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Graphs with vertex rainbow connection number two

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Abstract An edge colored graph G is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of a graph G, denoted by $\mathrm{rc}(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. A vertex colored graph G is vertex rainbow connected if any two vertices are connected by a path whose internal vertices have distinct colors. The vertex rainbow connection number of G, denoted by $\mathrm{rvc}(G)$, is the smallest number of colors that are needed in order to make G vertex rainbow connected. In 2011, Kemnitz and Schiermeyer considered graphs with $\mathrm{rc}(G) = 2$. We investigate graphs with $\mathrm{rvc}(G) = 2$. First, we prove that $\mathrm{rvc}(G) \leqslant 2$ if $|E(G)| \geqslant \binom{n-2}{2} + 2$, and the bound is sharp. Denote by $s(n, 2)$ the minimum number such that, for each graph G of order n, we have $\mathrm{rvc}(G) \leqslant 2$ provided $|E(G)| \geqslant s(n, 2)$. It is proved that $s(n, 2) = \binom{n-2}{2} + 2$. Next, we characterize the vertex rainbow connection numbers of graphs G with $|V(G)| = n, \mathrm{diam}(G) \geqslant 3$ and clique number $\omega(G) = n - s$ for $1 \leqslant s \leqslant 4$.

Keywords vertex-coloring, vertex rainbow connection number, clique number

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We refer to [1] for the graph-theoretic terms not described here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of edges $|E(G)|$ is called the *size* of the graph G. A *k-edge-coloring* of a graph G is a mapping $c : E(G) \to \{1, 2, \ldots, k\}, k \in \mathbb{N}$, in other words, an assignment of k colors to the edges of G. Let G be a nontrivial connected graph with an edge-coloring. The graph G is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. The *rainbow connection number* [2] of a graph G, denoted by $\mathrm{rc}(G)$, is the smallest number of colors that are needed in order to make G rainbow connected.

Similar to the concept of rainbow connection number, Krivelevich and Yuster [6] proposed the concept of vertex rainbow connection number. A *k-vertex-coloring* of a graph G is a mapping $c : V(G) \to \{1, 2, \ldots, k\}, k \in \mathbb{N}$, that is, an assignment of k colors to the vertices of G. Let G be a nontrivial connected graph with a vertex-coloring. A path P of G is called a *rainbow path* if its internal vertices have distinct colors. The graph G is *vertex rainbow connected* if any pair of distinct vertices is connected by a rainbow path. A vertex-coloring under which G is vertex rainbow connected is called a *rainbow coloring. The vertex rainbow connection number* of a connected graph G which is not a complete graph, denoted by $\mathrm{rvc}(G)$, is the smallest number of colors that are needed in order to make G vertex rainbow

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connected. Denote $\mathrm{rvc}(G) = 0$ if G is a complete graph. Let $\mathrm{diam}(G)$ denote the diameter of a connected graph G. Then $\mathrm{rvc}(G) \geqslant \mathrm{diam}(G) - 1$ with equality if the diameter is 1 or 2. An easy observation is that if G is a connected graph of order n then $\mathrm{rvc}(G) \leqslant n-2$. It was shown in [8] that $\mathrm{rvc}(G) = n-2$ if and only if G is a path of order n.

For the rainbow connection number and the vertex rainbow connection number, some examples were given to show that there is no upper bound for one of the parameters in terms of the other in [6]. Hence, the investigation of vertex rainbow connection number should be an interesting subject. Krivelevich and Yuster [6] proved that if G is a connected graph with n vertices and minimum degree $\delta(G)$, then $\mathrm{rvc}(G) < 11n/\delta(G)$. The computational complexity of vertex rainbow connection was studied in [4]. It was proved that the computation of $\mathrm{rvc}(G)$ is NP-hard. In fact it is already NP-complete to decide if $\mathrm{rvc}(G) = 2$. Chen et al. [3] investigated the Nordhaus-Gaddum-type result for the vertex rainbow connection number. Rainbow connection and vertex rainbow connection attract many attentions of the researchers. The reader can see [9] for a survey on this topic.

In this paper, we propose the following problem.

Problem 1.1. *Let G be a nontrivial connected graph of order n. For every integer $k, 0 \leqslant k \leqslant n-2$, compute and minimize the function $s(n,k)$ with the following property: If $|E(G)| \geqslant s(n,k)$, then* $\mathrm{rvc}(G) \leqslant k$.

The following theorem and Lemma 2.1 say that $s(n,2) = \binom{n-2}{2} + 2$.

Theorem 1.2. *Let G be a connected graph of order $n \geqslant 4$. If $|E(G)| \geqslant \binom{n-2}{2} + 2$, then* $\mathrm{rvc}(G) \leqslant 2$.

2 Proof of Theorem 1.2

The set of neighbours of a vertex u in a graph G is denoted by $N(u)$. Denote $|N(u)| = d(u)$ and by $\delta(G)$ the minimum degree of the vertices of G. A path or complete graph of order n is denoted by P_n or K_n. We first state a useful lemma which shows a lower bound for $s(n,k)$.

Lemma 2.1. $s(n,k) \geqslant \binom{n-k}{2} + k$.

Proof. It suffices to show that there exists a graph G such that $|E(G)| = \binom{n-k}{2} + k - 1$ and $\mathrm{rvc}(G) \geqslant k+1$. We construct a graph G as follows: Take a $K_{n-k} - e$ and denote the two vertices of degree $n-k-2$ with v_1 and v_2. Now take a path P_{k+1} with vertices labeled $u_1, u_2, \ldots, u_k, u_{k+1}$ and identify the vertices v_2 and u_1. The resulting graph G has order n and size $|E(G)| = \binom{n-k}{2} + k - 1$. It is easy to see that $\mathrm{diam}(G) = k+2$. Thus $\mathrm{rvc}(G) \geqslant \mathrm{diam}(G) - 1 = k+1$. □

An easy observation is that $s(n,0) = \binom{n}{2}$. By Lemma 2.1, we have the following simple result.

Lemma 2.2. (i) $s(n,1) = \binom{n-1}{2} + 1$; (ii) $s(n,n-2) = n-1$; (iii) $s(n,n-3) = n$.

Proof. (i) We claim that if $|E(G)| \geqslant \binom{n-1}{2} + 1$, then $\mathrm{diam}(G) \leqslant 2$. Suppose $\mathrm{diam}(G) = t \geqslant 3$. Consider a diameter path $P = u_1u_2u_3u_4 \cdots u_tu_{t+1}$. We must have $d(u_1) + d(u_4) \leqslant n-2$. Thus

$$|E(G)| \leqslant \binom{n}{2} - (2n - 3 - (n-2)) = \binom{n-1}{2},$$

a contradiction. Hence, $\mathrm{diam}(G) \leqslant 2$. Then $\mathrm{rvc}(G) \leqslant 1$. By Lemma 2.1, we have $s(n,1) = \binom{n-1}{2} + 1$.

(ii) If G is a nontrivial connected graph with $|E(G)| \geqslant n-1$, then $\mathrm{rvc}(G) \leqslant n-2$. Therefore, $s(n,n-2) = n-1$.

(iii) Let G be a nontrivial connected graph with $|E(G)| \geqslant n$. It was shown in [8] that if G is not a path, then $\mathrm{rvc}(G) \leqslant n-3$. Thus $s(n,n-3) = n$ by Lemma 2.1. □

With the above lemmas established, we are ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2. We shall prove by induction on n. For $n = 4$ or 5, the result follows by Lemma 2.2. Now assume $n \geqslant 6$. If $|E(G)| \geqslant \binom{n-1}{2} + 1$, then $\mathrm{rvc}(G) \leqslant 1$ by Lemma 2.2(i). Thus we may further assume $|E(G)| \leqslant \binom{n-1}{2}$. Then we must have $\delta(G) \leqslant \frac{(n-1)(n-2)}{n} < n-2$.

Case 1. $\delta(G) = 1$. Let u be a vertex satisfying $\delta(G) = d(u) = 1$, and set $H = G - u$. We have

$$|E(H)| \geqslant \binom{n-2}{2} + 2 - 1 = \binom{(n-1)-1}{2} + 1.$$

Thus $\mathrm{rvc}(H) \leqslant 1$ by Lemma 2.2(i). Color v with 2, where v is incident with u, and assign 1 to the remaining vertices. We obtain that G is vertex rainbow connected. Hence $\mathrm{rvc}(G) \leqslant 2$.

Case 2. $\delta(G) \geqslant 2$. Suppose there exist two vertices $w_1, w_2 \in V(G)$ with $w_1w_2 \notin E(G)$ satisfying $N(w_1) \cap N(w_2) = \emptyset$, and let $H = G - \{w_1, w_2\}$. Then $|E(H)| \geqslant \binom{n-2}{2} + 2 - (n-2) = \binom{(n-2)-1}{2} + 1$. Obviously, H is connected. Hence $\mathrm{rvc}(H) \leqslant 1$ by Lemma 2.2(i). We note the result in [7] that if $|E(G)| \geqslant \binom{n-2}{2} + 2$, then $\mathrm{diam}(G) \leqslant \mathrm{rc}(G) \leqslant 3$. Hence, there exists a $w_1 - w_2$ path $w_1u_1u_2w_2$. Color u_2, u_3 with 2, where u_3 is incident with w_1 and assign 1 to the remaining vertices. For any vertex $v \in V(H)$ with $u_2v \notin E(G)$, there must exist a path u_2uv, where $u \in V(H)$ and $u \neq u_3$, otherwise, it contradicts that $|E(H)| \geqslant \binom{(n-2)-1}{2} + 1$. It is easy to check that G is vertex rainbow connected. Hence $\mathrm{rvc}(G) \leqslant 2$. Now for any $w_1, w_2 \in V(G)$ with $w_1w_2 \notin E(G)$, we may assume $N(w_1) \cap N(w_2) \neq \emptyset$. Thus $\mathrm{diam}(G) = 2$, and so $\mathrm{rvc}(G) = 1$.

This completes the proof of the theorem. □

3 Vertex rainbow connection number and clique number

Kemnitz and Schiermeyer [5] considered graphs with $\mathrm{rc}(G) = 2$ and large clique number. In this section, we investigate graphs with $\mathrm{rvc}(G) = 2$ and large clique number.

Theorem 3.1. *Let G be a connected graph of order n and clique number $\omega(G)$. If $\omega(G) = n + 1 - i$ for $i \in \{1, 2\}$, then* $\mathrm{rvc}(G) = i - 1$. *If $\omega(G) = n - 2$, then $1 \leqslant \mathrm{rvc}(G) \leqslant 2$.*

Proof. If $i = 1$, then G is a complete graph which implies $\mathrm{rvc}(G) = 0$. If $i = 2$, then $\omega(G) = n - 1$. Thus $\binom{n-1}{2} + 1 \leqslant |E(G)| \leqslant \binom{n}{2} - 1$. It is easy to see that $\mathrm{rvc}(G) = 1$ by Lemma 2.2. If $\omega(G) = n - 2$, then $\binom{n-2}{2} + 2 \leqslant |E(G)| < \binom{n}{2} - 1$. Applying Theorem 1.2, we obtain $1 \leqslant \mathrm{rvc}(G) \leqslant 2$. □

Let $X \subseteq V(G)$, we say that $[X]$ is a subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. Recall that a *clique* of a graph is a set of mutually adjacent vertices, and that a *maximum clique* is a clique of the largest possible size in a given graph. *The clique number* $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. Let G be a connected graph and X be a maximum clique of G. We say that $N_X(u)$ is the set of neighbors of u in $[X]$ and $d_X(u) = |N_X(u)|$.

Theorem 3.2. *Let G be a connected graph of order n and* $\mathrm{diam}(G) \geqslant 3$. *If $\omega(G) = n - 3$ and X is a maximum clique of G, then either* $\mathrm{rvc}(G) = 2$, *or* $\mathrm{rvc}(G) = 3$ *and one of the following holds. Let $V(G) \setminus X = \{v_1, v_2, v_3\}$.*

(i) $[V(G) \setminus X] \cong P_3$ *and $d_X(v_2) = d_X(v_3) = 0$, where $v_1v_2, v_2v_3 \in E(G)$.*

(ii) $[V(G) \setminus X] \cong K_2 + K_1, d_X(v_2) = 0, d_X(v_3) \geqslant 1$ *and $N_X(v_1) \cap N_X(v_3) = \emptyset$, where $v_1v_2 \in E(G)$.*

(iii) $[V(G) \setminus X] \cong 3K_1$, *$d_X(v) = 1$ and $N_X(u) \cap N_X(v) = \emptyset$, where u and v are any two distinct vertices in $V(G) \setminus X$.*

Proof. Let $F = [V(G) \setminus X]$ and c be a vertex-coloring of G.

Case 1. $\mathrm{diam}(G) = 3$. Hence $\mathrm{rvc}(G) \geqslant \mathrm{diam}(G) - 1 = 2$. We prove Case 1 by analysing the structure of F.

Subcase 1.1. Suppose F is connected. Then $F \cong K_3$ or $F \cong P_3$. Let $V(F) = \{v_1, v_2, v_3\}$. We define the vertex-coloring c as follows: $c(v) = 2$ for $v \in \bigcup_{1 \leqslant i \leqslant 3} N_X(v_i)$, and color all other vertices with 1. It is easy to verify that G is vertex rainbow connected with colors 1 and 2.

Subcase 1.2. $F \cong K_2 + K_1$. Let $V(K_2) = \{v_1, v_2\}$, $V(K_1) = \{v_3\}$. Choose $c(v) = 2$ for any $v \in N_X(v_1) \cup N_X(v_2)$ and for the remaining vertices with 1. Assume $N_X(v_1) \cap N_X(v_3) \neq \emptyset$ or $N_X(v_2) \cap N_X(v_3) \neq \emptyset$. Without of loss generality, we may assume $N_X(v_2) \cap N_X(v_3) \neq \emptyset$ with $u \in N_X(v_2) \cap N_X(v_3)$.

Since $[X]$ is a complete graph, for any vertex $v \in X\backslash\{u\}$, there exists a rainbow path v_1v_2uv connecting v_1 and v. Similarly, there exists a rainbow path $v_1v_2uv_3$. For any $v \in X$, either v_3v is contained in $E(G)$, or there exists a rainbow path v_3uv. Thus G is vertex rainbow connected. Now we assume $N_X(v_1) \cap N_X(v_3) = \emptyset$ and $N_X(v_2) \cap N_X(v_3) = \emptyset$. Obviously, we obtain two rainbow paths $v_1u_1u_3v_3$ and $v_2u_2u_3v_3$, where $u_i \in N_X(v_i)$ ($u_1 = u_2$ is possible). For any $v \in X$, either v_iv is contained in $E(G)$, or there exists a rainbow path v_iu_iv, where $1 \leqslant i \leqslant 3$. This is a vertex-coloring of G with $rvc(G) = 2$.

Subcase 1.3. $F \cong 3K_1$. Let $V(F) = \{v_1, v_2, v_3\}$. For any two vertices $v_i, v_j \in V(F)$, without loss of generality, let v_1 and v_2 satisfy $N_X(v_1) \cap N_X(v_2) \neq \emptyset$ with $u \in N_X(v_1) \cap N_X(v_2)$. The following coloring c with colors 1 and 2 induces a vertex-coloring of G with $\mathrm{rvc}(G) = 2 : c(u) = 2$ and $c(v) = 1$, where $v \in V(G) \setminus \{u\}$.

Now we assume that for any two vertices in F there is no common neighbour in $[X]$. Let $d_X(v_1) = d_X(v_2) = d_X(v_3) = 1$. Assume that $\mathrm{rvc}(G) = 2$ and $N_X(v_i) = \{u_i\}$. This implies $c(u_1) \neq c(u_2) \neq c(u_3)$, which is not possible. Therefore, $\mathrm{rvc}(G) \geqslant 3$. Assign 2 to u_2, 3 to u_3 and 1 to all other vertices. This is a vertex-coloring of G with $\mathrm{rvc}(G) = 3$. We may assume $|N_X(v_1) \cup N_X(v_2) \cup N_X(v_3)| \geqslant 4$. Without loss of generality, let $|N_X(v_1)| \geqslant 2$, say $\{u_1, u_1'\} \subseteq N_X(v_1)$, $u_2 \in N_X(v_2)$ and $u_3 \in N_X(v_3)$. Color u_1, u_2 with 2 and the remaining vertices with 1. For any $v \in X$, either v_iv is contained in $E(G)$, or there exists a rainbow path v_iu_iv, where $1 \leqslant i \leqslant 3$. It is easy to see that there also exist three rainbow paths $v_1u_1'u_2v_2$, $v_1u_1u_3v_3$ and $v_2u_2u_3v_3$. Then $\mathrm{rvc}(G) = 2$.

Case 2. $\mathrm{diam}(G) \geqslant 4$. We must have $F \cong P_3$ or $F \cong K_2 + K_1$. Let $V(F) = \{v_1, v_2, v_3\}$. Assume $F \cong P_3$. Since $\mathrm{diam}(G) \geqslant 4$, without loss of generality, we obtain $d_X(v_1) \geqslant 1$ and $d_X(v_2) = d_X(v_3) = 0$, where $v_1v_2, v_2v_3 \in E(G)$. Let $u_1 \in N_X(v_1)$. We assign colors to G as follows: $c(v_1) = 2, c(u_1) = 3$ and color all other vertices with 1. For any vertex $v \in X \setminus \{u_1\}$, there exists a rainbow path $v_3v_2v_1u_1v$.

Assume $F \cong K_2 + K_1$. Without loss of generality, we obtain $d_X(v_1) \geqslant 1, d_X(v_2) = 0, d_X(v_3) \geqslant 1$ and $N_X(v_1) \cap N_X(v_3) = \emptyset$, where $v_1v_2 \in E(G)$. Let $u_1 \in N_X(v_1)$ and $u_3 \in N_X(v_3)$. We assign colors to G as follows: $c(u_1) = 2, c(u_3) = 3$ and color the remaining vertices with 1. Obviously, there exists a rainbow path $v_2v_1u_1u_3v_3$.

Therefore, G is vertex rainbow connected with colors $1, 2$ and 3. Since $\mathrm{diam}(G) \geqslant 4$, then $\mathrm{rvc}(G) = 3$. □

Theorem 3.3. *Let G be a connected graph of order n, $\mathrm{diam}(G) = 3$ and clique number $\omega(G) = n - 4$. Let X be a maximum clique of G and $V(G) \setminus X = \{v_1, v_2, v_3, v_4\}$. Then either $\mathrm{rvc}(G) = 2$, or one of the following holds*:

(i) $[V(G) \setminus X] \cong P_4$, $d_X(v_2) = d_X(v_3) = 1, N_X(v_2) = N_X(v_3)$ *and* $N_X(v_1) = N_X(v_4) = \emptyset$.

(ii) $[V(G) \setminus X] \cong K_2 + 2K_1$, $d_X(v_1) = 0, d_X(v_3) = d_X(v_4) = 1, N_X(v_3) \neq N_X(v_4)$ *and* $N_X(v_3) \cup N_X(v_4) \subseteq N_X(v_2)$, *where* $v_1v_2 \in E(G)$.

(iii) $[V(G) \setminus X] \cong K_2 + 2K_1$, $d_X(v_2) = d_X(v_3) = d_X(v_4) = 1, N_X(v_1) \cap N_X(v_2) \neq \emptyset$ *and for any two vertices in* $\{v_2, v_3, v_4\}$, *there is no common neighbour in* $[X]$, *where* $v_1v_2 \in E(G)$.

(iv) $[V(G) \setminus X] \cong K_2 + 2K_1$, $d_X(v_2) = d_X(v_3) = d_X(v_4) = 1$ *and for any two vertices in* $V(G) \setminus X$, *there is no common neighbour in* $[X]$, *where* $v_1v_2 \in E(G)$.

(v) $[V(G) \setminus X] \cong 4K_1$, $d_X(v_2) = d_X(v_3) = d_X(v_4) = 1, N_X(v_1) \cap N_X(v_2) \neq \emptyset$ *and for any two vertices in* $\{v_2, v_3, v_4\}$, *there is no common neighbour in* $[X]$.

(vi) $[V(G) \setminus X] \cong 4K_1$, $d_X(v_1) \geqslant 2, d_X(v_2) = d_X(v_3) = d_X(v_4) = 1$ *and for any two vertices in* $V(G) \setminus X$, *there is no common neighbour in* $[X]$.

(vii) $[V(G)\setminus X] \cong 4K_1$, $d_X(v_1) = d_X(v_2) = d_X(v_3) = d_X(v_4) = 1$ *and for any two vertices in* $V(G)\setminus X$, *there is no common neighbour in* $[X]$.

Moreover, we have $\mathrm{rvc}(G) = 3$ *from Case* (i) *to Case* (vi) *and* $\mathrm{rvc}(G) = 4$ *for Case* (vii).

Proof. Let $F = [V(G) \setminus X]$ and c be a vertex-coloring of G. Since $\mathrm{diam}(G) = 3$, thus $\mathrm{rvc}(G) \geqslant \mathrm{diam}(G) - 1 = 2$. We prove the theorem by analysing the structure of F.

Case 1. F is connected and $F \cong K_4, C_4, C_4 + e_1, K_{1,3}, K_{1,3} + e_2$ or P_4 (see Figure 1). Let $V(F) \cong \{v_1, v_2, v_3, v_4\}$. Suppose $F \cong K_4, C_4$ or $C_4 + e_1$. The following coloring c with colors 1 and 2 induces a vertex-coloring of G with $\mathrm{rvc}(G) = 2$: Choose $c(v) = 2$ for any $v \in \bigcup_{1 \leqslant i \leqslant 4} N_X(v_i)$ and for all other vertices with 1.

Suppose $F \cong K_{1,3}$ or $K_{1,3} + e_2$. If $N_X(v_3) \neq \emptyset$, say $u \in N_X(v_3)$, then we color u with 2 and all other vertices with 1. Now we may assume $N_X(v_3) = \emptyset$. Choose $c(v) = 2$ for any $v \in N_X(v_1) \cup N_X(v_2) \cup N_X(v_4)$, $c(v_3) = 2$ and for the remaining vertices with 1.

Suppose $F \cong P_4$. Let $N_X(v_2) = N_X(v_3) = \{u\}$ and $N_X(v_1) = N_X(v_4) = \emptyset$. It is easy to verify that $rvc(G) \geqslant 3$. Color v_2 with 2, u with 3 and all other vertices with 1. Then G is vertex rainbow connected. Hence $\mathrm{rvc}(G) = 3$. Since $\mathrm{diam}(G) = 3$, up to isomorphism, we may only need to consider the following three subcases. Let G^* be a connected spanning subgraph of the graph G.

Let $E(G^*) = E([X]) \cup E(F) \cup \{v_2u_2, v_3u_3\}$, where $\{u_2, u_3\} \subseteq X$ and $u_2 \neq u_3$. Color u_2, v_3 with 2 and the remaining vertices with 1.

Let $E(G^*) = E([X]) \cup E(F) \cup \{v_1u_1, v_3u_3\}$, where $\{u_1, u_3\} \subseteq X$ and $u_1 = u_3$ is possible. Color u_1, u_3, v_2 with 2 and the remaining vertices with 1.

Let $E(G^*) = E([X]) \cup E(F) \cup \{v_1u_1, v_4u_4\}$, where $\{u_1, u_4\} \subseteq X$ and $u_1 = u_4$ is possible. We define the vertex-coloring c as follows: $c(u_1) = c(u_4) = c(v_3) = 2$ and color the remaining vertices with 1.

It is easy to check that G^* is vertex rainbow connected. Then $\mathrm{rvc}(G) \leqslant \mathrm{rvc}(G^*) \leqslant 2$. Hence $\mathrm{rvc}(G) = 2$.

Case 2. $F \cong K_3 + K_1$ or $P_3 + K_1$. First we assume $F \cong K_3 + K_1$. Let $V(K_3) = \{v_1, v_2, v_3\}$ and $V(K_1) = \{v_4\}$. Suppose there exists a vertex v_i $(1 \leqslant i \leqslant 3)$ satisfying $N_X(v_i) \cap N_X(v_4) \neq \emptyset$, say $u \in N_X(v_i) \cap N_X(v_4)$. Color u with 2 and all other vertices with 1. Suppose $|(N_X(v_1) \cap N_X(v_4)) \cup (N_X(v_2) \cap N_X(v_4)) \cup (N_X(v_3) \cap N_X(v_4))| = 0$. For $1 \leqslant i \leqslant 3$, we must have $N_X(v_i) \neq \emptyset$, say $u_i \in N_X(v_i)$. Choose $c(u_1) = c(u_2) = c(u_3) = 2$ and for the remaining vertices with 1.

Next we assume $F \cong P_3 + K_1$. Let $V(P_3) = \{v_1, v_2, v_3\}$ and $V(K_1) = \{v_4\}$. We assign colors to G as follows: $c(v) = 2$ for $v \in N_X(v_4)$ and color the remaining vertices with 1. This is a vertex-coloring of G with $\mathrm{rvc}(G) = 2$.

Case 3. $F \cong 2K_2$ or $K_2 + 2K_1$. First we assume $F \cong 2K_2$. Let $V(2K_2) = \{v_1, v_2, v_3, v_4\}$, where $v_1v_2 \in E(F)$ and $v_3v_4 \in E(F)$. The following coloring c with colors 1 and 2 induces a vertex-coloring of G with $\mathrm{rvc}(G) = 2$: $c(v) = 2$ for $v \in N_X(v_3) \cup N_X(v_4)$ and color all other vertices with 1.

Next we assume $F \cong K_2 + 2K_1$. Let $V(K_2) = \{v_1, v_2\}$ and $V(2K_1) = \{v_3, v_4\}$. Since G is connected, we obtain $N_X(v_3) \neq \emptyset$ and $N_X(v_4) \neq \emptyset$. We can prove the case by the following two subcases.

Subcase 3.1. Up to isomorphism, let $N_X(v_1) = \emptyset$. Suppose $N_X(v_2) \cap N_X(v_3) \cap N_X(v_4) \neq \emptyset$, say $u \in N_X(v_2) \cap N_X(v_3) \cap N_X(v_4)$. Choose $c(u) = 2$ and for the remaining vertices with 1. Suppose $N_X(v_2) \cap N_X(v_3) \cap N_X(v_4) = \emptyset$. Since $\mathrm{diam}(G) = 3$, we must have $N_X(v_2) \cap N_X(v_3) \neq \emptyset$ and $N_X(v_2) \cap N_X(v_4) \neq \emptyset$, say $u_3 \in N_X(v_2) \cap N_X(v_3)$ and $u_4 \in N_X(v_2) \cap N_X(v_4) (u_3 \neq u_4)$. Let $d_X(v_3) = d_X(v_4) = 1$. Assume that $\mathrm{rvc}(G) = 2$ with $c(u_3) = c(u_4)$. This implies that there is not a rainbow path between v_3 and v_4. Assume that $\mathrm{rvc}(G) = 2$ with $c(u_3) \neq c(u_4)$. Without loss of generality, let $c(u_3) = c(v_2)$. This implies that there is not a rainbow path between v_1 and v_3. Therefore, $\mathrm{rvc}(G) \geqslant 3$. Color u_3 with 2, u_4 with 3 and all other vertices with 1. Then G is vertex rainbow connected. Hence $\mathrm{rvc}(G) = 3$. Now we may assume $\max\{d_X(v_3), d_X(v_4)\} \geqslant 2$. Color u_3, u_4 with 2 and color the remaining vertices with 1. Then G is vertex rainbow connected, and so $\mathrm{rvc}(G) = 2$.

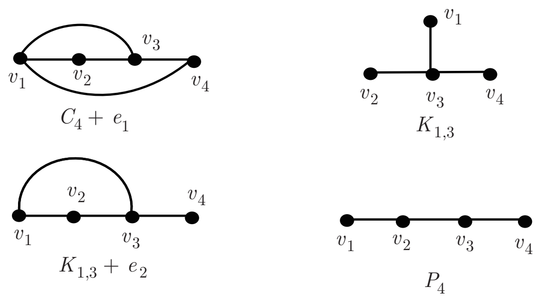

Figure 1 $e_1 = v_1v_3$ and $e_2 = v_1v_2$

Subcase 3.2. $N_X(v_1)\neq\emptyset$ and $N_X(v_2)\neq\emptyset$. Suppose $N_X(v_3)\cap N_X(v_4)\neq\emptyset$, say $u\in N_X(v_3)\cap N_X(v_4)$. Color u with 2 and color all other vertices with 1.

Suppose $N_X(v_1)\cap N_X(v_2)\neq\emptyset$ and $N_X(v_3)\cap N_X(v_4)=\emptyset$, say $u\in N_X(v_1)\cap N_X(v_2)$. Let $d_X(v_2)=d_X(v_3)=d_X(v_4)=1$. If $(N_X(v_2)\cap N_X(v_3))\cup(N_X(v_2)\cap N_X(v_4))\neq\emptyset$, without loss of generality, let $N_X(v_2)\cap N_X(v_3)\neq\emptyset$. Color u with 2 and color all other vertices with 1. If $(N_X(v_2)\cap N_X(v_3))\cup(N_X(v_2)\cap N_X(v_4))=\emptyset$. It is easy to check that G is not vertex rainbow connected with colors 1 and 2. Let $u_3\in N_X(v_3)$. Color u with 2, u_3 with 3 and the remaining vertices with 1. It is easy to verify that G is vertex rainbow connected. Hence $\mathrm{rvc}(G)=3$. Now we may assume $\max\{d_X(v_3),d_X(v_4)\}\geqslant 2$ or $\min\{d_X(v_1),d_X(v_2)\}\geqslant 2$. Assume that $\max\{d_X(v_3),d_X(v_4)\}\geqslant 2$. Without loss of generality, let $d_X(v_3)\geqslant 2$, say $\{u_3,u_3'\}\in N_X(v_3)$ and $u_4\in N_X(v_4)$ satisfying $u_3\neq u$. We define the vertex-coloring c as follows: $c(u_3)=c(u_4)=2$ and color all other vertices with 1. Assume that $\min\{d_X(v_1),d_X(v_2)\}\geqslant 2$. Let $u_4\in N_X(v_4)$. If $u\in N_X(v_3)$, then we color u with 2 and all other vertices with 1. If $u\notin N_X(v_3)$, then we assign colors to G as follows: $c(u)=c(u_4)=2$ and color the remaining vertices with 1.

Suppose $(N_X(v_1)\cap N_X(v_3))\cup(N_X(v_1)\cap N_X(v_4))\cup(N_X(v_2)\cap N_X(v_3))\cup(N_X(v_2)\cap N_X(v_4))\neq\emptyset$ and $N_X(v_1)\cap N_X(v_2)=N_X(v_3)\cap N_X(v_4)=\emptyset$. Without loss of generality, let $N_X(v_2)\cap N_X(v_4)\neq\emptyset$, say $u\in N_X(v_2)\cap N_X(v_4)$. Let $u_1\in N_X(v_1)$. Suppose $u_1\in N_X(v_3)$, color u with 2 and the remaining vertices with 1. Suppose $u_1\notin N_X(v_3)$, color u,u_1 with 2 and the remaining vertices with 1.

At last, we assume that for any two vertices in F there is no common neighbour in $[X]$, where $V(F)=\{v_1,v_2,v_3,v_4\}$. Let $d_X(v_2)=d_X(v_3)=d_X(v_4)=1$. We have $\mathrm{rvc}(G)\geqslant 3$. Let $u_3\in N_X(v_3)$ and $u_4\in N_X(v_4)$. Color u_3 with 2, u_4 with 3 and all other vertices with 1. It is easy to check that G is vertex rainbow connected. Hence $\mathrm{rvc}(G)=3$. Now we may assume $\max\{d_X(v_3),d_X(v_4)\}\geqslant 2$ or $\min\{d_X(v_1),d_X(v_2)\}\geqslant 2$. Assume that $\max\{d_X(v_3),d_X(v_4)\}\geqslant 2$. Without loss of generality, let $d_X(v_4)\geqslant 2$, say $u_3\in N_X(v_3)$ and $u_4\in N_X(v_4)$. Color u_3,u_4 with 2 and the remaining vertices with 1. Assume that $\min\{d_X(v_1),d_X(v_2)\}\geqslant 2$. Let $u_i\in N_X(v_i)$, where $1\leqslant i\leqslant 4$. The following coloring c with colors 1 and 2 induces a vertex-coloring of G with $\mathrm{rvc}(G)=2: c(u_1)=c(u_2)=c(u_4)=2$ and color all other vertices with 1.

Case 4. $F\cong 4K_1$. Let $V(F)\cong\{v_1,v_2,v_3,v_4\}$. Since $\mathrm{diam}(G)=3$, we must have $\bigcap_{1\leqslant i\leqslant 4}N_X(v_i)=\emptyset$. Up to isomorphism, we only need to consider the following three subcases.

Assume that $N_X(v_1)\cap N_X(v_2)\cap N_X(v_3)\neq\emptyset$, say $u\in N_X(v_1)\cap N_X(v_2)\cap N_X(v_3)$. Color u with 2 and color the remaining vertices with 1. This is a vertex-coloring of G with $\mathrm{rvc}(G)=2$.

Assume that $N_X(v_1)\cap N_X(v_2)\neq\emptyset$, and for any three vertices in $V(G)\setminus X$, there is no common neighbour in $[X]$. Suppose $N_X(v_3)\cap N_X(v_4)\neq\emptyset$, say $u\in N_X(v_1)\cap N_X(v_2)$ and $v\in N_X(v_3)\cap N_X(v_4)$. Choose $c(v)=2$ and for all other vertices with 1. Suppose $N_X(v_3)\cap N_X(v_4)=\emptyset$. Let $d_X(v_2)=d_X(v_3)=d_X(v_4)=1$ and $\bigcup_{2\leqslant i\neq j\leqslant 4}(N_X(v_i)\cap N_X(v_j))=\emptyset$. Then G is not vertex rainbow connected with two colors. Let $u_3\in N_X(v_3)$ and $u_4\in N_X(v_4)$. Color u_3 with 2, u_4 with 3 and all other vertices with 1. It is easy to check that G is vertex rainbow connected. Hence $\mathrm{rvc}(G)=3$. Now we may assume that $\max\{d_X(v_3),d_X(v_4)\}\geqslant 2$ or $\min\{d_X(v_1),d_X(v_2)\}\geqslant 2$. If $\max\{d_X(v_3),d_X(v_4)\}\geqslant 2$, without loss of generality, let $d_X(v_3)\geqslant 2$, then we color u_3,u_4 with 2 and color the remaining vertices with 1. If $\min\{d_X(v_1),d_X(v_2)\}\geqslant 2$, then the following coloring c with colors 1 and 2 induces a vertex-coloring of G with $\mathrm{rvc}(G)=2: c(u)=c(u_4)=2$ and color the remaining vertices with 1.

Assume that for any two vertices in F there is no common neighbour in $[X]$, where $V(F)=\{v_1,v_2,v_3,v_4\}$. Let $u_i\in N_X(v_i)$, where $1\leqslant i\leqslant 4$. Let $d_X(v_1)=d_X(v_2)=d_X(v_3)=d_X(v_4)=1$, we obtain $\mathrm{rvc}(G)\geqslant 4$. Color u_2 with 2, u_3 with 3, u_4 with 4 and the remaining vertices with 1. Then $\mathrm{rvc}(G)=4$. Let $d_X(v_2)=d_X(v_3)=d_X(v_4)=1$ and $d_X(v_1)\geqslant 2$. Then $\mathrm{rvc}(G)\geqslant 3$. Color u_1,u_2 with 2, u_3 with 3 and all other vertices with 1. Thus $\mathrm{rvc}(G)=3$. Without loss of generality, we may assume $d_X(v_1)\geqslant 2$ and $d_X(v_2)\geqslant 2$. We define the vertex-coloring c with $\mathrm{rvc}(G)=2$ as follows: Color $c(u_1)=c(u_2)=c(u_3)=2$ and all other vertices with 1.

This completes the proof of the theorem. □

Theorem 3.4. *Let G be a connected graph of order n, $\mathrm{diam}(G)\geqslant 4$ and cliquenumber $\omega(G)=n-4$. If X is a maximum clique of G, then either $\mathrm{rvc}(G)=3$, or $\mathrm{rvc}(G)=4$ and one of the following holds.*

Let $V(G) \setminus X = \{v_1, v_2, v_3, v_4\}$.

(i) $[V(G) \setminus X] \cong P_4$ *and* $d_X(v_2) = d_X(v_3) = d_X(v_4) = 0$.

(ii) $[V(G) \setminus X] \cong P_3 + K_1, d_X(v_2) = d_X(v_3) = 0$ *and* $N_X(v_1) \cap N_X(v_4) = \emptyset$, *where* $v_1v_2, v_2v_3 \in E(G)$.

(iii) $[V(G) \setminus X] \cong 2K_2, d_X(v_2) = d_X(v_4) = 0$ *and* $N_X(v_1) \cap N_X(v_3) = \emptyset$, *where* $v_1v_2, v_3v_4 \in E(G)$.

(iv) $[V(G) \setminus X] \cong K_2 + 2K_1, d_X(v_1) = 0, d_X(v_2) = d_X(v_3) = d_X(v_4) = 1$ *and for any two vertices in* $\{v_2, v_3, v_4\}$, *there is no common neighbour in* $[X]$, *where* $v_1v_2 \in E(G)$.

Proof. Let $F = [V(G) \setminus X]$ and c be a vertex-coloring of G. Since $\mathrm{diam}(G) \geqslant 4$, thus $\mathrm{rvc}(G) \geqslant \mathrm{diam}(G) - 1 \geqslant 3$. We prove the theorem by analysing the structure of F.

Case 1. F is connected and $F \cong K_{1,3}, K_{1,3} + e_2, P_4, C_4$ or $C_4 + e_1$ (see Figure 1). Let $V(F) \cong \{v_1, v_2, v_3, v_4\}$.

Suppose $F \cong K_{1,3}$. Then there exists a connected spanning subgraph G^* of the graph G such that $G^* \cong G_1$ and $E(G_1) = E([X]) \cup E(F) \cup \{v_1u_1\}$, where $u_1 \in X$. Color v_1 with 2, u_1 with 3 and the remaining vertices with 1. It is easy to check that G_1 is vertex rainbow connected. Then $\mathrm{rvc}(G) \leqslant \mathrm{rvc}(G^*) = \mathrm{rvc}(G_1) \leqslant 3$. Hence $\mathrm{rvc}(G) = 3$. Suppose $F \cong K_{1,3} + e_2$. Thus G must have a connected spanning subgraph G^* such that $G^* \cong G_1$, and so $\mathrm{rvc}(G) = 3$.

Suppose $F \cong P_4$. Let $d_X(v_2) = d_X(v_3) = d_X(v_4) = 0$. We can prove that $diam(G) = 5$. Let $u_1 \in N_X(v_1)$. We assign colors to G as follows: $c(v_2) = 2, c(v_1) = 3, c(u_1) = 4$ and color all other vertices with 1. This is a vertex-coloring of G with $\mathrm{rvc}(G) = 4$. Now we may assume that there exists a connected spanning subgraph G^* of the graph G such that $G^* \cong G_2$ and $E(G_2) = E([X]) \cup E(F) \cup \{v_2u_2\}$, where $u_2 \in X$. The following coloring c with colors 1, 2 and 3 induces a vertex-coloring of G_2 with $\mathrm{rvc}(G_2) = 3 : c(v_2) = 2, c(u_2) = 3$ and color all other vertices with 1. Thus $\mathrm{rvc}(G) = 3$.

Suppose $F \cong C_4$ or $C_4 + e_1$. Then G must have a connected spanning subgraph G^* such that $G^* \cong G_2$. Hence $\mathrm{rvc}(G) = 3$.

Case 2. $F \cong K_3 + K_1$ or $P_3 + K_1$.

Suppose $F \cong K_3 + K_1$. Let $V(K_3) = \{v_1, v_2, v_3\}$ and $V(K_1) = \{v_4\}$. There exists a connected spanning subgraph G^* of the graph G such that $E(G^*) = E([X]) \cup E(F) \cup \{v_1u_1, v_4u_4\}$, where $u_1, u_4 \in X$ and $u_1 \neq u_4$. Choose $c(u_1) = 2, c(u_4) = 3$ and for the remaining vertices with 1. This is a vertex-coloring of G^* with $\mathrm{rvc}(G^*) = 3$. Hence $\mathrm{rvc}(G) = 3$.

Suppose $F \cong P_3 + K_1$. Let $V(P_3) = \{v_1, v_2, v_3\}$ and $V(K_1) = \{v_4\}$, where $v_1v_2, v_2v_3 \in E(G)$. Assume that $N_X(v_2) \cap N_X(v_4) \neq \emptyset$, we have $\mathrm{diam}(G) \leqslant 3$, a contradiction. Without loss of generality, we assume $N_X(v_1) \cap N_X(v_4) \neq \emptyset$, say $u \in N_X(v_1) \cup N_X(v_4)$. Thus there exists a connected spanning subgraph G^* of the graph G such that $E(G^*) = E([X]) \cup E(F) \cup \{v_1u, v_4u\}$. We assign colors to G^* as follows: $c(v_1) = 2, c(u) = 3$ and color the remaining vertices with 1. It is easy to check that G^* is vertex rainbow connected. Hence $\mathrm{rvc}(G) = 3$. Now we may assume that $N_X(v_i) \cap N_X(v_4) = \emptyset$, where $1 \leqslant i \leqslant 3$. Let $d_X(v_2) = d_X(v_3) = 0$. Then $\mathrm{diam}(G) = 5$. Let $u_1 \in N_X(v_1)$ and $u_4 \in N_X(v_4)$. Color v_1 with 2, u_1 with 3, u_4 with 4 and all other vertices with 1. This is a vertex-coloring of G with $\mathrm{rvc}(G) = 4$. For the remaining cases, we only need to assign 2 to the vertices of $\bigcup_{1 \leqslant i \leqslant 3} N_X(v_i)$, 3 to the vertices of $N_X(v_4)$ and color all other vertices with 1. Thus $\mathrm{rvc}(G) = 3$.

Case 3. $F \cong 2K_2$ or $K_2 + 2K_1$.

Suppose $F \cong 2K_2$. Let $V(2K_2) = \{v_1, v_2, v_3, v_4\}$, where $v_1v_2 \in E(F)$ and $v_3v_4 \in E(F)$. Assume that $N_X(v_i) \cap N_X(v_j) \neq \emptyset$ with $u \in N_X(v_i) \cap N_X(v_j)$, where $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant 4$. Thus there exists a connected spanning subgraph G^* of the graph G such that $E(G^*) = E([X]) \cup E(F) \cup \{v_iu, v_ju\}$. The following coloring c with colors 1, 2 and 3 induces a vertex-coloring of G^* with $\mathrm{rvc}(G^*) = 3 : c(v_i) = 2, c(u) = 3$ and color all other vertices with 1. Hence $\mathrm{rvc}(G) = 3$.

Assume that $N_X(v_i) \cap N_X(v_j) = \emptyset$, where $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant 4$. Let $d_X(v_2) = d_X(v_4) = 0$. Then $\mathrm{diam}(G) = 5$. Let $u_1 \in N_X(v_1)$ and $u_3 \in N_X(v_3)$. Color u_1 with 2, u_3 with 3, v_3 with 4 and all other vertices with 1. This is a vertex-coloring of G with $\mathrm{rvc}(G) = 4$. Without loss of generality, now we may assume $N_X(v_1) \neq \emptyset, N_X(v_2) \neq \emptyset$ and $N_X(v_3) \neq \emptyset$. Let $u_2 \in N_X(v_2)(u_1 = u_2$ is possible). Choose $c(u_1) = c(u_2) = 2, c(u_3) = 3$ and for the remaining vertices with 1. This is a vertex-coloring of G with $\mathrm{rvc}(G) = 3$.

Suppose $F \cong K_2 + 2K_1$. Let $V(K_2) = \{v_1, v_2\}$ and $V(2K_1) = \{v_3, v_4\}$. Since G is connected, we have $N_X(v_3) \neq \emptyset$ and $N_X(v_4) \neq \emptyset$. Since $\mathrm{diam}(G) \geqslant 4$, without loss of generality, let $N_X(v_1) = \emptyset$ and $N_X(v_2) \neq \emptyset$. For any two vertices v_i, v_j in $\{v_2, v_3, v_4\}$, let $N_X(v_i) \cap N_X(v_j) \neq \emptyset$, say $u \in N_X(v_i) \cap N_X(v_j)$. We assign colors to G as follows: $c(v) = 2$, for any $v \in \bigcup_{2 \leqslant i \leqslant 4} N_X(v_i) \setminus \{u\}$, $c(u) = 3$ and color the remaining vertices with 1. It is easy to check that G is vertex rainbow connected.

For any two vertices in $\{v_2, v_3, v_4\}$, there is no common neighbour in $[X]$. Let $d_X(v_2) = d_X(v_3) = d_X(v_4) = 1$. Assume that $\mathrm{rvc}(G) = 3$. Then there is not a rainbow path connecting v_i and v_j, where $i, j \in \{1, 3, 4\}$ with $i \neq j$. Thus $\mathrm{rvc}(G) \geqslant 4$. Let $u_i \in N_X(v_i)$ with $2 \leqslant i \leqslant 4$. We assign colors to G as follows: $c(u_2) = 2, c(u_3) = 3, c(u_4) = 4$ and color all other vertices with 1. This is a vertex-coloring of G with $\mathrm{rvc}(G) = 4$. Now we may assume $\max\{d_X(v_2), d_X(v_3), d_X(v_4)\} \geqslant 2$. Without loss of generality, let $d_X(v_2) \geqslant 2$, say $\{u_2, u_2'\} \subseteq N_X(v_2)$. The following coloring c with colors 1, 2 and 3 induces a vertex-coloring of G with $\mathrm{rvc}(G) = 3 : c(u_2) = 2, c(u_2') = 3, c(u_3) = 2, c(u_4) = 3$ and color all other vertices with 1.

This completes the proof of the theorem. □

4 Conclusions

In this paper, we first show a lower bound for $s(n, k)$, and obtain $s(n, 2) = \binom{n-2}{2} + 2$. The readers could try to consider the precise values of $s(n, k)$, where $k \geqslant 3$.

Next, we mainly characterize graphs G with $|V(G)| = n, \mathrm{diam}(G) \geqslant 3$ and $\omega(G) = n - s$ for $3 \leqslant s \leqslant 4$. It would be possible to characterize all connected graphs of order n, diameter 3 and vertex rainbow connection number 2 with clique number $n - s$ for $s \geqslant 5$. However, the case analysis will enlarge extensively since the number of exceptional graph classes with $|V(G)| = n, \mathrm{diam}(G) = 3, \omega(G) = n - s$, but vertex rainbow connection number $\mathrm{rvc}(G) > 2$ increases.

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