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GRAPHS WITH 4-RAINBOW INDEX 3 AND $N-1$

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Abstract

Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A tree T in G is called a *rainbow tree* if no two edges of T receive the same color. For a vertex set $S \subseteq V(G)$, a tree that connects S in G is called an *S-tree*. The minimum number of colors that are needed in an edge-coloring of G such that there is a rainbow S-tree for every set S of k vertices of $V(G)$ is called the *k-rainbow index* of G, denoted by $rx_k(G)$. Notice that a lower bound and an upper bound of the k-rainbow index of a graph with order n is $k-1$ and $n-1$, respectively. Chartrand et al. got that the k-rainbow index of a tree with order n is $n-1$ and the k-rainbow index of a unicyclic graph with order n is $n-1$ or $n-2$. Li and Sun raised the open problem of characterizing the graphs of order n with $rx_k(G) = n-1$ for $k \geq 3$. In early papers we characterized the graphs of order n with 3-rainbow index 2 and $n-1$. In this paper, we focus on $k = 4$, and characterize the graphs of order n with 4-rainbow index 3 and $n-1$, respectively.

Keywords: rainbow S-tree, k-rainbow index.

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1. INTRODUCTION

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *rainbow path* if any two edges of the path have distinct colors. G is *rainbow connected* if any two vertices of G are connected by a rainbow path. The minimum number of colors required to make G rainbow connected is called its *rainbow connection number*, denoted by $rc(G)$. Results on the rainbow connectivity can be found in [2, 3, 4, 5, 6, 10, 11].

These concepts were introduced by Chartrand et al. in [4]. In [7], they generalized the concept of rainbow path to rainbow tree. A tree T in G is called a *rainbow tree* if no two edges of T receive the same color. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree that connects S. Given a fixed integer k with $2 \leq k \leq n$, the edge-coloring c of G is called a *k-rainbow coloring* of G if, for every set S of k vertices of G, there exists a rainbow S-tree, and we say that G is *k-rainbow connected*. The *k-rainbow index* $rx_k(G)$ of G is the minimum number of colors that are needed in a *k-rainbow coloring* of G. Clearly, when $k = 2$, $rx_2(G)$ is nothing new but the rainbow connection number $rc(G)$ of G. For every connected graph G of order n, it is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$.

The *Steiner distance* $d_G(S)$ of a set S of vertices in G is the minimum size (number of edges) of a tree in G that connects S. Such a tree is called a *Steiner S-tree* or simply an *S-tree*. The *k-Steiner diameter* $sdiam_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G. Then there is a simple upper bound and lower bound for $rx_k(G)$.

Observation 1.1 [7]**.** *For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, we have $k-1 \leq sdiam_k(G) \leq rx_k(G) \leq n-1$.*

It is easy to get the following observations.

Observation 1.2 [7]**.** *Let G be a connected graph of order n containing two bridges e and f. For each integer k with $2 \leq k \leq n$, every k-rainbow coloring of G must assign distinct colors to e and f.*

Observation 1.3 [8]**.** *Let G be a connected graph of order n, and H be a connected spanning subgraph of G. Then $rx_k(G) \leq rx_k(H)$.*

The following is an immediate consequence of the observations above. Namely, trees attain the upper bound of k-rainbow index, regardless of the value of k.

Proposition 1.4 [7]**.** *Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$, $rx_k(T) = n-1$.*

In [7], they also showed that the k-rainbow index of a unicyclic graph is $n-1$ or $n-2$.

Theorem 1.5 [7]**.** *If G is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then*

$$rx_k(G) = \begin{cases} n-2, & k=3 \text{ and } g \geq 4; \\ n-1, & g=3 \text{ or } 4 \leq k \leq n. \end{cases} \tag{1}$$

Notice that a lower bound and an upper bound of the k-rainbow index of a graph with order n is $k-1$ and $n-1$, respectively. In [10], the authors raised an open problem: for $k \geq 3$, characterize the graphs of order n with $rx_k(G) = n-1$. It is not easy to settle down the problem for general k. In [8] and [12], we characterized the graphs of order n with 3-rainbow index 2 and $n-1$, respectively. In this paper we mainly deal with the 4-rainbow index of graphs with order n. More specifically, characterize the graphs of order n whose 4-rainbow index is 3 and $n-1$, respectively.

2. CHARACTERIZATION OF GRAPHS WITH $rx_4(G)$= 3

First we give a necessary and sufficient condition for $rx_4(G) = 3$. Note that if a connected graph of order 4 has three colors, then it has a rainbow spanning tree. Thus, the following lemma holds.

Lemma 2.1. *Let G be a connected graph of order n $(n \geq 4)$. Then $rx_4(G) = 3$ if and only if each induced subgraph of G with order 4 is connected and has three different colors.*

Next we give some necessary conditions for $rx_4(G) = 3$. By Lemma 2.1, it is easy to get the following proposition.

Proposition 2.2. *Let G be a graph of order n with $rx_4(G) = 3$, where $n \geq 5$. Then $\delta(G) \geq n-3$ and $\Delta(\overline{G}) \leq 2$. In other words, $\overline{G}$ is the union of some paths (may be trivial) and cycles.*

For fixed integers p, q, an edge-coloring of a complete graph K_n is called a (p,q)-*coloring* if the edges of every $K_p \subseteq K_n$ are colored with at least q distinct colors. Clearly, $(p,2)$-colorings are the classical Ramsey colorings without monochromatic K_p as subgraphs. Let $f(n,p,q)$ be the minimum number of colors needed for a (p,q)-coloring of K_n. In [9], Erdös and Gyárfás got that $f(10,4,3) = 4$, and so the following proposition holds.

Proposition 2.3. *Let G be a graph of order n with $rx_4(G) = 3$. Then $n \leq 9$.*

By Lemma 2.1 and Theorem 1.5, we get the following proposition.

Proposition 2.4. *Let G be a connected graph of order n ($n \geq 4$) with $rx_4(G) = 3$. Then $\overline{G}$ contains neither C_4 nor C_5.*

When G is a graph of order 4, it is obvious that $rx_4(G) = 3$ if and only if G is connected. Hence, for the remaining values of n with $5 \leq n \leq 9$ we distinguish five cases.

Lemma 2.5. *Let G be a connected graph of order 5. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of P_5 or $K_2 \cup K_3$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, it is easy to check that if $\overline{G}$ is not a subgraph of P_5 or $K_2 \cup K_3$, then $\overline{G}$ is isomorphic to C_4 or C_5, a contradiction by Proposition 2.4.

Conversely, by Observation 1.3, we need to provide an edge-coloring $C : E \to \{1,2,3\}$ of G when $\overline{G}$ is isomorphic to P_5 or $K_2 \cup K_3$. Suppose $\overline{G}$ is isomorphic to P_5, denote $V(\overline{G}) = \{v_1, \cdots, v_5\}$ and $E(\overline{G}) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$. Set $c(v_1v_3) = 2$, $c(v_1v_4) = 1$, $c(v_1v_5) = 3$, $c(v_2v_4) = 3$, $c(v_2v_5) = 2$, $c(v_3v_5) = 1$. Suppose $\overline{G}$ is isomorphic to $K_2 \cup K_3$, denote $V(\overline{G}) = \{v_1, \cdots, v_5\}$ and $E(\overline{G}) = \{v_1v_2, v_2v_3, v_1v_3, v_4v_5\}$. Set $c(v_1v_4) = 1$, $c(v_1v_5) = 2$, $c(v_2v_4) = 2$, $c(v_2v_5) = 3$, $c(v_3v_4) = 3$, $c(v_3v_5) = 1$. It is easy to show that the two edge-colorings make G 4-rainbow connected. ■

Lemma 2.6. *Let G be a graph of order 6. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of C_6 or $2K_3$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, if $\overline{G}$ is not a subgraph of C_6 or $2K_3$, then $\overline{G}$ contains C_4 or C_5, a contradiction by Proposition 2.4.

Conversely, by Observation 1.3, we need to provide an edge-coloring $C : E \to \{1,2,3\}$ of G when $\overline{G}$ is isomorphic to C_6 or $2K_3$. Suppose $\overline{G}$ is isomorphic to

C_6, denote $V(\overline{G}) = \{v_1, \cdots, v_6\}$ and $E(\overline{G}) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$. Set $c(v_1v_3) = 2$, $c(v_1v_4) = 3$, $c(v_1v_5) = 1$, $c(v_2v_4) = 1$, $c(v_2v_5) = 2$, $c(v_2v_6) = 3$, $c(v_3v_5) = 3$, $c(v_3v_6) = 1$, $c(v_4v_6) = 2$. Suppose $\overline{G}$ is isomorphic to $2K_3$, denote $V(\overline{G}) = \{v_1, \cdots, v_6\}$ and $E(\overline{G}) = \{v_1v_2, v_1v_3, v_2v_3, v_4v_5, v_4v_6, v_5v_6\}$. Set $c(v_1v_4) = 3$, $c(v_1v_5) = 2$, $c(v_1v_6) = 1$, $c(v_2v_4) = 1$, $c(v_2v_5) = 3$, $c(v_2v_6) = 2$, $c(v_3v_4) = 2$, $c(v_3v_5) = 1$, $c(v_3v_6) = 3$. It is easy to show that the two edge-colorings make G 4-rainbow connected. ∎

It is a tedious work to check whether a graph is 4-rainbow connected when $7 \le n \le 9$. Hence we introduce an algorithm with the idea of backtracking to deal with such cases. Given a graph $G = (V(G), E(G))$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, we color $E(G)$ with colors $\{1,2,3\}$ in a proper order: at the beginning, consider the edge of the subgraph induced by $\{v_1, v_2\}$, namely the edge v_1v_2, and color it with 1 initially. Once all edges of the subgraph induced by $\{v_1, v_2, \ldots, v_s\}$ are colored, we come to deal with the new edges of the larger subgraph by adding v_{s+1} to the former one. For a new edge e, we color it with 1, 2 or 3, and if the subgraph induced by the vertices incident with already colored edges is 4-rainbow connected, we go on to the next edge of e. Otherwise if all 1, 2 and 3 are not available, we go back to the former edge of e and give it a new color and repeat the procedure. Clearly, the procedure always terminates. We should point out that the algorithm has a good performance when $n \le 9$, although the time complexity is not polynomial. In fact, we need the algorithm only to test whether four graphs have 4-rainbow colorings in the following three lemmas.

Algorithm The 4-rainbow coloring of a graph

Input: a graph $G = (V, E)$ with $V = \{v_1, v_2, ..., v_n\}$, $E = \{e_1, e_2, ..., e_m\}$.
Output: give a 4-rainbow coloring $colorlist[m]$ of G, or verify that G has no 4-rainbow coloring.

1. reorder the edge sequence $e_1, e_2, ..., e_m$, to make sure $E(G[v_1, ..., v_t]) = \{e_1, ..., e_s\}$, where s denotes the number of edges of $G[v_1, ..., v_t]$, where $1 \le t \le n$.
2. fix the color of e_1 with 1. Initialize $i = 2$ and $colorlist = [1, 0, 0, ..., 0]$;
3. while $i \ge 2$
    - if $i > m$
        - show *colorlist*; stop;
        - $colorlist[i] = colorlist[i] + 1$;
    - if $colorlist[i] > 3$
        - $colorlist[i] = 0$; $i - -$;
    - else if **Boolean CHECK**(e_i)
        - $i + +$;
4. there is no 4-rainbow coloring; stop.

Boolean CHECK(e_s)
Input: a graph $G = (V, E)$ with $V = \{v_1, v_2, ..., v_n\}$, $E = \{e_1, e_2, ..., e_m\}$ with the order described above. Set $e_s = (v_p, v_q)$, where $p < q$. Give a coloring of the first s edges of $E(G)$.
Output: determine whether the given coloring is not 4-rainbow.
1. for $i = 1$ up to $q - 2$ and $i \neq p$
    for $j = i + 1$ up to $q - 1$ and $j \neq p$
        if all edges of the induced subgraph $G[v_i, v_j, v_p, v_q]$ are colored but $G[v_i, v_j, v_p, v_q]$ is not 4-rainbow colored.
            return $false$; stop;
2. return $true$; stop.

Lemma 2.7. *Let G be a graph of order 7. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, if $\overline{G}$ is not a subgraph of C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$, then by Proposition 2.4, $\overline{G}$ is isomorphic to $P_4 \cup P_3$ or $P_4 \cup K_3$ or P_7 or C_7. By Observation 1.3, we need only to verify that $rx_4(G) \neq 3$ when $\overline{G}$ is isomorphic to $P_4 \cup P_3$. By the algorithm, $rx_4(G) \neq 3$.

Conversely, by Observation 1.3 again, we need to provide an edge-coloring of G when $\overline{G}$ is isomorphic to C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$. The four colorings are shown in Figure 1. It is easy to show that these four colorings make G 4-rainbow connected. ∎

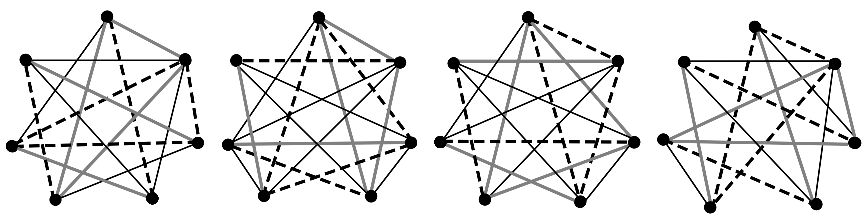

Figure 1. Graphs for Lemma 2.7 (lines of the same type have the same color).

Lemma 2.8. *Let G be a graph of order 8. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of $K_2 \cup 2K_3$ or $P_6 \cup K_2$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, if $\overline{G}$ is not a subgraph of $K_2 \cup 2K_3$ or $P_6 \cup K_2$, then by Proposition 2.4, it is easy to check that either $\overline{G}$ contains $P_4 \cup P_3 \cup K_1$ or $\overline{G}$ is isomorphic to $C_6 \cup 2K_1$. By Observation 1.3, we need to verify that $rx_4(G) \neq 3$ when $\overline{G}$ is isomorphic to $P_4 \cup P_3 \cup K_1$ or $\overline{G}$ is isomorphic to $C_6 \cup 2K_1$. If $\overline{G}$ is isomorphic to $P_4 \cup P_3 \cup K_1$, then by Lemma 2.7, $rx_4(G) \neq 3$. If $\overline{G}$ is isomorphic to $C_6 \cup 2K_1$, by the algorithm, $rx_4(G) \neq 3$.

Conversely, by Observation 1.3 again, we need to provide an edge-coloring of G when $\overline{G}$ is isomorphic to $K_2 \cup 2K_3$ or $P_6 \cup K_2$. The two edge-colorings are shown in the first two graphs of Figure 2. It is easy to show that the two edge-colorings make G 4-rainbow connected. ■

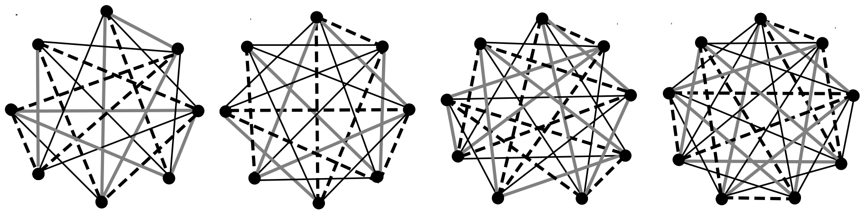

Figure 2. Graphs for Lemma 2.8, 2.9.

Lemma 2.9. *Let G be a graph of order 9. Then $rx_4(G) = 3$ if and only if $\overline{G}$ is a subgraph of $3K_3$ or $P_3 \cup 3K_2$.*

Proof. Let G be a graph with $rx_4(G) = 3$. By Proposition 2.2, if $\overline{G}$ is not a subgraph of $3K_3$ or $P_3 \cup 3K_2$, then by Proposition 2.4, it is easy to check that either $\overline{G}$ contains P_4 or $\overline{G}$ is isomorphic to $K_3 \cup 3K_2$. By Observation 1.3, we need to verify that $rx_4(G) \neq 3$ when $\overline{G}$ is isomorphic to P_4 or $K_3 \cup 3K_2$, by the algorithm, in each case, $rx_4(G) \neq 3$.

Conversely, by Observation 1.3 again, we need only to provide an edge-coloring of G when $\overline{G}$ is isomorphic to $3K_3$ or $P_3 \cup 3K_2$. The two edge-colorings are shown in the last two graphs of Figure 2. It is easy to show that the two edge-colorings make G 4-rainbow connected. ■

Combining the preceding five lemmas, we are ready to characterize the graphs whose 4-rainbow index is 3.

Theorem 2.10. *Let G be a connected graph of order $n \geq 4$. Then $rx_4(G) = 3$ if and only if G is one of the following graphs:* (1) *G is a connected graph of order 4;* (2) *G is of order 5 and $\overline{G}$ is a subgraph of P_5 or $K_2 \cup K_3$;* (3) *G is of order 6 and $\overline{G}$ is a subgraph of C_6 or $2K_3$;* (4) *G is of order 7 and $\overline{G}$ is a subgraph of C_6 or $2K_2 \cup K_3$ or $P_5 \cup K_2$ or $2K_3$;* (5) *G is of order 8 and $\overline{G}$ is a subgraph of $K_2 \cup 2K_3$ or $P_6 \cup K_2$;* (6) *G is of order 9 and $\overline{G}$ is a subgraph of $3K_3$ or $P_3 \cup 3K_2$.*

3. Characterization of graphs with $rx_4(G) = \mathbf{n-1}$

First of all, we need some notation and basic results.

Definition 3.1. Let G be a connected graph with n vertices and m edges. Define the *cyclomatic number* of G as $c(G) = m - n + 1$. A graph G with $c(G) = k$ is called a *k-cyclic* graph. According to this definition, if a graph G meets $c(G) = 0$, 1, 2 or 3, then G is called acyclic (or a tree), unicyclic, bicyclic, or tricyclic, respectively.

Definition 3.2. For a subgraph H of a connected graph G and $v \in V(G)$, let $d(v, H) = min\{d_G(v, x) : x \in V(H)\}$.

Let G be a connected graph. To *contract* an edge $e = uv$ is to delete e and replace its ends by a single vertex incident to all the edges which were incident to either u or v. Let $G^{'}$ be the graph obtained by contracting some edges of G and suppose that the resulting graph $G^{'}$ is a simple graph. Given a rainbow coloring of $G^{'}$, when it comes back to G, every modified edge takes the following operation: assign the color of uv to uw and a fresh color to the edge wv if an edge uv of $G^{'}$ is expanded into two edges uw, wv between the ends of the contracted edge. Then G can be made to be 4-rainbow connected if G' is 4-rainbow connected. Hence, the following lemma holds.

Lemma 3.3. *Let G be a connected graph, and $G^{'}$ be a connected graph by contracting some edges of G. Then $rx_4(G) \leq rx_4(G^{'}) + |V(G)| - |V(G^{'})|$.*

The Θ-*graph* is a graph consisting of three internally disjoint paths with common end vertices and of lengths a, b, and c, respectively, such that $a \leq b \leq c$. It follows that if a Θ-graph has order n, then $a + b + c = n + 1$.

Let G be a connected graph of order n, to *subdivide* an edge e is to delete e, add a new vertex x, and join x to the ends of e. We will first give some sufficient conditions to make sure that the 4-rainbow index of G never attains the upper bound $n - 1$.

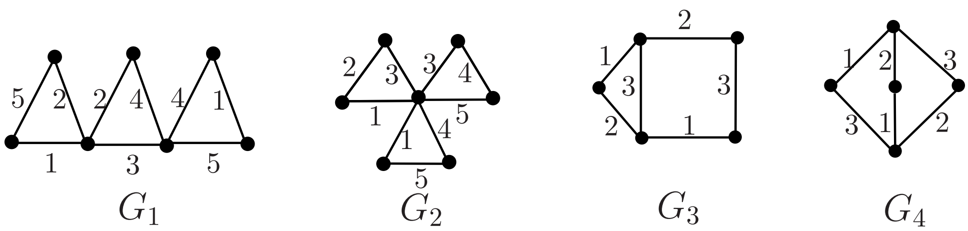

Figure 3. Graphs for Lemma 3.4.

Lemma 3.4. *Let G be a connected graph of order n. If G contains three edge-disjoint cycles, or a Θ-graph of order at least 5 as subgraphs, then $rx_4(G) \leq n-2$.*

Proof. Consider two graphs G_1, G_2 in Figure 3, and by checking the given edge-coloring in the figure, we have $rx_4(G_i) \leq |V(G_i)|-2$, $i = 1, 2$. Then if G contains three edge-disjoint cycles C_1, C_2, C_3, we can extend the three triangles of G_1 or G_2 to C_1, C_2 and C_3 respectively by a sequence of operations of subdivision. Then add to the cycles an additional set of edges, to get a spanning subgraph G' of G. By Observation 1.3 and Lemma 3.3, we have $rx_4(G) \leq rx_4(G') \leq rx_4(G_i) + |V(G')| - |V(G_i)| \leq n-2$.

Let $\mathcal{G}$ be the set of Θ-graphs whose order is exactly 5. Then $\mathcal{G} = \{G_3, G_4\}$ (see Figure 3). By checking the given edge-coloring, we have $rx_4(G_i) \leq |V(G_i)| - 2$, $i = 3, 4$. Similarly, $rx_4(G) \leq n-2$ follows. ■

A graph G is a *cactus* if every edge is part of at most one cycle in G.

Lemma 3.5. *Let G be a cactus of order n and $c(G) = 2$. Then $rx_4(G) = n-1$.*

Proof. Let the two cycles of G be C^1 and C^2, where $C^1 = v_1v_2\cdots v_\ell v_1$, $C^2 = v'_1v'_2\cdots v'_{\ell'}v'_1$, the unique path connecting the two cycles be $v_iPv'_j$, where the two end-vertices v_i and v'_j may coincide. Suppose we have a color set C and $|C| = n-2$. Set $C = \{1, 2, \cdots, n-2\}$ and E_i is the set of edges colored with i, $c_i = |E_i|$, $1 \leq i \leq n-2$. Without loss of generality, we always set $c_1 \geq c_2 \geq \cdots \geq c_{n-2}$. Notice that $\sum_{i=1}^{n-2} c_i = n+1$. We distinguish the following cases.

Case 1. $c_1 = 4$, $c_2 = c_3 = \cdots = c_{n-2} = 1$. We have the following claim.

Claim 1. No three edges of C^1 or C^2 have the same color.

Proof. Suppose $c(v_1v_2) = c(v_pv_{p+1}) = c(v_qv_{q+1})$, where v_1v_2, v_pv_{p+1}, v_qv_{q+1} are three distinct edges. Let $S = \{v_1, v_p, v_q\}$. It is easy to check that any tree connecting S contains at least two edges of v_1v_2, v_pv_{p+1} and v_qv_{q+1}, this contradiction proves the claim.

By Observation 1.2 and Claim 1, at least 3 edges of E_1 exist on cycles and each cycle has at most two of them. Suppose v_1v_2 and v_pv_{p+1} of C^1 have color 1, we distinguish two subcases: (1) there is a cut edge uu' in E_1. Suppose $d(u, C^1) \geq d(u', C^1)$ and $d(u, v_i) = d(u, C^1)$, where $2 \leq i \leq p$. Any tree connecting v_1 and u contains at least two edges colored with 1. (2) no cut edge has color 1. Then at least two edges, say $v'_1v'_2$ and $v'_qv'_{q+1}$ of C^2 have color 1, and the end-vertices of the path connecting C^1 and C^2 are v_i and v'_j, where $2 \leq i \leq p$, $2 \leq j \leq q$. Again, any tree connecting v_1 and v'_1 contains at least two edges in E_1.

Case 2. $c_1 = 3, c_2 = 2, c_3 = \cdots = c_{n-2} = 1$. We also have the following claim.

Claim 2. No four edges of a cycle can have only two colors.

Proof. Suppose otherwise four edges, v_1v_2, v_pv_{p+1}, v_qv_{q+1}, v_rv_{r+1} of C^1 have color a or b, where $a, b \in C$. Set $S = \{v_1, v_p, v_q, v_r\}$. It is easy to check that any tree connecting S contains at least three of the four edges above. By the Pigeon Hole Principle, one of the two colors occurs at least twice, a contradiction.

By Claim 2, at most three edges of $C^i(i = 1, 2)$ can have colors 1 and 2. Notice that $|E_1 \cup E_2| = 5$. Since no two cut edges can have the same color, there are the following possibilities: (1) three edges of $E_1 \cup E_2$ are in a cycle, say C^1. Then there exist cut edges in $E_1 \cup E_2$, or the other two edges of $E_1 \cup E_2$ are both in C^2. Similar to Case 1, we can choose three vertices such that no rainbow tree connects them. (2) two edges of $E_1 \cup E_2$ are in each cycle. Then a cut edge uu' exists in $E_1 \cup E_2$. There are two situations according to the positions of uu' and the other four edges of $E_1 \cup E_2$ in cycles. We can always find three vertices such that any tree connecting them contains at least three edges of $E_1 \cup E_2$. (3) two edges of $E_1 \cup E_2$ are in one cycle, and other two of them are cut edges. The argument is similar, and it also produces a contradiction.

Case 3. $c_1 = c_2 = c_3 = 2, c_4 = \cdots = c_{n-2} = 1$. In a number of subcases similar to those in Cases 1 and 2, a set S of vertices can be found such that a tree connecting them contains at least four edges from $E_1 \cup E_2 \cup E_3$. So by the Pigeon Hole Principle again, one of the three colors occurs at least twice.

By the analysis above, all the possibilities of an $(n-2)$-coloring lead to a contradiction, thus we have $rx_4(G) \geq n - 1$. On the other hand, by Observation 1.1, it follows that $rx_4(G) = n - 1$. ∎

To characterize all the graphs with 4-rainbow index $n - 1$, we need to introduce more graphs. Let $\mathcal{G}_1$ be the set of graphs by identifying each vertex of K_4 with an end-vertex of an arbitrary path, and $\mathcal{G}_2$ be the set of graphs by identifying each vertex of $K_4 - e$ with the root of an arbitrary tree.

Lemma 3.6. *Let G be a connected graph of order n. If $G \in \mathcal{G}_1 \cup \mathcal{G}_2$, then $rx_4(G) = n - 1$.*

Proof. Suppose $G \in \mathcal{G}_1$, and v_1, v_2, v_3 and v_4 are the four pendant vertices of G. We have $d_G(v_1, v_2, v_3, v_4) = n - 1$. Combining with Observation 1.1, we have $rx_4(G) = n - 1$. Let $G \in \mathcal{G}_2$. Denote by H the induced subgraph $K_4 - e$ of G, where $E(H) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_2v_4\}$ and denote by T_i the tree rooted at v_i, $i = 1, 2, 3, 4$. We have the following claim.

Claim 3. No three edges of H share colors with the cut edges.

Proof. Let $v_i'v_i''$, $1 \leq i \leq 3$, be the cut edges whose colors exist in H. We may assume that $d(v_i', H) \geq d(v_i'', H)$. Notice that the deletion of any three edges of H disconnects G, and we will get some components. Let v be an arbitrary vertex of H in the component different from the one containing v_1'. Set $S = \{v, v_1', v_2', v_3'\}$. There is no rainbow tree connecting S, which verifies Claim 3.

Now we are aiming to prove that H needs at least three fresh colors different from the $n-4$ colors of cut edges to make sure that G is 4-rainbow connected. Then we get the conclusion $rx_4(G) = n-1$. Since $rx_4(H) = 3$ and by Claim 3, one or two edges of H have the color of cut edges. Assume first that the colors of cut edges $v_1'v_1''$, $v_2'v_2''$ appear in H. Suppose $d(v_i', H) \geq d(v_i'', H)$, $i = 1, 2$. Since the deletion of two edges incident to a vertex of degree two disconnects H, the position of the two edges of H having the colors of cut edges may have the following possibilities: v_1v_4, v_2v_4 or v_1v_4, v_3v_4 or v_1v_2, v_3v_4. Notice that the remaining three edges can only have fresh colors. If only two colors are used, then at least two edges of H have the same color. It is easy to find two vertices v_i, v_j of H, such that no rainbow tree connects S, where $S = \{v_1', v_2', v_i, v_j\}$. Assume then only one edge of H has the color of cut edge, say $v_1'v_1''$ of T_i. Suppose $d(v_1', H) \geq d(v_1'', H)$. Then any tree connecting v_1' and the three vertices of H except v_i makes use of at least three edges of H, namely at least three new distinct colors are needed in H. Thus the result follows. ■

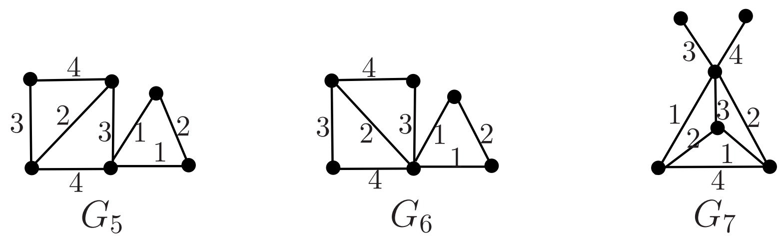

Figure 4. Graphs for Theorem 3.7.

Now we are prepared to characterize the graphs of order n whose 4-rainbow index is $n-1$.

Theorem 3.7. *Let G be a graph of order n. Then $rx_4(G) = n-1$ if and only if G is a tree, or a unicyclic graph, or a cactus with $c(G) = 2$, or $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.*

Proof. By Lemma 3.3, 3.4, 3.5, 3.6, we only need to prove the necessity. Let G be a graph with $rx_4(G) = n-1$. By Proposition 1.4, Theorem 1.5, Lemma 3.4 and Lemma 3.5, we know that if G is not a tree or a unicyclic graph or a cactus with $c(G) = 2$, then G contains a K_4 or $K_4 - e$ as an induced subgraph. Now suppose that G contains a K_4 or $K_4 - e$ but $G \notin \mathcal{G}_1 \cup \mathcal{G}_2$. Consider the three graphs G_5, G_6, G_7. By checking the given coloring in Figure 4, we have $rx_4(G_i) \leq n-2$, $i = 5, 6, 7$. Thus we can extend G_5, G_6 or G_7 to get a spanning subgraph G' of G, then $rx_4(G) \leq rx_4(G') \leq n-2$, a contradiction. ■

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