The skew-rank of oriented graphs*

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Abstract

An oriented graph G^σ is a digraph without loops and multiple arcs, where G is called the underlying graph of G^σ. Let $S(G^\sigma)$ denote the skew-adjacency matrix of G^σ. The rank of the skew-adjacency matrix of G^σ is called the *skew-rank* of G^σ, denoted by $sr(G^\sigma)$. The skew-adjacency matrix of an oriented graph is skew symmetric and the skew-rank is even. In this paper we consider the skew-rank of simple oriented graphs. Firstly we give some preliminary results about the skew-rank. Secondly we characterize the oriented graphs with skew-rank 2 and characterize the oriented graphs with pendant vertices which attain the skew-rank 4. As a consequence, we list the oriented unicyclic graphs, the oriented bicyclic graphs with pendant vertices which attain the skew-rank 4. Moreover, we determine the skew-rank of oriented unicyclic graphs of order n with girth k in terms of matching number. We investigate the minimum value of the skew-rank among oriented unicyclic graphs of order n with girth k and characterize oriented unicyclic graphs attaining the minimum value. In addition, we consider oriented unicyclic graphs whose skew-adjacency matrices are nonsingular.

Key words: Oriented graph; Skew-adjacency matrix; Skew-rank.
AMS Classifications: 05C20, 05C50, 05C75.

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1 Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* $A(G)$ of a graph G of order n is the $n \times n$ symmetric 0-1 matrix $(a_{ij})_{n\times n}$ such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0, otherwise. We denote by $Sp(G)$ the spectrum of $A(G)$. The rank of $A(G)$ is called to be the rank of G, denoted by $r(G)$. Let G^σ ba a graph with an orientation which assigns to each edge of G a direction so that G^σ becomes an oriented graph. The graph G is called the *underlying graph* of G^σ. The *skew-adjacency matrix* associated to the oriented graph G^σ is defined as the $n \times n$ matrix $S(G^\sigma) = (s_{ij})$ such that $s_{ij} = 1$ if there has an arc from v_i to v_j, $s_{ij} = -1$ if there has an arc from v_j to v_i and $s_{ij} = 0$ otherwise. Obviously, the skew-adjacency matrix is skew symmetric. The *skew-rank* of an oriented graph G^σ, denoted by $sr(G^\sigma)$, is defined as the rank of the skew-adjacency matrix $S(G^\sigma)$. The *skew-spectrum* $Sp(G^\sigma)$ of G^σ is defined as the spectrum of $S(G^\sigma)$. Note that $Sp(G^\sigma)$ consists of only purely imaginary eigenvalues and the skew-rank of an oriented graph is even.

Let $C_k^\sigma = u_1u_2\cdots u_ku_1$ be an even oriented cycle. The *sign* of the even cycle C_k^σ, denoted by $sgn(C_k^\sigma)$, is defined as the sign of $\prod_{i=1}^{k} s_{u_iu_{i+1}}$ with $u_{k+1} = u_1$. An even oriented cycle C_k^σ is called *evenly-oriented* (*oddly-oriented*) if its sign is positive (negative). If every even cycle in G^σ is evenly-oriented, then G^σ is called *evenly-oriented*. An oriented graph is called an *elementary oriented graph* if such an oriented graph is K_2 or a cycle with even length. An oriented graph $\mathscr{H}$ is called a *basic oriented graph* if each component of $\mathscr{H}$ is an elementary oriented graph.

The oriented graph G^σ is called *multipartite* if its underlying graph G is *multipartite*. An *induced subgraph* of G^σ is an induced subgraph of G and each edge preserves the original orientation in G^σ. For an induced subgraph H^σ of G^σ, let $G^\sigma - H^\sigma$ be the subgraph obtained from G_w by deleting all vertices of H_w and all incident edges. For $V' \subseteq V(G^\sigma)$, $G^\sigma - V'$ is the subgraph obtained from G^σ by deleting all vertices in V' and all incident edges. A vertex of a graph G^σ is called *pendant* if it is only adjacent to one vertex, and is called *quasi-pendant* if it is adjacent to a pendant vertex. A set M of edges in G^σ is a *matching* if every vertex of G^σ is incident with at most one edge in M. It is *perfect matching* if every vertex of G^σ is incident with exactly one edge in M. We denote by $m_{G^\sigma}(i)$ the number of matchings of G^σ with i edges and by $\beta(G^\sigma)$ the *matching number* of G^σ (i.e. the number of edges of a maximum matching in G^σ). For an oriented graph G^σ on at least two vertices, a vertex $v \in V(G^\sigma)$ is called *unsaturated* in G_w if there exists a maximum matching M of G^σ in which no edge is incident with v; otherwise, v is called *saturated* in G_w. Denote by P_n, S_n, C_n, K_n a path, a star, a cycle and a complete graph all of which are simple unoriented graphs of order n, respectively. $K_{n_1,n_2,\cdots,n_r}$ represents a complete r-partite unoriented graphs. A graph is called *trivial* if it has one vertex and no edges.

Recently the study of the skew-adjacency matrix of oriented graphs attracted some attentions. Cavers et al. [4] provided a paper about the skew-adjacency matrices in which authors considered the following topics: graphs whose skew-adjacency matrices are all cospectral; relations between the matching polynomial of a graph and the characteristic polynomial of its adjacency and skew-adjacency matrices; skew-spectral radii and an analogue of the Perron-Frobenius theorem; and the number of skew-adjacency matrices of a graph with distinct spectra. Anuradha and Balakrihnan [2] investigated skew spectrum of the Cartesian product of an oriented graph with a oriented Hypercube. Anuradha et. al [3] considered the skew spectrum of special bipartite graphs and solved a conjecture of Cui and Hou [7]. Hou et al [9] gave an expression of the coefficients of the characteristic polynomial of the skew-adjacency matrix $S(G^\sigma)$. As its applications, they present new combinatorial proofs of some known results. Moreover, some families of oriented bipartite graphs with $Sp(S(G^\sigma)) = iSp(G)$ were given. Gong et al [11] investigated the coefficients of weighted oriented graphs. In addition they established recurrences for the characteristic polynomial and deduced a formula for the matching polynomial of an arbitrary weighted oriented graph. Xu [18] established a relation between the spectral radius and the skew spectral radius. Also some results on the skew-spectral radius of an oriented graph and its oriented subgraphs were derived. As applications, a sharp upper bound of the skew-spectral radius of oriented unicyclic graphs was present. Some authors investigated the skew-energy of oriented graphs, one can refer to [1, 5, 10, 12, 13, 17, 19].

This paper is organized as follows. In Section 2, we list some preliminary results. In Section 3, we characterize the connected oriented graphs which attaining the skew-rank 2 and determine the oriented graphs with pendant vertex which attaining the skew-rank 4. As a consequence, we investigate oriented unicyclic graphs, oriented bicyclic graphs of order n with pendant vertices which attain the skew-rank 4, respectively. In Section 4, we determine the skew-rank of unicyclic graphs of order n with fixed girth in terms of matching number. Moreover we study the minimum value of skew-rank of the oriented unicyclic graphs of order n with fixed girth and characterize oriented graphs with the minimum skew-rank. In Section 5, we consider the non-singularity of the skew-adjacency matrices of oriented unicyclic graphs.

2 Preliminary Results

The following results can be derived from fundamental matrix theory.

Lemma 2.1 *(i). Let H^σ be an induced subgraph of G^σ. Then $sr(H^\sigma) \leq sr(G^\sigma)$.*

(ii). Let $G^\sigma = G_1^\sigma \cup G_2^\sigma \cup \cdots \cup G_t^\sigma$*, where* $G_1^\sigma, G_2^\sigma, \cdots, G_t^\sigma$ *are connected components of* G^σ*. Then* $sr(G^\sigma) = \sum_{i=1}^{t} sr(G_i^\sigma)$*.*

(iii). Let G^σ *be an oriented graph on* n *vertices. Then* $sr(G^\sigma) = 0$ *if and only if* G^σ *is a graph without edges (empty graph).*

As we know, the oriented tree and its underlying graph have the same spectrum [9, 14]. So the following is immediate from [6].

Lemma 2.2 *Let* T^σ *be an oriented tree with matching number* $\beta(T)$*. Then*

$$sr(T^\sigma) = r(T) = 2\beta(T).$$

The next result is an immediate result of Lemma 2.2.

Lemma 2.3 *Let* P_n^σ *be an oriented path of order* n*. Then* $sr(P_n^\sigma) = \begin{cases} n-1, & n \text{ is odd,} \\ n, & n \text{ is even.} \end{cases}$

Lemma 2.4 *[9][14] Let* C_n^σ *be an oriented cycle of order* n*. Then*

$$sr(C_n^\sigma) = \begin{cases} n, & C_n^\sigma \text{ is oddly-oriented,} \\ n-2, & C_n^\sigma \text{ is evenly-oriented,} \\ n-1, & \text{otherwise.} \end{cases}$$

Lemma 2.5 *Let* G^σ *be an oriented graph containing a pendant vertex* v *with the unique neighbor* u*. Then* $sr(G^\sigma) = sr(G^\sigma - u - v) + 2$*.*

Proof. Assume that all vertices in $V(G^\sigma)$ are indexed by $\{v_1, v_2, \cdots, v_n\}$ with $v_1 = v$, $v_2 = u$. Then the skew-adjacency matrix can be expressed as

$$S(G^\sigma) = \begin{pmatrix} 0 & s_{12} & 0 & \cdots & 0 \\ s_{21} & 0 & s_{23} & \cdots & s_{2n} \\ 0 & s_{32} & 0 & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & s_{n2} & s_{n3} & \cdots & 0 \end{pmatrix},$$

where the first two rows and columns are labeled by v_1, v_2. So it follows that

$$\begin{aligned} sr(G^\sigma) &= r\begin{pmatrix} 0 & s_{12} & 0 & \cdots & 0 \\ s_{21} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & s_{n3} & \cdots & 0 \end{pmatrix} \\ &= r\begin{pmatrix} 0 & s_{12} \\ s_{21} & 0 \end{pmatrix} + r\begin{pmatrix} 0 & \cdots & s_{3n} \\ \vdots & \ddots & \vdots \\ s_{n3} & \cdots & 0 \end{pmatrix} \\ &= r\begin{pmatrix} 0 & s_{12} \\ s_{21} & 0 \end{pmatrix} + sr(G^\sigma - v_1 - v_2) \\ &= 2 + sr(G^\sigma - u - v). \end{aligned}$$

■

Remark. In fact the result also holds for the unoriented graph, one can refer to Corollary 1 (pp.234) [6].

For convenience, the transformation in Lemma 2.5 is called $\delta-$*transformation*. The skew-rank of some graph can be derived by finite steps of $\delta-$transformation.

Let w be a common neighbor of two nonadjacent vertices u, v. The edges among u, v and w have the *uniform orientations* if the arcs is from u, v to w or from w to u, v. The edges among u, v and w have the *opposite orientations* if one arc is from u (v) to w and the another is from w to v (u).

Two nonadjacent vertices u, v of an oriented graph G^σ are called *uniform (opposite) twins* if $N(u) = N(v)$ and the corresponding edges among u, v and each neighbor have the uniform (opposite) orientations.

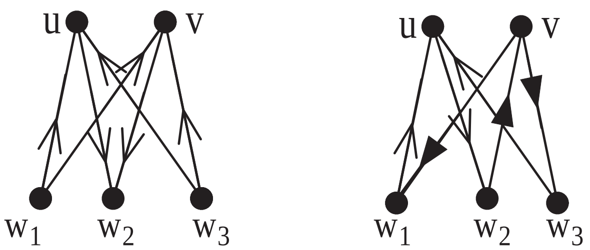

Figure 1: *Uniform twins u, v in the left figure, but opposite twins in the right figure.*

Example 2.6 *Two graphs shown in Fig. 1 contain uniform, opposite twins. u, v are uniform twins in the left graph and opposite twins in the right graph.*

For an oriented graph G^σ, the uniform (opposite) twins in $S(G^\sigma)$ correspond the identical (opposite) rows and columns. Hence deleting or adding a uniform (opposite) twin vertex does not change the skew-rank of an oriented graph. Hence we have

Lemma 2.7 *Let u, v be uniform (opposite) twins of an oriented graph G^σ. Then $sr(G^\sigma) = sr(G^\sigma - u) = sr(G^\sigma - v)$.*

Two pendant vertices are called *pendant twins* in G^σ if they have the same neighbor in G^σ. By Lemma 2.7, we have

Lemma 2.8 *Let u, v be pendant twins of an oriented graph G^σ. Then $sr(G^\sigma) = sr(G^\sigma - u) = sr(G^\sigma - v)$.*

By the definitions of uniform (opposite) twins and evenly-oriented graph, we can derive the following results.

Lemma 2.9 *Let G^σ be an oriented complete multipartite graph. If all its 4-vertex cycles are evenly-oriented, then all vertices in the same vertex partite set are uniform or opposite twins.*

3 Oriented graphs with small skew-rank

According to Lemmas 2.1 and 2.3, it is obvious that $sr(G^\sigma) \geq 2$ if G is a simple non-empty graph. A natural problem is to characterize the extremal connected oriented graphs whose skew-ranks attain the lower bound 2 and the second lower bound 4.

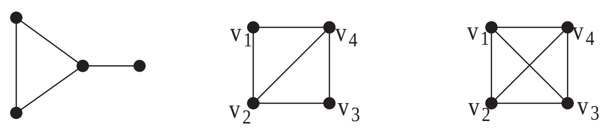

Figure 2: *Three graphs G_1, $K_{1,1,2}$ and K_4*

Let G_1 be the graph obtained from K_3 by adding a pendant edge to some vertex in K_3 (as depicted in Fig. 2). Let G^σ be an oriented graph. Let v be a vertex of G^σ and $V' \subset V(G^\sigma)$. The notation $N(v)$ represents the neighborhood of v in G^σ. $G^\sigma[V']$ denotes the induced subgraph of G^σ on the vertices in V' including the orientations of edges.

Theorem 3.1 *Let G^σ be a connected oriented graph of order n $(n = 2, 3, 4)$. $sr(G^\sigma) = 2$ if and only if G^σ satisfies one of the following statements:*

1. *If $n = 2$, G^σ is an oriented path P_2^σ with arbitrary orientation.*

2. If $n=3$, then G^σ is K_3^σ or P_3^σ. Each edge has any orientation in G^σ.

3. If $n=4$, then G^σ is one of the following oriented graphs with some properties:

(a) Evenly-oriented cycle C_4^σ.

(b) $K_{1,3}^\sigma$ and each edge has any orientation.

(c) Evenly-oriented graph $K_{1,1,2}^\sigma$.

Proof. If $n=2,3$, the results can be easily verified from Lemmas 2.4 and 2.3.

If $n=4$, then all 4-vertex connected unoriented graphs are $K_{1,3}$, C_4, P_4, $K_{1,1,2}$, K_4, G_1 (as depicted in Fig. 2). By Lemmas 2.3 and 2.5 the oriented graphs with P_4 or G_1 as the underlying graph have skew-rank 4. And $sr(C_4^\sigma)=4$ if C_4^σ is an oddly-oriented cycle from Lemma 2.4, but the value is 2 if it is evenly-oriented cycle. If the underlying graph G is isomorphic to $K_{1,3}$, then $sr(G^\sigma)=2$ and each edge has any orientation. Next we shall consider the skew-rank of oriented graphs with $K_{1,1,2}$ or K_4 as their underlying graphs.

For convenience, all vertices of $K_{1,1,2}$ are labeled by $\{v_1,v_2,v_3,v_4\}$ (as depicted in Fig. 2). Then the skew-adjacency matrix of the oriented graph $K_{1,1,2}^\sigma$ can be expressed as

$$S(K_{1,1,2}^\sigma)=\begin{pmatrix} 0 & s_{12} & 0 & s_{14} \\ -s_{12} & 0 & s_{23} & s_{24} \\ 0 & -s_{23} & 0 & s_{34} \\ -s_{14} & -s_{24} & -s_{34} & 0 \end{pmatrix}.$$

Then

$$sr(K_{1,1,2}^\sigma)=r\begin{pmatrix} 0 & s_{12} & 0 & 0 \\ -s_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{34}+s_{23}\cdot\frac{s_{14}}{s_{12}} \\ 0 & 0 & -s_{34}-s_{23}\cdot\frac{s_{14}}{s_{12}} & 0 \end{pmatrix}.$$

So $sr(K_{1,1,2}^\sigma)=2$ if and only if $s_{34}+s_{23}\cdot\frac{s_{14}}{s_{12}}=0$, i.e., $s_{12}s_{34}+s_{14}s_{23}=0$ which implies that the subgraph C_4^σ with vertex set $\{v_1,v_2,v_3,v_4\}$ of $K_{1,1,2}^\sigma$ is evenly-oriented.

The skew-adjacency matrix of the oriented graph K_4^σ can be expressed as

$$S(K_4^\sigma)=\begin{pmatrix} 0 & s_{12} & s_{13} & s_{14} \\ -s_{12} & 0 & s_{23} & s_{24} \\ -s_{13} & -s_{23} & 0 & s_{34} \\ -s_{14} & -s_{24} & -s_{34} & 0 \end{pmatrix}.$$

Then

$$sr(K_4^\sigma) = r\begin{pmatrix} 0 & s_{12} & 0 & 0 \\ -s_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{34}+s_{23}\cdot\frac{s_{14}}{s_{12}}-s_{24}\cdot\frac{s_{13}}{s_{12}} \\ 0 & 0 & -s_{34}-s_{23}\cdot\frac{s_{14}}{s_{12}}+s_{24}\cdot\frac{s_{13}}{s_{12}} & 0 \end{pmatrix}.$$

Assume that $s_{34}+s_{23}\cdot\frac{s_{14}}{s_{12}}-s_{24}\cdot\frac{s_{13}}{s_{12}}=0$. It is equivalent to $s_{12}s_{34}+s_{14}s_{23}=s_{13}s_{24}$. Obviously the value of the left side is 0, 2 or -2. But the value of the right side is 1 or -1. So $s_{34}+s_{23}\cdot\frac{s_{14}}{s_{12}}-s_{24}\cdot\frac{s_{13}}{s_{12}}\neq 0$. Therefore $sr(K_4^\sigma)=4$. ■

Next we give a lemma which plays a key role in our proof of Theorem 3.3.

Lemma 3.2 *[16] A connected graph is not a complete multipartite graph if and only if it contains P_4, G_1 (as depicted in Fig. 2) or two copies of P_2 as an induced subgraph.*

Theorem 3.3 *Let G^σ be a connected oriented graph of order $n\geq 5$. Then $sr(G^\sigma)=2$ if and only if the underlying graph of G^σ is a complete bipartite or tripartite graph and all 4-vertex cycles are evenly-oriented in G^σ.*

Proof. Sufficiency:

Assume that G^σ is a complete bipartite graph K_{n_1,n_2} and all its 4-vertex cycles are evenly-oriented. Then all vertices in the same partite vertex set are uniform or opposite twins by Lemma 2.9. Let X_1, X_2 be two partite vertex sets of K_{n_1,n_2}. Suppose that $n_1\geq 2$. Let x_1, x_2 be two arbitrary vertices in X_1. By Lemma 2.7, we have $sr(K^\sigma_{n_1,n_2}) = sr(K^\sigma_{n_1,n_2}-x_1)=sr(K^\sigma_{n_1,n_2}-x_2)=sr(P_2^\sigma)=2$.

Similarly, $sr(K^\sigma_{n_1,n_2,n_3}) = sr(K_3^\sigma) = 2$ if all 4-vertex cycles are evenly-oriented in $K^\sigma_{n_1,n_2,n_3}$.

Necessity:

Assume that the underlying graph G is not a complete multipartite graph. Then G must contain P_4, G_1 (as depicted in Fig. 2) or two copies of P_2 as an induced subgraph by Lemma 3.2. This implies that $sr(G^\sigma)\geq 4$ which is a contradiction.

Combining the above discussion, we infer that G is a complete multipartite graph. Assume that the underlying graph G is a complete t-partite graph $K_{n_1,n_2,\cdots,n_t}$. Suppose that $t\geq 4$. Then G^σ must contain an induced subgraph K_4^σ. From the proof of Theorem 3.1, $sr(G^\sigma)\geq sr(K_4^\sigma)=4$. So $t=2$ or 3.

Case 1. $t=2$.

Let X_1, X_2 be the two partite vertex sets of K_{n_1,n_2}. If the cardinality of one of them is one, the G^σ is an oriented star $K^\sigma_{1,n-1}$ and each edge has arbitrary orientation. Assume that the cardinality of every partite vertex set is more than one. If $K^\sigma_{n_1,n_2}$ contains an oddly-oriented cycle C_4^σ as an induced subgraph, then $sr(K_{n_1,n_2}^\sigma) \geq sr(C_4^\sigma) = 4$. So all 4-vertex cycles in K_{n_1,n_2}^σ are evenly-oriented.

Case 2. $t = 3$.

Similarly to the above discussion, we conclude that all 4-vertex cycles in K_{n_1,n_2,n_3}^σ are evenly-oriented. ■

Theorem 3.4 *Let G^σ be an oriented graph with pendant vertex of order n. Then $sr(G^\sigma) = 4$ if and only if G^σ is one of the following oriented graphs with some properties:*

1. *Graphs obtained by inserting some edges with arbitrary orientation between the center of $S_{n-n_1-n_2}^\sigma$ $(n_1+n_2 \geq 2)$ and some vertices (maybe partial or all) of a complete bipartite oriented graph K_{n_1,n_2}^σ such that all 4-vertex cycles in K_{n_1,n_2}^σ are evenly-oriented.*

2. *Graphs obtained by inserting some edges with arbitrary orientation between the center of $S_{n-n_1-n_2-n_3}^\sigma$ $(n_1+n_2+n_3 \geq 3)$ and some vertices (maybe partial or all) of a complete tripartite oriented graph K_{n_1,n_2,n_3}^σ such that all 4-vertex cycles in K_{n_1,n_2,n_3}^σ are evenly-oriented.*

Proof. Sufficiency: It is easy to verify that the results hold by Lemma 2.5 and Theorem 3.3.

Necessity: Assume that $sr(G^\sigma) = 4$. Let x be a pendant vertex in G^σ and $N(x) = y$. Suppose that $G^\sigma - x - y = G_{11}^\sigma \cup G_{12}^\sigma \cup \cdots \cup G_{1t}^\sigma$ where $G_{11}^\sigma, G_{12}^\sigma, \cdots, G_{1t}^\sigma$ are connected components of $G^\sigma - x - y$. If each G_{1i}^σ $(i = 1, 2, \cdots, t)$ is trivial, then $G^\sigma - x - y$ is an oriented star. So $sr(G^\sigma) = 2$ which is a contradiction. Next we shall verify that there exists exactly one nontrivial connected components in $G^\sigma - x - y$.

Assume that there exist two nontrivial connected components in $G^\sigma - x - y$. Without loss of generality, we denote them by G_{11}, G_{12}.

By Lemma 2.5, we have

$$\begin{aligned} sr(G^\sigma) &= 2 + sr(G^\sigma - x - y) \\ &= 2 + \sum_{j=1}^{2} sr(G_{1j}^\sigma) \\ &\geq 2 + \sum_{j=1}^{2} 2 \qquad \text{since } sr(G_{1j}^\sigma) \geq 2 \\ &= 6. \end{aligned}$$

This is a contradiction.

So there exits exactly one nontrivial connected component in $G^\sigma - x - y$. Without loss of generality, assume that G^σ_{11} is nontrivial. So $G^\sigma - x - y = G^\sigma_{11} \cup (n - |G^\sigma_{11}| - 2)K_1$. Hence $sr(G^\sigma) = sr(G^\sigma_{11}) + 2 \geq 4$ with the equality holding if and only if $sr(G^\sigma_{11}) = 2$. So G^σ_{11} is one of the graphs as described in Theorem 3.3. It is evident that the subgraph induced by x, y and all isolated vertices in $G^\sigma - x - y$ is an oriented star $S^\sigma_{n-|G^\sigma_{11}|}$. Therefore G^σ can be obtained by inserting some edges with any orientation between the center of $S^\sigma_{n-|G^\sigma_{11}|}$ and some vertices (maybe partial or all) of G^σ_{11}. ■

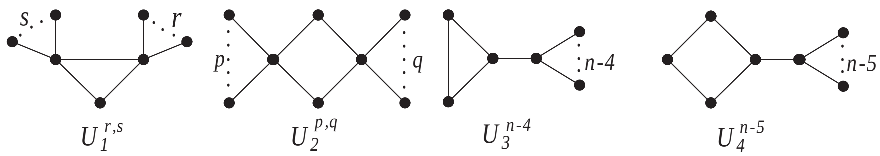

Figure 3: *Four unoriented unicyclic graphs* $U_1^{r,s}$, $U_2^{p,q}$, U_3^{n-4}, U_4^{n-5}

By Lemma 2.4 and Theorem 3.4, we have

Theorem 3.5 *Let U^σ be an oriented unicyclic graph of order n and C^σ be the oriented cycle in U^σ. Then $sr(U^\sigma) = 4$ if and only if U^σ is one of the following graphs with some properties:*

1. *The oddly-oriented cycle C^σ_4,or the evenly-oriented cycle C^σ_6, or the oriented cycle C_5 with any orientation.*

2. *The oriented graphs with $U_1^{r,s}$ ($r + s = n - 3$), $U_2^{p,q}$ ($p + q = n - 4$) or U_3^{n-4} (as depicted in Fig. 3) as the underlying graph and each edge has any orientation in U^σ.*

3. *The oriented graphs with U_4^{n-5} (as depicted in Fig. 3) as the underlying graph in which C^σ_4 is an evenly-oriented cycle.*

Theorem 3.6 *Let B^σ be an oriented bicyclic graph of order n with pendant vertices. Then $sr(B^\sigma) = 4$ if and only if B^σ is one of the following graphs with some properties:*

1. *The oriented graphs with B_1, B_2 or B_3 (as depicted in Fig. 4) as the underlying graph in which each edge has any orientation.*

2. *The oriented graphs with B_4 or B_5 (as depicted in Fig. 4) as the underlying graph in which the subgraph induced by vertices u_i ($i = 1, 2, 3, 4$) is an even-oriented cycle.*

3. *The oriented graphs with B_6 or B_7 (as depicted in Fig. 4) as the underlying graph such that all 4-vertex cycles induced by four vertices among w_i ($i = 1, 2$) and v_j ($j = 1, 2, 3$) are evenly-oriented.*

4. *The oriented graphs with B_8 or B_9 (as depicted in Fig. 4) as the underlying graph such that the induced subgraph $K^\sigma_{1,1,2}$ is evenly-oriented.*

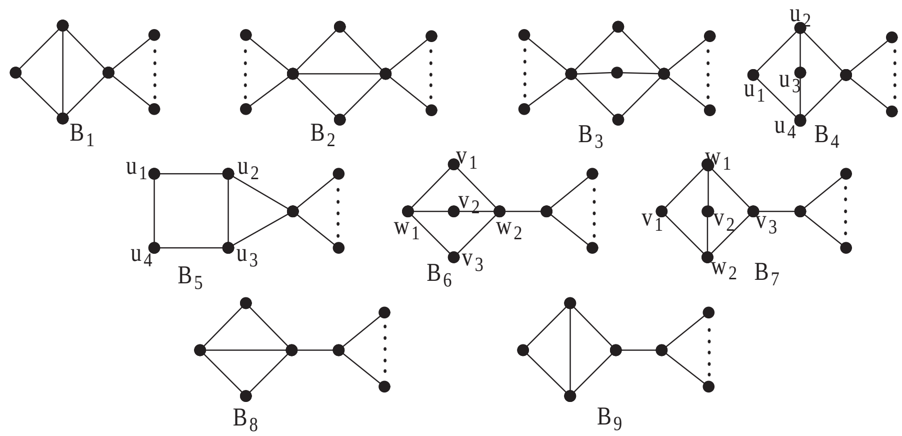

Figure 4: *Nine unoriented bicyclic graphs B_i's* $(i = 1, 2, \cdots, 9)$

4 Skew-rank of oriented unicyclic graphs

In this section we determine the skew-rank of the oriented unicyclic graphs of order n with girth k in terms of matching number. Moreover, we investigate the minimum value of the skew-rank among oriented unicyclic graphs of order n with girth k and characterize the extremal oriented unicyclic graphs.

Lemma 4.1 *[9, 11] Let G^σ be an oriented graph of order n with skew adjacency matrix $S(G^\sigma)$ and its characteristic polynomial*

$$\phi(G^\sigma, \lambda) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-i} = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + (-1)^{n-1}a_{n-1}\lambda + (-1)^n a_n.$$

Then

$$a_i = \sum_{\mathscr{H}} (-1)^{c^+} 2^c$$

if i is even, where the summation is over all basic oriented subgraphs $\mathscr{H}$ of G^σ having i vertices and c^+, c are the numbers of evenly-oriented even cycles and even cycles contained in $\mathscr{H}$, respectively. In particular, $a_i = 0$ if i is odd.

Theorem 4.2 *Let G^σ be an oriented unicyclic graph of order n with girth k and matching number $\beta(G^\sigma)$. Then*

$$sr(G^\sigma) = \begin{cases} 2\beta(G^\sigma) - 2, & \text{if } C^\sigma_k \text{ is evenly-oriented and } \beta(G^\sigma) = 2\beta(G^\sigma - C^\sigma_k), \\ 2\beta(G^\sigma), & \text{ortherwise.} \end{cases}$$

Proof. If $i > \beta(G^\sigma)$, G^σ contains no basic oriented subgraphs with $2i$ vertices and $a_{2i} = 0$. Suppose that $i \le \beta(G^\sigma)$. Note that $\lambda^{n-2\beta(G^\sigma)}$ is a factor of the characteristic polynomial $\phi(G^\sigma, \lambda)$ of $S(G^\sigma)$, which implies $sr(G^\sigma) \le 2\beta(G^\sigma)$. So we consider the coefficient $a_{2\beta(G^\sigma)}$. Next we divide into three cases to verify this result.

Case 1. k is odd.

Note that there does not exist even cycle in every basic oriented subgraph $\mathscr{H}$. So $a_{2\beta(G^\sigma)} = \sum_{\mathscr{H}}(-1)^0 2^0 = \sum_{\mathscr{H}} 1 \neq 0$. It yields $sr(G^\sigma) = 2\beta(G^\sigma)$.

Case 2. k is even and C_k^σ is oddly-oriented.

There exists an even cycle in some basic oriented subgraph, but no evenly-oriented cycle in any basic oriented subgraph. So $a_{2\beta(G^\sigma)} \neq 0$ which implies $sr(G^\sigma) = 2\beta(G^\sigma)$.

Case 3. k is even and C_k^σ is evenly-oriented.

Let $\mathcal{H}$ be the set of basic oriented subgraphs on $2\beta(G^\sigma)$ vertices. Let $\mathcal{H}_1$ be the set of basic oriented subgraphs on $2\beta(G^\sigma)$ vertices which contain only $\beta(G^\sigma)$ copies of K_2. Let $\mathcal{H}_2$ be the set of basic oriented subgraphs on $2\beta(G^\sigma)$ vertices which contain C_k^σ and $\beta(G^\sigma) - \frac{k}{2}$ copies of K_2. Obviously, $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. Thus

$$\begin{aligned} a_{2\beta(G^\sigma)} &= \sum_{\mathscr{H}\in\mathcal{H}_1} (-1)^0 \cdot 2^0 + \sum_{\mathscr{H}\in\mathcal{H}_2} (-1)^1 \cdot 2^1 \\ &= \beta(G^\sigma) - 2\beta(G^\sigma - C_k^\sigma). \end{aligned}$$

It is evident that $sr(G^\sigma) = 2\beta(G^\sigma)$ if $\beta(G^\sigma) - 2\beta(G^\sigma - C_k^\sigma) \neq 0$ and $sr(G^\sigma) < 2\beta(G^\sigma)$ if $\beta(G^\sigma) - 2\beta(G^\sigma - C_k^\sigma) = 0$. In what follows we shall verify $sr(G^\sigma) = 2\beta(G^\sigma) - 2$, i.e. $a_{2\beta(G^\sigma)-2} \neq 0$, if $\beta(G^\sigma) - 2\beta(G^\sigma - C_k^\sigma) = 0$. Let $\mathcal{H}_1'$ be the set of basic oriented subgraphs on $2\beta(G^\sigma) - 2$ vertices which contain only $\beta(G^\sigma) - 1$ copies of K_2. Let $\mathcal{H}_2'$ be the set of basic oriented subgraphs on $2\beta(G^\sigma) - 2$ vertices which contain C_k^σ and $\beta(G^\sigma) - \frac{k}{2} - 1$ copies of K_2. By Lemma 4.1, we have

$$\begin{aligned} a_{2\beta(G^\sigma)-2} &= \sum_{\mathscr{H}\in\mathcal{H}_1'} (-1)^0 \cdot 2^0 + \sum_{\mathscr{H}\in\mathcal{H}_2'} (-1)^1 \cdot 2^1 \\ &= m_{G^\sigma}\big(\beta(G^\sigma) - 1\big) - 2m_{G^\sigma - C_k^\sigma}\big(\beta(G^\sigma - C_k^\sigma) - 1\big). \end{aligned}$$

For convenience, we introduce three notations.

$\mathcal{S}_1$: the set of $(\beta(G^\sigma) - 1)$-matchings of G^σ;

$\mathcal{S}_2$: the set of $(\beta(G^\sigma - C_k^\sigma) - 1)$-matchings of $G^\sigma - C_k^\sigma$;

$\mathcal{S}_3 = \{M' \mid M' = C_k^\sigma \cup M,\ M \in \mathcal{S}_2\}$.

It is evident that $|\mathcal{S}_1| \ge 2|\mathcal{S}_2|$ and $|\mathcal{S}_2| = |\mathcal{S}_3|$. Next we shall verify that $m_{G^\sigma}\big(\beta(G^\sigma) - 1\big) - 2m_{G^\sigma - C_k^\sigma}\big(\beta(G^\sigma - C_k^\sigma) - 1\big) \neq 0$. Since $|\mathcal{S}_1| = m_{G^\sigma}(\beta(G^\sigma) - 1)$ and $|\mathcal{S}_2| = m_{G^\sigma - C_k^\sigma}\big(\beta(G^\sigma - C_k^\sigma) - 1\big)$, so we only verify that $|\mathcal{S}_1| > 2|\mathcal{S}_2|$. Note that C_k^σ has exactly two perfect matchings M_1, M_2 with $\frac{k}{2}$ edges. Suppose that $\mathcal{S}^* = \{M_1 \cup M | M \in \mathcal{S}_2\} \cup \{M_2 \cup M | M \in \mathcal{S}_2\}$.

So $|\mathcal{S}^*| = 2|\mathcal{S}_2| = 2|\mathcal{S}_3|$ and $|\mathcal{S}^*| \leq |\mathcal{S}_1|$. It is evident that there exists a $(\beta(G^\sigma)-1)$-matching M^*, which is the union of a matching of $G^\sigma - C_k^\sigma$ with $\beta(G^\sigma) - \frac{k}{2}$ edges and a matching of C_k^σ with $\frac{k}{2} - 1$ edges, such that $M^* \in \mathcal{S}_1$ and $M^* \notin \mathcal{S}^*$. It follows that $|\mathcal{S}_1| \geq |\mathcal{S}^*| + 1 = 2|\mathcal{S}_2| + 1 > 2|\mathcal{S}_2|$. Thus the result follows. ■

Let $H_{n,k}$ be an underlying graph obtained from C_k by attaching $n-k$ pendant edges to some vertex on C_k.

Theorem 4.3 *Let G^σ be an oriented unicyclic graph of order n with girth k $(n > k)$. Then*

$$sr(G^\sigma) \geq \begin{cases} k, & k \textit{ is even,} \\ k+1, & k \textit{ is odd.} \end{cases}$$

This bound is sharp.

Proof. Since G^σ must contain $H_{k+1,k}^\sigma$ as an induced subgraph, so $sr(H_{k+1,k}^\sigma) \leq sr(G^\sigma)$ by Lemma 2.1. By Lemmas 2.3 and 2.5, we have

$$sr(H_{k+1,k}^\sigma) = \begin{cases} k, & k \text{ is even,} \\ k+1, & k \text{ is odd.} \end{cases}$$

Note that all oriented graphs with $H_{n,k}$ as the underlying graph have the same skew rank as $H_{k+1,k}^\sigma$. So the result holds. ■

The following results can be derived by similar method in Theorems 3.1 and 3.3 in [8].

Lemma 4.4 *Let T^σ be an oriented tree with $u \in V(T^\sigma)$ and G_0^σ be an oriented graph different from T^σ. Let G^σ be a graph obtained from G_0^σ and T^σ by joining u with certain vertices of G_0^σ. Then the following statements hold:*

1. *If u is saturated in T^σ, then*

$$sr(G^\sigma) = sr(G_0^\sigma) + sr(T^\sigma).$$

2. *If u is unsaturated in T^σ, then*

$$sr(G^\sigma) = sr(T^\sigma - u) + sr(G_0^\sigma + u),$$

where $G_0^\sigma + u$ is the subgraph of G^σ induced by the vertices of G_0^σ and u.

Let G^σ be an oriented unicyclic graph and C^σ be the unique oriented cycle of G^σ. Let G_0^σ be the graph obtained from G^σ by deleting the two neighbors of v on C^σ and let $G^\sigma\{v\}$ be the component of G_0^σ containing v. Then $G^\sigma\{v\}$ is an oriented tree and an induced subgraph of G^σ.

By the above result, we have

Theorem 4.5 *Let G^σ be an oriented unicyclic graph and C^σ be the unique oriented cycle in G^σ. Then the following statements hold:*

1. *If there exists a vertex $v \in V(C^\sigma)$ which is saturated in $G^\sigma\{v\}$, then*

$$sr(G^\sigma) = sr(G^\sigma\{v\}) + sr(G^\sigma - G^\sigma\{v\}),$$

   where $G^\sigma\{v\}$ is an oriented tree rooted at v and containing v.

2. *If there does not exit a vertex $v \in V(C^\sigma)$ which is saturated in $G^\sigma\{v\}$, then*

$$sr(G^\sigma) = sr(C^\sigma) + sr(G^\sigma - C^\sigma).$$

Let U^* be an underlying graph which is obtained from a cycle C_k and a star S_{n-k} by inserting an edge between a vertex on C_k and the center of S_{n-k}.

Theorem 4.6 *Let G^σ be an oriented unicyclic graph of order n and C_k^σ be the unique oriented cycle in G^σ. Assume that $sr(G^\sigma) = \begin{cases} k, & k \text{ is even,} \\ k+1, & k \text{ is odd.} \end{cases}$ Then the following statements hold:*

1. *If there exists a vertex $v \in V(C_k^\sigma)$ which is saturated in $G^\sigma\{v\}$, then $G^\sigma\{v\}$ is an oriented star, $\beta(G^\sigma - G\{v\}) = \begin{cases} \frac{k-2}{2}, & k \text{ is even,} \\ \frac{k-1}{2}, & k \text{ is odd.} \end{cases}$ and G^σ has any orientation;*

2. *If there does not exist a vertex $v \in V(C_k^\sigma)$ which is saturated in $G^\sigma\{v\}$, then*

   (a) If k is odd, then $G \cong U^*$ and G^σ has any orientation;

   (b) If k is even, then $G \cong U^*$ and C_k^σ is evenly-oriented.

Proof. Assume that there exists a vertex $v \in V(C_k^\sigma)$ which is saturated in $G^\sigma\{v\}$. Note that $G^\sigma\{v\}$ and $G^\sigma - G^\sigma\{v\}$ are two trees. If k is even, by Lemmas 2.2 and 4.5 we have

$$\begin{aligned} sr(G^\sigma) &= sr(G^\sigma\{v\}) + sr(G^\sigma - G^\sigma\{v\}) \\ &= 2\beta(G^\sigma\{v\}) + 2\beta(G^\sigma - G^\sigma\{v\}) = k \end{aligned}$$

Since $\beta(G^\sigma\{v\}) \geq 1$, $\beta(G^\sigma - G^\sigma\{v\}) \geq \frac{k-2}{2}$, so $\beta(G^\sigma\{v\}) = 1$ and $\beta(G^\sigma - G^\sigma\{v\}) = \frac{k-2}{2}$, which implies $G^\sigma\{v\}$ is an oriented star. From the above process, we can find that this result is independent of the orientations of edges. So G^σ has any orientation.

Similarly the result holds for the case that k is odd.

Suppose that there does not exist a vertex $v \in V(C_k^\sigma)$ which is saturated in $G^\sigma\{v\}$. By Theorem 4.5, we have

$$sr(G^\sigma) = sr(C_k^\sigma) + 2\beta(G^\sigma - C_k^\sigma).$$

Next we deal with the following three cases.

Case 1. k is odd.

By Lemma 2.4 and the above equality, we have $k+1 = k-1+2\beta(G^\sigma - C_k^\sigma)$. It follows that $\beta(G^\sigma - C_k^\sigma) = 1$, i.e. $G^\sigma - C_k^\sigma$ is a star, and G^σ has any orientation.

Case 2. k is even and C_k^σ is oddly-oriented.

By the discussion in Case 1, we have $\beta(G^\sigma - C_k^\sigma) = 0$. This contradicts to the fact that there does not exist a vertex $v \in V(C_k^\sigma)$ which is saturated in $G^\sigma\{v\}$. So this case can not happen.

Case 3. k is even and C_k^σ is evenly-oriented.

By the above discussion, we have $\beta(G^\sigma - C_k^\sigma) = 1$, i.e. $G^\sigma - C_k^\sigma$ is an oriented star. ■

5 Non-singularity of skew-adjacency matrices of oriented unicyclic graphs

Let $\mathscr{U}_{n,k}$ be the set of oriented unicyclic graphs of order n with girth k. Let $\mathscr{U}_1$ be the set of oriented unicyclic graphs of order n with girth k which can be changed to be an empty (null) graph by finite steps of δ-transformation. Let $\mathscr{U}_2$ be the set of oriented unicyclic graphs of order n with girth k which can be changed to be an oriented cycle C_k^σ or the union of isolated vertices and C_k^σ by finite steps of δ-transformation. Obviously, $\mathscr{U}_{n,k} = \mathscr{U}_1 \cup \mathscr{U}_2$.

Theorem 5.1 *Let G^σ be an oriented unicyclic graph of order n with girth k ($k < n$). Then*

1. *If $G^\sigma \in \mathscr{U}_1$, then* $sr(G^\sigma) \leq \begin{cases} n, & n \text{ is even}, \\ n-1, & n \text{ is odd}. \end{cases}$

2. *If $G^\sigma \in \mathscr{U}_2$, then* $sr(G^\sigma) \leq \begin{cases} n-1, & n \text{ is odd, } k \text{ is odd}, \\ n-2, & n \text{ is even, } k \text{ is odd}, \\ n, & n \text{ is even and } C_k^\sigma \text{ is oddly-oriented}, \\ n-1, & n \text{ is odd and } C_k^\sigma \text{ is oddly-oriented}, \\ n-2, & n \text{ is even and } C_k^\sigma \text{ is evenly-oriented}, \\ n-3, & n \text{ is odd and } C_k^\sigma \text{ is evenly-oriented}. \end{cases}$

Proof. If $G^\sigma \in \mathscr{U}_1$, then by at most $\lfloor \frac{n}{2} \rfloor$ steps of $\delta-$transformation G^σ can be changed to an empty (null) graph. By Lemma 2.5, $sr(G^\sigma) \leq 2 \cdot \lfloor \frac{n}{2} \rfloor$.

If $G^\sigma \in \mathscr{U}_2$, then by at most $\lfloor \frac{n-k}{2} \rfloor$ steps of $\delta-$transformation G^σ can be changed to be oriented cycle C_k^σ or the union of isolated vertices and C_k^σ. By Lemma 2.5, $sr(G^\sigma) \leq 2 \cdot \lfloor \frac{n-k}{2} \rfloor + sr(C_k^\sigma)$. The result holds by Lemma 2.4. ■

In what follows we consider the non-singularity of skew-adjacency matrices of oriented unicyclic graphs. As we know, if the order n is odd, then the oriented unicyclic graph must be singular. So we only need consider the oriented unicyclic graph with even order. By Theorem 5.1, we have

Theorem 5.2 *Let G^σ be an oriented unicyclic graph with even order n. Then $S(G^\sigma)$ is nonsingular if and only if $G^\sigma \in \mathscr{U}_1$ and G^σ has a perfect matching, or $G^\sigma \in \mathscr{U}_2$, C_k^σ is oddly-oriented and $G^\sigma - C_k^\sigma$ has a perfect matching.*

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