma)$. Then we have the following simple observation.

Lemma 2.3. *Let R be a finite group. If there exist $\iota \in \mathsf{Aut}(R)$ and $S \subseteq R$ such that $S \cap S^\iota = \emptyset$ and $S \cap S^\iota = R \backslash \{1\}$, then the Cayley digraph $\mathrm{Cay}(R, S)$ is self-complementary.*

Example 2.4. Let R be an additive abelian group of odd order $n > 1$. Take $S \subset R$ with $R \setminus \{0\} = S \cup (-S)$ and $S \cap (-S) = \emptyset$. Consider the map $\iota : R \to R$; $x \mapsto -x$. Then ι is an automorphism of R, and $S^\iota = -S$. Thus, by Lemma 2.3, the Cayley digraph $\mathrm{Cay}(R, S)$ is self-complementary.

The foregoing argument yields the follows result, see also [13].

Theorem 2.5. *There exists a vertex-transitive self-complementary digraph on n vertices if and only if n is odd.*

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ be an integer, where $p_1, \ldots, p_s$ are distinct odd primes, and $e_i \geq 1$ for all i. Muzychuk [18] showed that if $\varGamma$ is a vertex-transitive self-complementary graph of order n, then $p_i^{e_i} \equiv 1 \,(\mathsf{mod}\ 4)$ and $\varGamma$ has an induced subgraph which is vertex-transitive and self-complementary. Assume that $\varGamma = (V, A)$ is a vertex-transitive self-complementary digraph of order n. Then, by a similar argument as in [18], there exists a Sylow p_i-subgroup P of $\mathsf{Aut}\varGamma$ and a complementary may $\iota : \varGamma \to \overline{\varGamma}$ such that $P^\iota = P$ and P has an orbit U with $|U| = p_i^e$. Thus we can get a self-complementary digraph $[U] := (U, A \cap U^{(2)})$.

Proposition 2.6. *Let $\Gamma = (V, A)$ be a vertex-transitive self-complementary digraph of order $p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$, where p_i's are distinct odd primes, and $e_i \geq 1$ for all i. Then for each i, the digraph Γ has an induced subdigraph of order $p_i^{e_i}$ that is both vertex-transitive and self-complementary.*

3. Arc-transitive self-complementary digraphs

Recall that a digraph graph $\varGamma = (V, A)$ is *arc-transitive*, if $\mathsf{Aut}\varGamma$ acts transitively on both V and A. We first give an example of arc-transitive self-complementary digraphs.

Example 3.1. Let R be the additive group of the finite field $\mathbb{F}_{p^f}$ of order p^f, and let a be a generator of the multiplicative group of $\mathbb{F}_{p^f}$, where p is an odd prime with $p^f \equiv 3 \,(\mathsf{mod}\ 4)$. Set $S = \{a^{2i+1} \mid 1 \leq i \leq \frac{p^f-1}{2}\}$ and $\varGamma = \mathrm{Cay}(R, S)$. Let σ be the Frobenius automorphism of $\mathbb{F}_{p^f}$, that is, $x^\sigma = x^p$ for $x \in \mathbb{F}_{p^f}$. Then σ has order f. For b, $c \in \mathbb{F}_{p^f}$ with $b \neq 0$, define

$\tau_{b,c}: \mathbb{F}_{p^f} \to \mathbb{F}_{p^f}$, $x \mapsto bx+c$. We have $\mathrm{A\Gamma L}(1,p^f) = \langle \sigma, \tau_{a,0}, \tau_{1,c} \mid c \in \mathbb{F}_{p^f}\rangle$, and $\Gamma\mathrm{L}(1,p^f) = \langle \sigma, \tau_{a,0}\rangle \leq \mathsf{Aut}(R)$. It is easily shown that $\sigma, \tau_{a^2,0}, \tau_{1,c} \in \mathsf{Aut}\mathit{\Gamma}$. Set $G = \langle \sigma, \tau_{a^2,0}, \tau_{1,c} \mid c \in \mathbb{F}_{p^f}\rangle$. Then G has index 2 in $\mathrm{A\Gamma L}(1,p^f)$ and acts transitively on the arcs of $\mathit{\Gamma}$. Noting that $R \setminus \{0\} = S \cup S^{\tau_{a,0}}$ and $S \cap S^{\tau_{a,0}} = \emptyset$, by Lemma 2.3, $\mathit{\Gamma}$ is also self-complementary.

We need some basic properties of Cayley digraphs for further argument. Let R be a finite group. Consider the action of R on R induced by the right multiplication. Denote by $\widehat{R}$ the resulting permutation group from this action. Then it is well-known that

$$\mathbf{N}_{\mathrm{Sym}(R)}(\widehat{R}) = \widehat{R}\mathsf{Aut}(R), \tag{3.1}$$

where $\mathrm{Sym}(R)$ is the symmetric group on R, and $\mathsf{Aut}(R)$ acts naturally on R. For a subset S of $R \setminus \{1\}$ and $\mathit{\Gamma} = \mathrm{Cay}(R,S)$, we have

$$\mathbf{N}_{\mathsf{Aut}\mathit{\Gamma}}(\widehat{R}) = \mathbf{N}_{\mathsf{Aut}\mathit{\Gamma}}(\widehat{R}) \cap \mathbf{N}_{\mathrm{Sym}(R)}(\widehat{R}) = \mathbf{N}_{\mathsf{Aut}\mathit{\Gamma}}(\widehat{R}) \cap \widehat{R}\mathsf{Aut}(R).$$

It implies that

$$\mathbf{N}_{\mathsf{Aut}\mathit{\Gamma}}(\widehat{R}) = \widehat{R}\mathsf{Aut}(R,S), \tag{3.2}$$

where $\mathsf{Aut}(R,S) = \{\sigma \in \mathsf{Aut}(R) \mid S^\sigma = S\}$. The Cayley digraph $\mathit{\Gamma} = \mathrm{Cay}(G,S)$ is called *normal* if $\mathsf{Aut}\mathit{\Gamma} = \widehat{R}\mathsf{Aut}(R,S)$, i.e., $\widehat{R}$ is a normal subgroup of $\mathsf{Aut}\mathit{\Gamma}$.

In the following we shall show that an arc-transitive self-complementary digraph either is a graph or it automorphism group has a normal regular subgroup, and so it is a normal Cayley digraph. It is well-known that a vertex-transitive digraph is isomorphic to a Cayley digraph if and only if its automorphism group has a regular subgroup. In fact, for a digraph $\mathit{\Gamma} = (V,A)$ such that $\mathsf{Aut}\mathit{\Gamma}$ has a regular subgroup R, it is easily shown that $R \to V, x \mapsto u^x$ is an isomorphism from $\mathrm{Cay}(R,S)$ to $\mathit{\Gamma}$, where u is a given vertex and $S = \{x \in R \mid (u,u^x) \in A\}$.

Assume that $\mathit{\Gamma} = (V,A)$ is an arc-transitive self-complementary digraph. Recall that $A^* = \{(u,v) \mid (v,u) \in A\}$. If $A^* = A$ then $\mathit{\Gamma}$ can be viewed as a symmetric graph by identifying each pair (u,v) and (v,u) of arcs as an edge $\{u,v\}$, and then one can get $\mathit{\Gamma}$ from [19]. Thus we deal with the case where $A^* \neq A$ in the following theorem.

Theorem 3.2. *Assume that $\mathit{\Gamma} = (V,A)$ is an arc-transitive digraph self-complementary. If $A^* \neq A$ then $|V| = p^f \equiv 3 \,(\mathsf{mod}\ 4)$ for an odd prime p, $\mathsf{Aut}\mathit{\Gamma}$ is isomorphic to a subgroup of $\mathrm{A\Gamma L}(1,p^f)$ with index 2, and $\mathit{\Gamma}$ is isomorphic to the normal Cayley digraph constructed in Example* 3.1.

Proof. Since $\mathit{\Gamma}$ is arc-transitive, either $A^* = A$ or $A^* \cap A = \emptyset$ and $V^{(2)} = A \cup A^*$. Assume that $A^* \neq A$. Then $\mathsf{Aut}\mathit{\Gamma}$ is 2-homogenous but not 2-transitive on V. By [7], we may let $\mathrm{ASL}(1,p^f) \leq \mathsf{Aut}\mathit{\Gamma} \leq \mathrm{A\Gamma L}(1,p^f)$ and

V be the underlying set of the field $\mathbb{F}_{p^f}$, where p is a prime and f is positive integer with $p^f \equiv 3 \pmod 4$. In particular, Γ is isomorphic to a normal Cayley digraph of the additive group R of $\mathbb{F}_{p^f}$.

Let a be a generator of the multiplicative group of $\mathbb{F}_{p^f}$. Then a lies in $\Gamma^+(0)$ or $\overline{\Gamma}^+(0)$. Since Γ is self-complementary, up to isomorphism of digraphs, we may chose $a \in \Gamma^+(0)$. Since $\mathsf{Aut}\Gamma$ is transitive on A, we conclude that the stabilizer $(\mathsf{Aut}\Gamma)_0$ of 0 in $\mathsf{Aut}\Gamma$ acts transitively on $\Gamma^+(0)$, which has odd size $\frac{p^f-1}{2}$. Thus $\Gamma^+(0) = \{a^g \mid g \in (\mathsf{Aut}\Gamma)_0\}$.

Clearly, $(\mathsf{Aut}\Gamma)_0 \leq \Gamma\mathrm{L}(1, p^f)$. Recall that $\Gamma\mathrm{L}(1, p^f) = \langle \sigma, \tau_{a,0} \rangle$, where σ is the Frobenius automorphism of $\mathbb{F}_{p^f}$, and $\tau_{a,0}$ is defined as Example 3.1. Then $\Gamma\mathrm{L}(1, p^f)$ is the semidirect product of $\langle \tau_{a,0} \rangle$ by $\langle \sigma \rangle$, which have order $p^f - 1$ and f respectively. Since $p^f \equiv 3 \pmod 4$, and so f is odd, $|\Gamma\mathrm{L}(1, p^f)|$ is not divisible by 4. Moreover, $\Gamma\mathrm{L}(1, p^f) = \langle \tau_{-1,0} \rangle \times \langle \tau_{a^2,0}, \sigma \rangle$, which contains a unique element of order 2. (We remark that $a^{\frac{p^f-1}{2}} = -1$.)

Note that $\mathsf{Aut}\Gamma$ has a normal regular subgroup $\mathrm{ASL}(1, q) = \{\tau_{1,c} \mid c \in \mathbb{F}_{p^f}\}$. Then $(x, y) \in A$ if and only if $(0^{\tau_{1,x}}, y) \in A$, which is equivalent to $(0, y-x) \in A$. It implies that $(x, y) \in A$ if and only if $y - x \in \Gamma^+(0)$. Thus, if $\tau_{-1,0} \in (\mathsf{Aut}\Gamma)_0$ then $(x, y) \in A$ shall yield $(y, x) \in A$, and so $A = A^*$, a contradiction. Then we have, $\tau_{-1,0} \notin (\mathsf{Aut}\Gamma)_0$; in particular, $(\mathsf{Aut}\Gamma)_0$ has odd order. It implies that $(\mathsf{Aut}\Gamma)_0 \leq \langle \tau_{a^2,0}, \sigma \rangle$, and hence $\Gamma^+(0) = \{a^g \mid g \in (\mathsf{Aut}\Gamma)_0\} \subseteq \{a^g \mid g \in \langle \tau_{a^2,0}, \sigma \rangle\} = \{a^{2i+1} \mid 1 \leq i \leq \frac{p^f-1}{2}\}$. Recalling that $|\Gamma^+(0)| = \frac{p^f-1}{2}$, we get $\Gamma^+(0) = \{a^{2i+1} \mid 1 \leq i \leq \frac{p^f-1}{2}\}$, and then $\Gamma^+(x) = \{x + a^{2i+1} \mid 1 \leq i \leq \frac{p^f-1}{2}\}$ for each $x \in V$. Therefore, Γ is isomorphic to the digraph given in Example 3.1. □

4. Digraphs of order a product of two primes

We begin this section with a fact that every (vertex-transitive) self-complementary digraph is properly contained in some (vertex-transitive) self-complementary digraph.

For a digraph Σ with vertex set U and a digraph Δ with vertex set W, the *lexicographic product* $\Sigma[\Delta]$ is the digraph with vertex set $U \times W$ such that the vertex (u, w) is adjacent to (u', w') if and only if either u is adjacent to u' in Σ or $u = u'$ and w is adjacent to w' in Δ. It is easily shown that $\overline{\Sigma[\Delta]} = \overline{\Sigma}[\overline{\Delta}]$. Thus, if both Σ and Δ are self-complementary then $\Sigma[\Delta]$ is also self-complementary. Note that $\mathsf{Aut}\Sigma[\Delta]$ has a subgroup $\mathsf{Aut}\Sigma \times \mathsf{Aut}\Delta$, which acts on $U \times W$ by

$$(u, w)^{(x,y)} = (u^x, w^y),\ u \in U,\ w \in W,\ x \in \mathsf{Aut}\Sigma,\ y \in \mathsf{Aut}\Delta.$$

Then the next simple fact follows.

Lemma 4.1. *If Σ and Δ are (vertex-transitive) self-complementary digraphs, then $\Sigma[\Delta]$ is also a (vertex-transitive) self-complementary digraph.*

Let G be a finite transitive permutation group on V. A nonempty subset B of V is a *block* of G if $B^g = B$ or $B \cap B^g = \emptyset$ for all $g \in G$. We refer the reader to [4, Sections 1.5 and 1.6] for notation and basic properties about blocks. Let B be a block of G on V. Then the *setwise stabilizer* $G_B := \{g \in G \mid B^g = B\}$ acts transitively on B with kernel $G_{(B)} = \cap_{u\in B} G_u$. Set $\mathcal{B} = \{B^g \mid g \in G\}$. Then G induces naturally a transitive permutation group on $\mathcal{B}$, denoted by $G^{\mathcal{B}}$, and $G^{\mathcal{B}} \cong G/G_{(\mathcal{B})}$, where $G_{(\mathcal{B})} := \cap_{g\in G} G_B^g$ is the kernel of G acting on $\mathcal{B}$. A block B of G is nontrivial if it has size a proper divisor of $|V|$. Recall that G is *imprimitive* if G has nontrivial blocks, and *primitive* otherwise.

Lemma 4.2. *Let G and $\mathcal{B}$ be as above, and let $K = G_{(\mathcal{B})}$. Take $B \in \mathcal{B}$ and set $\mathcal{B}_1 = \{C \in \mathcal{B} \mid K_{(C)} = K_{(B)}\}$. Then $\mathcal{B}_1$ is a block of $G^{\mathcal{B}}$ acting on $\mathcal{B}$; in particular, if $G^{\mathcal{B}}$ is primitive then either K is faithful on each $C \in \mathcal{B}$, or $K_{(B)}$ acts nontrivially on every $C \in \mathcal{B} \setminus \{B\}$.*

Proof. For $g \in G$, we have $K_{(B^g)} = G_{(B^g)} \cap K = G_{(B)}^g \cap K = (G_{(B)} \cap K)^g = K_{(B)}^g$. Then $K_{(B^g)} = K_{(B)}$ if and only if $g \in \mathbf{N}_G(K_{(B)})$, and thus $\mathcal{B}_1 = \{B^g \mid g \in \mathbf{N}_G(K_{(B)})\}$; in particular, $\mathcal{B}_1$ contains B and is an orbit of $\mathbf{N}_G(K_{(B)})$ acting on $\mathcal{B}$. Note that, for $g \in G_B$, we have $K_{(B)} = K_{(B^g)} = K_{(B)}^g$. This yields that $G_B \leq \mathbf{N}_G(K_{(B)})$. It follows that $\mathcal{B}_1$ is a block of $G^{\mathcal{B}}$, refer to [4, Theorem 1.5A].

Suppose that $G^{\mathcal{B}}$ is primitive. Then $\mathcal{B}_1 = \{B\}$ or $\mathcal{B}$. If $\mathcal{B}_1 = \{B\}$ then $K_{(C)} \neq K_{(B)}$ for $C \in \mathcal{B}$, and so $K_{(B)}$ acts nontrivially on C. Let $\mathcal{B}_1 = \mathcal{B}$. Then $K_{(B)}$ fixes V point-wise, yielding $K_{(B)} = 1$. Thus $K_{(B^g)} = K_{(B)}^g = 1$ for all $g \in G$. It follows that K is faithful on each $C \in \mathcal{B}$. □

Let $\Gamma = (V, A)$ is a digraph and $G \leq \mathsf{Aut}\Gamma$. Assume that G is transitive on V with a block B. Let $\mathcal{B} = \{B^g \mid g \in G\}$. Define a digraph $\Gamma_{\mathcal{B}}$ on $\mathcal{B}$ such that, for $B_1, B_2 \in \mathcal{B}$, the ordered pair (B_1, B_2) is an arc whenever there are $u_1 \in B_1$ and $u_2 \in B_2$ with $(u_1, u_2) \in A$. Then the digraph $\Gamma_{\mathcal{B}}$ is well-defined, which is called the *quotient digraph* of Γ over (or modulo) $\mathcal{B}$. It is easy to show that $G^{\mathcal{B}} \leq \mathsf{Aut}\Gamma_{\mathcal{B}}$, and thus $\Gamma_{\mathcal{B}}$ is vertex-transitive. For the case where Γ is self-complementary, by Lemma 2.2, we have the following lemma, see also [13, Theorem 1.2].

Lemma 4.3. *Let $\Gamma = (V, A)$ be a vertex-transitive self-complementary digraph. Let G be a transitive subgroup of $\mathsf{Aut}\Gamma$ such that $\mathbf{N}_{\mathrm{Sym}(V)}(G)$ contains a complementary map $\iota : \Gamma \to \overline{\Gamma}$, and set $X = G\langle\iota\rangle$. Let B be an X-block on V, and $\mathcal{B} = \{B^g \mid g \in \mathsf{Aut}\Gamma\}$. Then the subdigraph $[B]$ induced by B is a vertex-transitive self-complementary digraph, and $G^{\mathcal{B}}$*

is a transitive subgroup of the automorphism of some self-complementary digraph on $\mathcal{B}$ with a complementary map induced by ι. If further $\Gamma \cong \Gamma_{\mathcal{B}}[[B]]$ then $\Gamma_{\mathcal{B}}$ is self-complementary.

Proof. It suffices to show the last part of this lemma. Assume that $\Gamma \cong \Gamma_{\mathcal{B}}[[B]]$. Then $\overline{\Gamma} \cong \overline{\Gamma_{\mathcal{B}}}[\overline{[B]}]$. Thus yields that $\Gamma_{\mathcal{B}} \cong \overline{\Gamma}_{\mathcal{B}} \cong \overline{\Gamma_{\mathcal{B}}}$, and the lemma follows. □

The next lemma follows from [1, 2].

Lemma 4.4. *Let $\Gamma = (V, A)$ be a vertex-transitive self-complementary digraph of prime order p. Then there is an even order subgroup $\langle r\rangle$ of $\mathbb{Z}_p^*$, the multiplicative group of $\mathbb{Z}_p$, such that*

(1) *$\Gamma \cong \mathrm{Cay}(\mathbb{Z}_p, S)$, where S consists $\frac{p-1}{|\langle r\rangle|}$ cosets of $\langle r^2\rangle$ in $\mathbb{Z}_p^*$; and*
(2) *$\mathsf{Aut}(\mathrm{Cay}(\mathbb{Z}_p, S)) = \{\tau_{a,b} : \mathbb{Z}_p \to \mathbb{Z}_p, x \mapsto ax + b \mid a \in \langle r^2\rangle, b \in \mathbb{Z}_p\} \leq \mathrm{AGL}(1, p)$.*

Now we begin to give a characterization for self-complementary digraphs of order a product pq of two primes. We first deal with the case that $p = q$.

Theorem 4.5. *Let $\Gamma = (V, A)$ be a vertex-transitive self-complementary digraph. Assume that $|V| = p^2$ for some odd prime p. Then one of the following holds.*

(1) *$\Gamma \cong \Sigma[\Delta]$, where Σ and Δ are vertex-transitive self-complementary digraphs of order p;*
(2) *$\Gamma \cong \mathrm{Cay}(\mathbb{Z}_{p^2}, S)$, a Cayley digraph of the cyclic group $\mathbb{Z}_{p^2}$, and $\mathsf{Aut}(\mathrm{Cay}(\mathbb{Z}_{p^2}, S)) = \widehat{\mathbb{Z}_{p^2}}\mathsf{Aut}(\mathbb{Z}_{p^2}, S)$ with $\mathsf{Aut}(\mathbb{Z}_{p^2}, S) \cong \mathbb{Z}_d$, where d is such that $2d$ is a divisor of $p - 1$;*
(3) *$\mathsf{Aut}\Gamma$ has a normal Sylow p-subgroup isomorphic to $\mathbb{Z}_p^2$, and either $\mathsf{Aut}\Gamma$ is primitive on V, or $\langle \mathsf{Aut}\Gamma, \iota\rangle$ is isomorphic to a subgroup of $\mathrm{AGL}(1, p) \times \mathrm{AGL}(1, p)$, where ι is an arbitrary isomorphism from Γ to $\overline{\Gamma}$.*

Proof. By [17], we may let Γ be a Cayley digraph $\mathrm{Cay}(R, S)$, where R is the additive abelian group $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p^2$. Take a complementary map $\iota : \Gamma \to \overline{\Gamma}$, set $X = \mathsf{Aut}\Gamma\langle\iota\rangle$.

Let P be a Sylow p-subgroup of $\mathsf{Aut}\Gamma$ with $\widehat{R} \leq P$. Then P is also a Sylow p-subgroup of X. Since ι normalizes $\mathsf{Aut}\Gamma$, we know that P^ι is a Sylow p-subgroup of $\mathsf{Aut}\Gamma$, and so P and P^ι are conjugate in $\mathsf{Aut}\Gamma$. Set $P = P^{\iota g}$ for some $g \in \mathsf{Aut}\Gamma$. Clearly, $\iota' := \iota g$ is also an isomorphism from Γ to $\overline{\Gamma}$. Since p is odd, replacing ι' by its an odd power if necessary, we may let ι' has order coprime to p. Note that for $a \in R$, $a^{\iota'} = a$ if and only if $0^{\widehat{a}\iota'\widehat{a}^{-1}} = 0$. Then, replacing ι' by its a conjugation under $\widehat{R}$ if necessary, we may let $0^{\iota'} = 0$.

Let $Q = \mathbf{Z}(P)$ be the center of P. Then $Q \neq 1$. Since P is transitive on V, Q is semiregular on V, refer to [4, Theorem 4.2A]. Thus $|Q| = p$ or p^2.

Assume that $|Q| = p$. Then $|P|$ is divisible by p^3; otherwise, P shall be abelian, and so $P = Q$, a contradiction. Recall that Q is the center of P and ι' normalizes P. This implies that Q is normal in $Y := P\langle\iota'\rangle$, and so each Q-orbit on V is a block of Y and has size p. Let $\mathcal{B}$ be the set of the Q-orbits on V, and let K be the kernel of P acting on $\mathcal{B}$. Then $|\mathcal{B}| = p$, and so $P/K \cong P^{\mathcal{B}} \cong \mathbb{Z}_p$ as $P^{\mathcal{B}}$ is a p-subgroup of $\mathrm{Sym}(\mathcal{B})$. Thus K has order divisible by p^2. Since $|B| = p$ and $K^B \leq \mathrm{Sym}(B)$, we know that $K_{(B)}$ has order divisible by p. It follows from Lemma 4.2 that $K_{(B)}$ acts transitively on each $C \in \mathcal{B}$. This yields that if (B, C) is an arc of $\Gamma_{\mathcal{B}}$ then $\{(u_1, u_2) \mid u_1 \in B, u_2 \in C\} \subseteq A$. Then $\Gamma \cong \Gamma_{\mathcal{B}}[[B]]$ for $B \in \mathcal{B}$, and so part (1) of this theorem follows from Lemma 4.3.

Assume next that Q has order p^2. Then Q is a regular subgroup of P, and so $P = QP_u = Q \times P_u$ for $u \in V$. In particular, P_u is normal in P. Since P is transitive on V, we have $P_u = 1$, and thus $P = Q = \widehat{R}$.

Suppose first that $R = \mathbb{Z}_{p^2}$. Since Γ is self-complementary, by [9, Theorem 8.2], we have $\mathsf{Aut}\Gamma = \widehat{R}\mathsf{Aut}(R, S)$ with $\mathsf{Aut}(R, S) \cong \mathbb{Z}_d$, where d is a divisor of $p - 1$. In particular, $(\mathsf{Aut}\Gamma)_0 = \mathsf{Aut}(R, S)$. Recall that $\iota' : \Gamma \to \overline{\Gamma}$ is a complementary map, and $0^{\iota'} = 0$. It yields that ι' normalizes $(\mathsf{Aut}\Gamma)_0 = \mathsf{Aut}(R, S)$. On the other hand, since ι' normalizes $P = \widehat{R}$, we have $\iota' \in \mathsf{Aut}(R)$ by (3.1). Since ι' has order coprime to p, we know that $\langle\iota', \mathsf{Aut}(R, S)\rangle$ is a subgroup of $\mathsf{Aut}(R)$ and has order not divisible by p. Since $\iota'^2 \in \mathsf{Aut}\Gamma$, we get $\iota'^2 \in \mathsf{Aut}(R, S)$. Clearly, $\iota' \notin \mathsf{Aut}(R, S)$. Then $\mathsf{Aut}(R, S)$ has index 2 in $\langle\iota', \mathsf{Aut}(R, S)\rangle$. Noting that $\mathsf{Aut}(R) \cong \mathbb{Z}_{p(p-1)}$, it implies that $\langle\iota', \mathsf{Aut}(R, S)\rangle$ is isomorphic a subgroup of $\mathbb{Z}_{p-1}$. Then we have $\langle\iota', \mathsf{Aut}(R, S)\rangle \cong \mathbb{Z}_{2d}$, and thus part (2) of this theorem follows.

Suppose now that $R = \mathbb{Z}_p^2$. In this case, we shall prove that part (3) of this theorem occurs. Note that $P = \widehat{R}$ is a Sylow p-subgroup of $\mathsf{Aut}\Gamma$. It suffices to show that P is normal in $\mathsf{Aut}\Gamma$. If X is primitive on V then, by [5, Theorem 1.3] and [8], P is normal in X and hence in $\mathsf{Aut}\Gamma$, and $\mathsf{Aut}\Gamma$ is primitive on V. Thus we assume that X is imprimitive on V. By [8, Proposition B], $\mathrm{soc}(X) = M_1 \times M_2$ is transitive on V, where $\mathrm{soc}(X)$ is generated by all minimal normal subgroups of X, each M_i is a simple normal subgroup of X with p orbits of length p. Since $|X : \mathsf{Aut}\Gamma| = 2$, it implies that $\mathsf{Aut}\Gamma \trianglelefteq X$, and either $M_i \leq \mathsf{Aut}\Gamma$ or $X = M_i\mathsf{Aut}\Gamma$. If $X = M_i\mathsf{Aut}\Gamma$, then $|X| = |M_i||\mathsf{Aut}\Gamma|/|M_i \cap \mathsf{Aut}\Gamma|$, yielding $|M_i : (M_i \cap \mathsf{Aut}\Gamma)| = 2$, which is impossible as M_i is simple and has order divisible by the odd prime p. Therefore, $M_i \leq \mathsf{Aut}\Gamma$, $i = 1, 2$. Let C_i be an M_i-orbit on V. Consider the induced subdigraph $[C_i]$. By Lemma 4.3, $[C_i]$ is a vertex-transitive complementary digraph of order p. Then $\mathsf{Aut}[C_i]$ is soluble by Lemma 4.4. Since M_i is simple and induces a subgroup of $\mathsf{Aut}[C_i]$, we have

$M_i \cong \mathbb{Z}_p$. Thus $P = \widehat{R} = \mathrm{M}_1 \times M_2$. By [3, Lemma 1] and [8, Proposition B], we conclude that X is isomorphic a subgroup of $\widehat{\mathbb{Z}_p}\mathsf{Aut}(\mathbb{Z}_p)\times\widehat{\mathbb{Z}_p}\mathsf{Aut}(\mathbb{Z}_p)$, and then we get part (3) of this theorem. □

Finally, we consider the digraphs which have order a product of two distinct odd primes.

Theorem 4.6. *Let $\Gamma = (V, A)$ be a vertex-transitive self-complementary digraph. Assume that $|V|$ is a product of two distinct odd primes. Then either*

(1) *$\Gamma \cong \Sigma[\Delta]$, where Σ and Δ are vertex-transitive self-complementary digraphs of prime order; or*
(2) *Γ is isomorphic to a normal Cayley digraph of $\mathbb{Z}_{pq}$, where p and q are distinct odd primes.*

Proof. Take a complementary map $\iota : \Gamma \to \overline{\Gamma}$, and set $G = \mathsf{Aut}\Gamma$ and $X = \mathsf{Aut}\Gamma\langle\iota\rangle$. Then $|X : G| = 2$. By [5, Theorem 1.3], we conclude that X is imprimitive on V.

Let B be a nontrivial block of X on V, and set $\mathcal{B} = \{B^x \mid x \in \mathsf{Aut}\Gamma\}$. Then $|B|$ and $|\mathcal{B}|$ are primes with $|V| = |B||\mathcal{B}|$. Set $|B| = p$ and $|\mathcal{B}| = q$. Note that for each $g \in G$, we have a complementary map $g\iota : \Gamma \to \overline{\Gamma}$, and thus $g\iota$ fixes a unique point in V by Lemma 2.1. It implies that $g\iota$ fixes (setwise) a unique block (in $\mathcal{B}$). Recalling that $\iota^2 \in \mathsf{Aut}\Gamma = G$, we conclude that $K := X_{(\mathcal{B})} \leq G$.

By Lemma 4.3, $G^{\mathcal{B}}$ is a transitive subgroup of the automorphism group of some vertex-transitive self-complementary digraph of order q. Then, by Lemma 4.4, we have $G^{\mathcal{B}} \cong \mathbb{Z}_q{:}\mathbb{Z}_f$ with $2f$ a divisor of $q-1$. Since $|X^{\mathcal{B}} : G^{\mathcal{B}}| = |X : G| = 2$, we know that $X^{\mathcal{B}}$ is a soluble transitive permutation group of prime degree q, it follows that $X^{\mathcal{B}} \cong \mathbb{Z}_q{:}\mathbb{Z}_{2f} \lesssim \mathrm{AGL}(1, q)$.

Suppose that $K = 1$. Then $G \cong G^{\mathcal{B}}$ and $X \cong X^{\mathcal{B}} \cong \mathbb{Z}_q{:}\mathbb{Z}_{2f}$. Since G is transitive on V, we know that pq is a divisor of $|G|$, and so p is a divisor of f. Then X has a normal subgroup $R \cong \mathbb{Z}_q{:}\mathbb{Z}_p$, which is regular on V. Clearly, ι centralizes a Sylow p-subgroup P of X. Since $|RP| = |R||P|/|R \cap P|$ and $RP \leq X$, we know that $qp|P|/|R \cap P|$ is a divisor of $2qf$. It implies that $|R \cap P|$ is divisible p. By Lemma 2.1, let $u^\iota = u$ for some $u \in V$. Then $(u^x)^\iota = u^x$ for $x \in R \cap P$; in particular, ι has at leat p fixed-points in V, which is impossible. Therefore, $K \neq 1$.

Assume first that $K_{(B)} \neq 1$. Noting that $G^{\mathcal{B}}$ is primitive on $\mathcal{B}$ as $|\mathcal{B}|$ is a prime, by Lemma 4.2, $K_{(B)}$ acts transitively on each $C \in \mathcal{B}$. This yields that, for an arc (B, C) of $\Gamma_{\mathcal{B}}$, we have $\{(u_1, u_2) \mid u_1 \in B, u_2 \in C\} \subseteq A$. Thus $\Gamma \cong \Gamma_{\mathcal{B}}[[B]]$ for $B \in \mathcal{B}$, and so we get part (1) of this theorem from Lemma 4.3.

Now let $K_{(B)} = 1$. Then K is a subgroup of $\mathsf{Aut}[B]$. By Lemmas 4.3 and 4.4, we have $K \leq \mathsf{Aut}[B] \cong \mathbb{Z}_p{:}\mathbb{Z}_d \lesssim \mathrm{AGL}(1,p)$, where d is such that $p-1$ is divisible by $2d$. Let P be the unique Sylow p-subgroup of K. Then P is a characteristic subgroup of K, and so $P \trianglelefteq X$ as $K \trianglelefteq X$. Thus $X/\mathbf{C}_X(P) = \mathbf{N}_X(P)/\mathbf{C}_X(P) \lesssim \mathsf{Aut}(P) \cong \mathbb{Z}_{p-1}$. Then $\mathbf{C}_X(P)K/\mathbf{C}_X(P) \lesssim \mathbb{Z}_{p-1}$, and so

$$\begin{aligned}(X/K)/(\mathbf{C}_X(P)K/K) &\cong X/\mathbf{C}_X(P)K \\ &\cong (X/\mathbf{C}_X(P))/(\mathbf{C}_X(P)K/\mathbf{C}_X(P)) \lesssim \mathbb{Z}_{p-1}.\end{aligned}$$

Recall that $X/K \cong X^{\mathcal{B}} \cong \mathbb{Z}_q{:}\mathbb{Z}_{2f} \lesssim \mathrm{AGL}(1,q)$. It follows that $\mathbf{C}_X(P)K/K \cong \mathbb{Z}_q{:}\mathbb{Z}_e$ for a divisor e of $2f$; in particular, the Sylow q-subgroup of X/K is contained in $\mathbf{C}_X(P)K/K$, which yields that $\mathbf{C}_X(P)K$ and hence $\mathbf{C}_X(P)$ acts transitively on $\mathcal{B}$. Noting that $\mathbf{C}_K(P) = P$, we have

$$\mathbf{C}_X(P)/P = \mathbf{C}_X(P)/(\mathbf{C}_X(P) \cap K) \cong \mathbf{C}_X(P)K/K \cong \mathbb{Z}_q{:}\mathbb{Z}_e.$$

It implies that $\mathbf{C}_X(P)/P$ has a normal Sylow q-subgroup $(P \times Q)/P \cong \mathbb{Z}_q$, where Q is a Sylow q-subgroup of $\mathbf{C}_X(P)$. Since $\mathbf{C}_X(P)/P \trianglelefteq X/P$, we have $(P \times Q)/P \trianglelefteq X/P$, yielding $P \times Q \trianglelefteq X$. Moreover, it is easily shown that $P \times Q$ is transitive and hence regular on V, and then (2) of this theorem follows. □

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H. Y. Chen, Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, China
E-mail address: 1120140003@mail.nankai.edu.cn

Z. P. Lu, Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, China
E-mail address: lu@nankai.edu.cn