# On the $q$-log-convexity conjecture of Sun

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**Abstract.** In the study of Ramanujan-Sato type series for $1/\pi$, Sun introduced a sequence of polynomials $S_n(q)$ as given by

$$S_n(q) = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k} q^k,$$

and he conjectured that the polynomials $S_n(q)$ are $q$-log-convex. Using the approach of Liu and Wang, we obtain a sufficient condition to ensure the $q$-log-convexity of self-reciprocal polynomials. Based on this criterion, we give an affirmative answer to Sun's conjecture.

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# 1 Introduction

The main objective of this paper is to prove a conjecture of Sun [12] on the $q$-log-convexity of the polynomials

$$S_n(q) = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k} q^k, \tag{1.1}$$

which arise in the study of Ramanujan-Sato type series for $1/\pi$.

Let us recall some definitions. A nonnegative sequence $\{a_n\}_{n\geq 0}$ is said to be log-concave if, for any $n \geq 1$,

$$a_n^2 \geq a_{n-1}a_{n+1};$$

and is said to be log-convex if, for any $n \geq 1$,

$$a_{n-1}a_{n+1} \geq a_n^2.$$

Many sequences arising in combinatorics, algebra and geometry, turn out to be log-concave or log-convex, see Brenti [1] or Stanley [11].

For a sequence of polynomials with real coefficients, Stanley introduced the notion of $q$-log-concavity. A polynomial sequence $\{f_n(q)\}_{n\geq 0}$ is said to be $q$-log-concave if, for any $n \geq 1$, the difference

$$f_n^2(q) - f_{n+1}(q)f_{n-1}(q)$$

has nonnegative coefficients. The $q$-log-concavity of polynomial sequences has been extensively studied, see Bulter [2], Krattenthaler [7], Leroux [8] and Sagan [10]. Similarly, a polynomial sequence $\{f_n(q)\}_{n\geq 0}$ is said to be $q$-log-convex if, for any $n \geq 1$, the difference

$$f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$$

has nonnegative coefficients. Liu and Wang [9] showed that many classical combinatorial polynomials are $q$-log-convex, see also [4, 5, 6]. It should be noted that Butler and Flanigan [3] introduced a different kind of $q$-log-convexity.

Sun posed six conjectures on the expansions of $1/\pi$ in terms of $S_n(q)$, one of which reads

$$\sum_{n=0}^{\infty} \frac{140n+19}{4624^n}\binom{2n}{n}S_n(64) = \frac{289}{3\pi}.$$

He also conjectured that the polynomials $S_n(q)$ are $q$-log-convex. It is easy to see that the coefficients of $S_n(q)$ are symmetric. Such polynomials are also said to be self-reciprocal. More precisely, a polynomial

$$f(q) = a_0 + a_1 q + \cdots + a_n q^n$$

is called a self-reciprocal polynomial of degree $n$ if $f(q) = q^n f(1/q)$.

In this paper, we give a proof of the $q$-log-convexity of the polynomials $S_n(q)$. Our proof is closely related to an approach of Liu and Wang [9]. Assume that

$$f_n(q) = \sum_{k=0}^{n} a(n,k)q^k. \tag{1.2}$$

Write the difference $f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$ as

$$\sum_{t=0}^{2n}\left[\sum_{k=0}^{\lfloor t/2\rfloor} \widetilde{\mathcal{L}}_t(a(n,k))\right]q^t,$$

where

$$\widetilde{\mathcal{L}}_t(a(n,k)) = \begin{cases} a(n+1,k)a(n-1,t-k) + a(n-1,k)a(n+1,t-k) & \\ \qquad -2a(n,k)a(n,t-k), & \text{if } 0 \leq k < \frac{t}{2}, \\ a(n+1,k)a(n-1,k) - a^2(n,k), & \text{if } t \text{ is even and } k = \frac{t}{2}. \end{cases} \tag{1.3}$$

Liu and Wang gave the following construction of $q$-log-convex polynomials.

**Theorem 1.1.** *Let $\{u_k\}_{k\geq 0}$ be a log-convex sequence of real numbers and let $\{f_n(q)\}_{n\geq 0}$ be a $q$-log-convex sequence of polynomials with nonnegative real coefficients as given in* (1.2)*. Let $\{g_n(q)\}_{n\geq 0}$ be a sequence of polynomials defined by*

$$g_n(q) = \sum_{k=0}^{n} a(n,k) u_k q^k. \tag{1.4}$$

*Assume that for any $n \geq 1$ and $0 \leq t \leq 2n$, there exists an integer $k'$ depending on $n$ and $t$ such that*

$$\widetilde{\mathcal{L}}_t(a(n,k)) \begin{cases} \geq 0, & \text{if } 0 \leq k \leq k', \\ \leq 0, & \text{if } k' < k \leq \frac{t}{2}. \end{cases}$$

*Then the polynomials $g_n(q)$ are $q$-log-convex.*

We shall make use of the above theorem for the polynomials $\{S_n(q)\}_{n\geq 0}$ by taking

$$u_k = \binom{2k}{k}, \quad a(n,k) = \binom{n}{k}\binom{2n-2k}{n-k}.$$

Numerical evidence indicates that $\widetilde{\mathcal{L}}_t(a(n,k))$ satisfies the criterion in the above theorem of Liu and Wang. Considering the symmetry of the coefficients of $S_n(q)$, we obtain an analogous criterion to Theorem 1.1 for the $q$-log-convexity of self-reciprocal polynomials. By using this criterion, we confirm the conjecture of Sun.

# 2 $q$-Log-convexity of self-reciprocal polynomials

Analogous to the criterion of Liu and Wang as given in Theorem 1.1, we find a sufficient condition for a sequence of self-reciprocal polynomials to be $q$-log-convex. For $0 \leq k \leq t/2$, we define

$$\begin{aligned}\mathcal{L}_t(a(n,k)) =& a(n+1,k)a(n-1,t-k) + a(n-1,k)a(n+1,t-k) \\ & - 2a(n,k)a(n,t-k).\end{aligned} \tag{2.1}$$

We obtain the following criterion which can be directly applied to $\{S_n(q)\}_{n\geq 0}$.

**Theorem 2.1.** *Given a log-convex sequence $\{u_k\}_{k\geq 0}$ and a $q$-log-convex sequence $\{f_n(q)\}_{n\geq 0}$ as defined in* (1.2)*, let $\{g_n(q)\}_{n\geq 0}$ be the polynomial sequence defined by* (1.4)*. Assume that the following two conditions are satisfied:*

*(C1) For any $n \geq 0$, the polynomial $g_n(q)$ is a self-reciprocal polynomial of degree $n$; and*

*(C2) For any $n \geq 1$ and $0 \leq t \leq n$, there exists an index $k'$ associated with $n, t$ such that*

$$\mathcal{L}_t(a(n,k)) \begin{cases} \geq 0, & \text{if } 0 \leq k \leq k', \\ \leq 0, & \text{if } k' < k \leq \frac{t}{2}. \end{cases}$$

*Then the polynomial sequence $\{g_n(q)\}_{n \geq 0}$ is $q$-log-convex.*

*Proof.* Under the assumption that each $g_n(q)$ is a self-reciprocal polynomial of degree $n$, it is easily checked that both $g_{n-1}(q)g_{n+1}(q)$ and $g_n^2(q)$ are self-reciprocal polynomials of degree $2n$. Moreover, the difference $g_{n-1}(q)g_{n+1}(q) - g_n^2(q)$ is also of degree $2n$ and hence it is self-reciprocal. Write $g_{n-1}(q)g_{n+1}(q) - g_n^2(q)$ as

$$\sum_{t=0}^{2n} B(n,t)q^t.$$

To prove the $q$-log-convexity of $\{g_n(q)\}_{n \geq 0}$, it suffices to show that $B(n,t)$ is nonnegative for any $0 \leq t \leq n$.

It can be verified that

$$B(n,t) = \begin{cases} \sum_{k=0}^{s} \mathcal{L}_t(a(n,k))u_k u_{t-k}, \text{ if } t = 2s+1, \\ \sum_{k=0}^{s-1} \mathcal{L}_t(a(n,k))u_k u_{t-k} + \frac{\mathcal{L}_t(a(n,s))}{2} u_s^2, \text{ if } t = 2s. \end{cases}$$

Based on the log-convexity of $\{u_k\}_{k \geq 0}$ and the $q$-log-convexity of $\{f_n(q)\}_{n \geq 0}$, we proceed to prove that $B(n,t)$ is nonnegative.

On one hand, write

$$f_{n-1}(q)f_{n+1}(q) - f_n^2(q) = \sum_{t=0}^{2n} A(n,t)q^t,$$

we have

$$A(n,t) = \begin{cases} \sum_{k=0}^{s} \mathcal{L}_t(a(n,k)), \text{ if } t = 2s+1, \\ \sum_{k=0}^{s-1} \mathcal{L}_t(a(n,k)) + \frac{\mathcal{L}_t(a(n,s))}{2}, \text{ if } t = 2s. \end{cases}$$

Since $\{f_n(q)\}_{n \geq 0}$ is $q$-log-convex, we deduce that $A(n,t) \geq 0$ for any $0 \leq t \leq 2n$.

On the other hand, by the log-convexity of $\{u_k\}_{k \geq 0}$, we have

$$u_0 u_t \geq u_1 u_{t-1} \geq \cdots \geq u_{k'} u_{t-k'} \geq \cdots \geq u_s u_{s+1} \geq 0, \quad \text{if } t = 2s+1, \tag{2.2}$$

$$u_0 u_t \geq u_1 u_{t-1} \geq \cdots \geq u_{k'} u_{t-k'} \geq \cdots \geq u_s^2 \geq 0, \quad \text{if } t = 2s. \tag{2.3}$$

To prove that $B(n,t) \geq 0$ for $0 \leq t \leq n$, we consider two cases.

If $t = 2s+1$, then by (2.2) and the condition (C2), we have

$$B(n,t) = \sum_{k=0}^{s} \mathcal{L}_t(a(n,k))u_k u_{t-k} \geq \sum_{k=0}^{s} \mathcal{L}_t(a(n,k))u_{k'} u_{t-k'}.$$

By the definition of $A(n,t)$, we get

$$B(n,t) = A(n,t)u_{k'}u_{t-k'},$$

which is nonnegative since $A(n,t) \geq 0$.

Similarly, when $t = 2s$, we have

$$\begin{aligned} B(n,t) &= \sum_{k=0}^{s-1} \mathcal{L}_t(a(n,k))u_k u_{t-k} + \frac{\mathcal{L}_t\,(a(n,s))}{2} u_s^2 \\ &\geq \sum_{k=0}^{s-1} \mathcal{L}_t(a(n,k))u_{k'} u_{t-k'} + \frac{\mathcal{L}_t\,(a(n,s))}{2} u_{k'} u_{t-k'}, \end{aligned}$$

which equals $A(n,t)u_{k'}u_{t-k'}$, and hence $B(n,t)$ is nonnegative. This completes the proof. ∎

# 3 The $q$-log-convexity of $S_n(q)$

In this section, we use Theorem 2.1 to prove Sun's conjecture on the $q$-log-convexity of $S_n(q)$. To this end, we need to establish the following log-convex property by using the technique of Liu and Wang as given in Theorem 1.1.

**Theorem 3.1.** *For $n \geq 0$, let*

$$f_n(q) = \sum_{k=0}^{n} \binom{n}{k}\binom{2n-2k}{n-k} q^k,$$

*then the sequence $\{f_n(q)\}_{n\geq 0}$ is $q$-log-convex.*

*Proof.* Let $h_n(q)$ denote the polynomial $q^n f_n(q^{-1})$, that is,

$$h_n(q) = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k} q^k.$$

Clearly, $\{f_n(q)\}_{n\geq 0}$ forms a $q$-log-convex sequence if and only if $\{h_n(q)\}_{n\geq 0}$ is $q$-log-convex. It is easily checked that $\{(1+q)^n\}_{n\geq 0}$ is $q$-log-convex and

$\{\binom{2k}{k}\}_{k\geq 0}$ is log-convex. By Theorem 1.1, to prove the $q$-log-convexity of $\{h_n(q)\}_{n\geq 0}$, it suffices to show that, for any $n \geq 1$ and $0 \leq t \leq 2n$, there exists $k'$ such that

$$\widetilde{\mathcal{L}}_t\left(\binom{n}{k}\right)\begin{cases} \geq 0, & \text{if } 0 \leq k \leq k', \\ \leq 0, & \text{if } k' < k \leq \frac{t}{2}, \end{cases}$$

where $\widetilde{\mathcal{L}}$ is defined by (1.3).

Let us consider $\mathcal{L}_t(\binom{n}{k})$ as defined by (2.1), which can be seen to have the same sign as $\widetilde{\mathcal{L}}_t(\binom{n}{k})$. For $n \geq 1$, $0 \leq t \leq 2n$ and $0 \leq k \leq t/2$, we have

$$\begin{aligned}\mathcal{L}_t\left(\binom{n}{k}\right) &= \binom{n+1}{k}\binom{n-1}{t-k} + \binom{n+1}{t-k}\binom{n-1}{k} - 2\binom{n}{t-k}\binom{n}{k} \\ &= \frac{1}{n(n+1)(n-k+1)}\binom{n}{k}\binom{n+1}{t-k}\varphi^{(n,t)}(k), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\varphi^{(n,t)}(k) =&(n+1)(n-k)(n-k+1) + (n+1)(n-t+k)(n-t+k+1) \\ &- 2n(n-k+1)(n-t+k+1).\end{aligned}$$

To determine the sign of $\varphi^{(n,t)}(k)$ for $0 \leq k \leq t/2$, we make use of the function $\varphi^{(n,t)}(x)$ on interval $[0, t/2]$. Taking the derivative of $\varphi^{(n,t)}(x)$ with respect to $x$, we obtain that

$$(\varphi^{(n,t)}(x))' = (4n+2)(2x-t) \leq 0.$$

Thus $\varphi^{(n,t)}(x)$ is decreasing on the interval $[0, t/2]$.

For any integers $n \geq 1$ and $0 \leq t \leq 2n$, we have $\varphi^{(n,t)}(0) = (n+1)(t^2-t) \geq 0$. Thus there is at most one sign change in the sequence $\{\varphi^{(n,t)}(k)\}_{0\leq k\leq \frac{t}{2}}$. It follows that there exists an integer $k'$ such that

$$\varphi^{(n,t)}(k)\begin{cases} \geq 0, & \text{if } 0 \leq k \leq k', \\ \leq 0, & \text{if } k' < k \leq \frac{t}{2}. \end{cases}$$

It is possible that $\{\varphi^{(n,t)}(k)\}_{0\leq k\leq \frac{t}{2}}$ are all nonnegative. In this case, we have that $k' = t/2$. So we get

$$\mathcal{L}_t\left(\binom{n}{k}\right)\begin{cases} \geq 0, & \text{if } 0 \leq k \leq k', \\ \leq 0, & \text{if } k' < k \leq \frac{t}{2}. \end{cases}$$

By Theorem 1.1, we deduce that $\{h_n(q)\}_{n\geq 0}$ is $q$-log-convex, and hence the proof is complete. ∎

For $0 \leq k \leq n$, let

$$a(n,k) = \binom{n}{k}\binom{2n-2k}{n-k}. \tag{3.2}$$

Based on the above theorem and the log-convexity of $\{\binom{2k}{k}\}_{k\geq 0}$, to prove the $q$-log-convexity of $\{S_n(q)\}_{n\geq 0}$, we only need to prove that the triangular array $\{a(n,k)\}_{0\leq k\leq n}$ satisfies condition (C2) in Theorem 2.1.

**Theorem 3.2.** *Let $\{a(n,k)\}_{0\leq k\leq n}$ be the triangular array defined by* (3.2)*. For any $n \geq 1$ and $0 \leq t \leq n$, there exists an integer $k'$ depending on $n, t$ such that*

$$\mathcal{L}_t(a(n,k)) \begin{cases} \geq 0, & \text{if } 0 \leq k \leq k', \\ \leq 0, & \text{if } k' < k \leq \frac{t}{2}. \end{cases}$$

To prove the above theorem, we need three lemmas.

**Lemma 3.3.** *For any $n \geq 1$ and $0 \leq t \leq n$, we have $\mathcal{L}_t(a(n,0)) \geq 0$.*

*Proof.* For $1 \leq n \leq 4$, it can be verified that $\mathcal{L}_t(a(n,0)) \geq 0$. So we may assume that $n \geq 5$. It can be checked that the sign of $\mathcal{L}_t(a(n,0))$ coincides with the sign of

$$\frac{\binom{2n}{n}\binom{n}{t}\binom{2n-2t}{n-t}\theta(t)}{n(n+1)(2n-1)(n-t+1)^2(2n-2t-1)},$$

where

$$\begin{aligned}\theta(x) =&(4n^2-1)x^4 - 2(2n-1)(2n^2+2n+1)x^3 + (4n^4+8n^3+8n^2-1)x^2 \\ &- 2n(n+1)(2n^2+4n-1)x + 2n(2n-1)(n+1)^2. \end{aligned} \tag{3.3}$$

To prove that $\mathcal{L}_t(a(n,0)) \geq 0$, we consider two cases:

Case 1: $t = n$. In this case, it suffices to show that $\theta(n) \leq 0$. But this is obvious for $n \geq 5$, since $\theta(n) = -n(n-1)(n-2)(n+1)$.

Case 2: $0 \leq t < n$. In this case, we need to show that $\theta(t) \geq 0$. To this end, treat $\theta(x)$ as a function of $x$ over the interval $[0, n-1]$. We have

$$\theta'(x) = 2(n-x)\theta_1(x),$$

where

$$\theta_1(x) =2(1-4n^2)x^2 + (2n-1)(2n^2+4n+3)x - (2n^3+6n^2+3n-1).$$

Moreover,

$$\theta_1'(x) = (2n-1)\theta_2(x),$$

where

$$\theta_2(x) = -4(2n+1)x + (2n^2+4n+3).$$

For $n \geq 5$,

$$\theta_2(0) = 2n^2+4n+3 > 0, \quad \theta_2(n-1) = -6n^2+8n+7 < 0.$$

Therefore, $\theta_2(x)$ decreases from a positive value to a negative value as $x$ increases from 0 to $n-1$. This implies that $\theta_1(x)$ first increases and then decreases over the interval $[0, n-1]$.

Observe that, for $n \geq 5$,

$$\begin{aligned} \theta_1(0) &= -2n^3 - 6n^2 - 3n + 1 < 0, \\ \theta_1(1) &= n(2(n-2)^2 - 9) > 0, \\ \theta_1(n-1) &= -4n^4 + 16n^3 - 16n^2 - 12n + 6 < 0. \end{aligned}$$

It follows that there exist $0 < x_1 < x_2 < n-1$ such that

$$\theta_1(x) \begin{cases} < 0, & \text{if } x \in [0, x_1), \\ \geq 0, & \text{if } x \in [x_1, x_2], \\ < 0, & \text{if } x \in (x_2, n-1]. \end{cases}$$

That is to say that $\theta(x)$ is decreasing on $[0, x_1)$, increasing on $[x_1, x_2]$, and decreasing on $(x_2, n-1]$.

It is easy to check that for $n \geq 5$,

$$\begin{aligned} \theta(0) &= 2n(2n-1)(n+1)^2 > 0, \\ \theta(1) &= 2n^2(2n-1)(n-1) > 0, \\ \theta(2) &= 2(n-2)(6n^3 - 13n^2 + 1) > 0, \\ \theta(n-1) &= -4 + 8n + 3n^4 - 10n^3 + 11n^2 > 0, \end{aligned}$$

and

$$\theta(0) > \theta(1) < \theta(2) > \theta(n-1).$$

So we see that $x_1 < 2$. If $x_2 > 2$, then $\theta(x)$ is increasing on $[2, x_2]$, and decreasing on $(x_2, n-1]$. If $x_2 \leq 2$, then $\theta(x)$ decreases on $(2, n-1]$. In either case, we obtain that $\theta(x) > 0$ for $x \in [2, n-1]$. Since $\theta(0) > 0$ and $\theta(1) > 0$, we deduce that $\theta(t) > 0$ for any integer $0 \leq t \leq n-1$. This completes the proof. ∎

**Lemma 3.4.** *Given $n \geq 2$ and $0 \leq t \leq n-1$, there exists an integer $k'$ depending on $n$ and $t$ such that*

$$\mathcal{L}_t(a(n,k)) \begin{cases} \geq 0, & \textit{if } 1 \leq k \leq k', \\ \leq 0, & \textit{if } k' < k \leq \frac{t}{2}. \end{cases}$$

*Proof.* For $n \geq 2$, $0 \leq t \leq n$ and $0 \leq k \leq t/2$, we have

$$\begin{aligned}\mathcal{L}_t(a(n,k)) =& \binom{n+1}{k}\binom{2n-2k+2}{n-k+1}\binom{n-1}{t-k}\binom{2n-2t+2k-2}{n-t+k-1}\\ &+\binom{n-1}{k}\binom{2n-2k-2}{n-k-1}\binom{n+1}{t-k}\binom{2n-2t+2k+2}{n-t+k+1}\\ &-2\binom{n}{k}\binom{2n-2k}{n-k}\binom{n}{t-k}\binom{2n-2t+2k}{n-t+k}.\end{aligned}$$

Write

$$\begin{aligned}\mathcal{L}_t(a(n,k)) =& \frac{1}{(n-k+1)^2(n-t+k+1)^2(2n-2k-1)(2n-2t+2k-1)}\\ &\times\frac{1}{n}\binom{n}{k}\binom{2n-2k}{n-k}\binom{n}{t-k}\binom{2n-2t+2k}{n-t+k}\psi^{(n,t)}(k), \quad (3.4)\end{aligned}$$

where

$$\begin{aligned}\psi^{(n,t)}(x) =& (n+1)(n-x)^2(n-x+1)^2(2n-2t+2x+1)(2n-2t+2x-1)\\ &+(n+1)(n-t+x)^2(n-t+x+1)^2(2n-2x-1)(2n-2x+1)\\ &-2n(n-x+1)^2(n-t+x+1)^2(2n-2x-1)(2n-2t+2x-1).\end{aligned} \quad (3.5)$$

Clearly, for $n \geq 2$, $0 \leq t \leq n-1$ and $1 \leq k \leq t/2$, the sign of $\mathcal{L}_t(a(n,k))$ coincides with that of $\psi^{(n,t)}(k)$. By (3.4) and Lemma 3.3, we see that $\psi^{(n,t)}(0) \geq 0$ when $0 \leq t \leq n-1$. Therefore, it suffices to show that there exists $0 \leq t_0 \leq t/2$ such that $\psi^{(n,t)}(x)$, regarded as a function of $x$, is increasing on the interval $[0, t_0)$ and decreasing on the interval $[t_0, t/2]$.

The derivative of $\psi^{(n,t)}(x)$ can be expressed as

$$(\psi^{(n,t)}(x))' = 2(2x-t)\psi_1^{(n,t)}(x),$$

where

$$\begin{aligned}\psi_1^{(n,t)}(x) =& 12(2n+1)x^4 - 24t(2n+1)x^3\\ &-2(16n^3-8(2t-1)n^2-2(7t^2+3t+1)n-(8t^2-4t+3))x^2\\ &+2t(16n^3-8(2t-1)n^2-2(t^2+3t+1)n-(2t^2-4t+3))x\\ &+\left(8n^5-4(4t-1)n^4+4(t^2-t-3)n^3+4(-t^2+5t+t^3-2)n^2\right.\\ &\left.+(4t^3-10t^2-1+11t)n-(2t^2-3t+1)\right).\end{aligned}$$

Moreover, we have

$$(\psi_1^{(n,t)}(x))' = 2(2x-t)\psi_2^{(n,t)}(x), \tag{3.6}$$

where

$$\begin{aligned}\psi_2^{(n,t)}(x) =& 12(2n+1)x^2 - 12t(2n+1)x - 16n^3 + 8(2t-1)n^2 \\ &+ 2(t^2+3t+1)n + (2t^2-4t+3).\end{aligned}$$

Notice that the quadratic function $\psi_2^{(n,t)}(x)$ is symmetric with respect to $x = t/2$. It follows that $\psi_2^{(n,t)}(x)$ decreases as $x$ increases from 0 to $t/2$.

It is routine to check that, for $n \geq 1$ and $0 \leq t < n$,

$$\psi_2^{(n,t)}(-\infty) > 0,$$

$$\psi_2^{(n,t)}\left(\frac{t}{2}\right) = -4n(2n-t)^2 - (4n-t-1)(2n-t) - 3(t-1) < 0,$$

then there exists a real zero $x_0$ of $\psi_2^{(n,t)}(x)$ on the interval $(-\infty, t/2]$.

If $x_0 \leq 0$, then we see that for $0 \leq x \leq t/2$, $\psi_2^{(n,t)}(x) \leq 0$, that is to say, $\psi_1^{(n,t)}(x)$ is increasing on $[0, t/2]$.

If $x_0 > 0$,

$$\psi_2^{(n,t)}(x) \begin{cases} > 0, & \text{if } 0 \leq x < x_0, \\ < 0, & \text{if } x_0 < x < t/2, \end{cases}$$

that is to say,

$$(\psi_1^{(n,t)}(x))' \begin{cases} < 0, & \text{if } 0 \leq x < x_0, \\ > 0, & \text{if } x_0 < x < t/2, \end{cases}$$

then $\psi_1^{(n,t)}(x)$ is decreasing on $[0, x_0]$ and increasing on $[x_0, t/2]$.

Using Maple, we find that for $n \geq 4$ and $0 \leq t < n$,

$$\begin{aligned}\psi_1^{(n,t)}\left(\frac{t}{2}\right) =& 8n^5 - 16n^4t + 12n^3t^2 - 4n^2t^3 + \frac{1}{2}nt^4 + 4n^4 - 4n^3t + nt^3 - \frac{1}{4}t^4 \\ &- 12n^3 + 20n^2t - 11nt^2 + 2t^3 - 8n^2 + 11nt - \frac{7}{2}t^2 - n + 3t - 1 \\ =& \left(\frac{1}{2}n - \frac{1}{4}\right)(2n-t)^4 + (n-2)(2n-t)^3 + \left(n - \frac{7}{2}\right)(2n-t)^2 \\ &+ 3(n-1)(2n-t) + 5n - 1 > 0,\end{aligned}$$

for $n = 2, 3$ and $0 \le t < n$,

$$\psi_1^{(2,t)}\left(\frac{t}{2}\right) = \frac{3}{4}\left((4-t)^2-1\right)^2 + 3(4-t) + \frac{33}{4} > 0,$$

$$\psi_1^{(3,t)}\left(\frac{t}{2}\right) = \frac{5}{4}(6-t)^4 + \left(\frac{11}{2}-t\right)(6-t)^2 + 6(6-t) + 14 > 0.$$

As can be seen, $\psi_1^{(n,t)}(t/2)$ is positive. Considering the value of $x_0$ and the sign of $\psi_1^{(n,t)}(0)$, there are three cases concerning the monotonicity of $\psi^{(n,t)}(x)$:

Case 1: $x_0 \le 0$ and $\psi_1^{(n,t)}(0) \ge 0$. In this case, $\psi_1^{(n,t)}(x)$ increases from a nonnegative value to a positive value as $x$ increases from 0 to $t/2$. Thus, $(\psi^{(n,t)}(x))'$ takes only nonpositive values on $[0, t/2]$. That is to say, $\psi^{(n,t)}(x)$ is decreasing on the interval $[0, t/2]$.

Case 2: $x_0 \le 0$ and $\psi_1^{(n,t)}(0) < 0$. In this case, $\psi_1^{(n,t)}(x)$ increases from a negative value to a positive value as $x$ increases from 0 to $t/2$. Therefore, there exists $0 < t_0 < t/2$ such that

$$\psi_1^{(n,t)}(x) \begin{cases} \le 0, & \text{if } 0 \le x \le t_0, \\ \ge 0, & \text{if } t_0 < x \le t/2. \end{cases}$$

Hence, we have

$$(\psi^{(n,t)}(x))' \begin{cases} \ge 0, & \text{if } 0 \le x \le t_0, \\ \le 0, & \text{if } t_0 < x \le t/2. \end{cases}$$

This implies that $\psi^{(n,t)}(x)$ is increasing on $[0, t_0]$ and decreasing on $[t_0, t/2]$.

Case 3: $0 < x_0 < t/2$. In this case, we claim that $\psi_1^{(n,t)}(0) < 0$. Based on this claim, we can deduce the monotonicity of $\psi^{(n,t)}(x)$ on $[0, t/2]$ by using the same argument as in case 2. To prove the claim, we note that the condition $0 < x_0 < t/2$ implies that $\psi_2^{(n,t)}(0) > 0$. So we proceed to prove that $\psi_1^{(n,t)}(0) < 0$ by using the positivity of $\psi_2^{(n,t)}(0)$. Using Maple, we find that

$$\begin{aligned} \psi_1^{(n,t)}(0) =& (n+1)\big(4nt^3 + 2(2n^2-4n-1)t^2 - (16n^3-12n^2-8n-3)t \\ & + (8n^4-4n^3-8n^2-1)\big), \\ \psi_2^{(n,t)}(0) =& 2(n+1)t^2 + 2(8n^2+3n-2)t - (2n-1)(8n^2+8n+3). \end{aligned}$$

By the assumption $0 \le t \le n-1$, we may regard $\psi_1^{(n,t)}(0)/(n+1)$ as a polynomial in $t$ over $[0, n-1]$. Denote this polynomial by $\xi(t)$. Similarly, treat $\psi_2^{(n,t)}(0)$ as a polynomial in $t$ and denote it by $\eta(t)$. We wish to show that $\xi(t) < 0$ for any $t$ satisfying $\eta(t) > 0$.

We claim that if $\eta(t) > 0$, then $n \geq 4$ and $t > 3n/4$. In fact, it is routine to check that $\eta(t) < 0$ (i.e., $\psi_2^{(n,t)}(0) < 0$) if

$$(n,t) \in \{(2,0),(2,1),(3,0),(3,1),(3,2)\}.$$

So $\eta(t) > 0$ implies $n \neq 2, 3$.

Moreover, we prove that $\eta(t) < 0$ for any $t \in [0, 3n/4]$.

The quadratic function $\eta(t)$ is symmetric with respect to

$$t = -\frac{8n^2 + 3n - 2}{2(n+1)} < 0,$$

which means that $\eta(t)$ is increasing on $[0, \frac{3}{4}n]$. Since

$$\eta(0) = -16n^3 - 8n^2 + 2n + 3 < 0,$$

$$\eta\left(\frac{3}{4}n\right) = -\frac{23}{8}n^3 - \frac{19}{8}n^2 - n + 3 < 0,$$

we see that $\eta(t) < 0$ on $[0, 3n/4]$, so $\eta(t) > 0$ implies $n \geq 4$ and $t > 3n/4$.

Now we show that for any integer $n \geq 4$, the polynomial $\xi(t)$ takes only negative values on the interval $(\frac{3}{4}n, n-1]$.

Consider the first order derivative and the second order derivative of $\xi(t)$ with respect to $t$,

$$\xi'(t) = 12nt^2 + (8n^2 - 16n - 4)t + (12n^2 - 16n^3 + 8n + 3),$$

$$\xi''(t) = 24nt + (8n^2 - 16n - 4).$$

Since $\xi''(\frac{3}{4}n) = 26n^2 - 16n - 4 > 0$, we have $\xi''(t) > 0$ for any $3n/4 < t \leq n-1$. Thus $\xi'(t)$ is strictly increasing on $(\frac{3}{4}n, n-1]$. Noting that

$$\xi'\left(\frac{3}{4}n\right) = -\frac{13}{4}n^3 + 5n + 3 < 0,$$

we deduce that there exists $3n/4 \leq t_1 \leq n-1$ such that

$$\xi'(t) \begin{cases} \leq 0, & \text{if } \frac{3}{4}n \leq t \leq t_1, \\ > 0, & \text{if } t_1 < t \leq n-1. \end{cases}$$

In view of

$$\xi\left(\frac{3}{4}n\right) = -\frac{1}{64}\left(4n^2(n-4)^2 + 136\left(n - \frac{9}{17}\right)^2 + \frac{440}{17}\right) < 0,$$

$$\xi(n-1) = -(4n-18)n^2 - 13n - 6 < 0,$$

we obtain that $\xi(t) < 0$ for any $t \in (\frac{3}{4}n, n-1]$.

Combining Cases 1, 2 and 3, we complete the proof. ∎

The above lemma is the key step in the proof of Theorem 3.2.

**Lemma 3.5.** *Given $n \geq 2$, there exists $k'$ depending on $n$ such that*

$$\mathcal{L}_n(a(n,k)) \begin{cases} \geq 0, & \text{if } 1 \leq k \leq k', \\ \leq 0, & \text{if } k' < k \leq \frac{n}{2}. \end{cases}$$

*Proof.* By (3.4) and (3.5), we obtain that for $n \geq 2$ and $1 \leq k \leq n/2$, the sign of $\mathcal{L}_n(a(n,k))$ coincides with that of

$$\begin{aligned}\psi^{(n,n)}(k) =& 8(2n+1)k^6 - 24n(2n+1)k^5 + 2(26n^3 - 2n + 12n^2 + 3)k^4 \\ & - 4n(3 + 6n^3 + 2n^2 - 2n)k^3 + 2(4n^2 + 2n - 1 - 4n^3 + 2n^5)k^2 \\ & + 2n(n-1)(2n-1)(n+1)k - n(n-1)(n-2)(n+1)^2.\end{aligned}$$

Since $\psi^{(2,2)}(1) = 8$, the lemma holds for $n = 2$. We now assume that $n \geq 3$. To determine the sign of $\psi^{(n,n)}(k)$, let us consider the derivative of $\psi^{(n,n)}(x)$ with respect to $x$. Using Maple, we get

$$(\psi^{(n,n)}(x))' = 2(2x-n)\psi_1^{(n,n)}(x),$$

where

$$\begin{aligned}\psi_1^{(n,n)}(x) =& 12(1+2n)x^4 - 24n(1+2n)x^3 + 2(6n^2 - 2n + 3 + 14n^3)x^2 \\ & - 2n(2n^3 + 3 - 2n)x - (n-1)(2n-1)(n+1).\end{aligned}$$

We also need to consider the derivative of $\psi_1^{(n,n)}(x)$ with respect to $x$:

$$(\psi_1^{(n,n)}(x))' = 2(2x-n)\psi_2^{(n,n)}(x),$$

where

$$\psi_2^{(n,n)}(x) = 12(1+2n)x^2 - 12n(1+2n)x + 2n^3 + 3 - 2n.$$

Note that the the quadratic function $\psi_2^{(n,n)}(x)$ is symmetric with respect to $x = n/2$. For $n \geq 3$,

$$\psi_2^{(n,n)}(0) = 2n^3 - 2n + 3 > 0,$$

$$\psi_2^{(n,n)}(n/2) = -4n^3 - 3n^2 - 2n + 3 < 0.$$

Thus, $\psi_2^{(n,n)}(x)$ decreases from a positive value to a negative value as $x$ increases from 0 to $n/2$. Hence, there exists $0 < x_0 < n/2$ such that

$$(\psi_1^{(n,n)}(x))' \begin{cases} \leq 0, & \text{if } 0 \leq x \leq x_0, \\ \geq 0, & \text{if } x_0 < x \leq n/2. \end{cases}$$

Noting that

$$\psi_1^{(n,n)}(0) = -n^2(n-1) - n(n^2-2) - 1 < 0,$$

$$\psi_1^{(n,n)}(n/2) = \frac{1}{4}(2n^3(n^2-2) + n^2(3n^2-2) + 4(2n-1)) > 0,$$

there exists $0 < x_1 < n/2$ such that

$$\psi_1^{(n,n)}(x) \begin{cases} \leq 0, & \text{if } 0 \leq x \leq x_1, \\ \geq 0, & \text{if } x_1 < x \leq n/2. \end{cases}$$

Therefore,

$$(\psi^{(n,n)}(x))' \begin{cases} \geq 0, & \text{if } 0 \leq x \leq x_1, \\ \leq 0, & \text{if } x_1 < x \leq n/2, \end{cases}$$

and hence $\psi^{(n,n)}(x)$ is increasing on $[0, x_1]$ and decreasing on $(x_1, n/2]$.

Moreover, for $n \geq 3$, we have

$$\psi^{(n,n)}(1) = (n-1)((3n-16)n^3 + (21n^2 + 8n - 12)) > 0,$$

$$\psi^{(n,n)}(n/2) = -\frac{1}{8}n(n-1)(n^2-n-4)(n+2)^2 < 0.$$

Thus there exists $1 < x_2 < n/2$ such that

$$\psi^{(n,n)}(x) \begin{cases} \geq 0, & \text{if } 1 \leq x \leq x_2, \\ \leq 0, & \text{if } x_2 < x \leq n/2. \end{cases}$$

Since for $n \geq 2$ and $1 \leq k \leq n/2$, $\mathcal{L}_n(a(n,k))$ has the same sign as $\psi^{(n,n)}(k)$, there exists $k'$ depending on $n$ such that $\mathcal{L}_n(a(n,k)) \geq 0$ for $1 \leq k \leq k'$ and $\mathcal{L}_n(a(n,k)) \leq 0$ for $k' < k \leq n/2$. This completes the proof. ∎

We are now ready to prove Theorem 3.2.

*Proof of Theorem 3.2.* By Lemma 3.3, for any $n \geq 1$ and $0 \leq t \leq n$, we have $\mathcal{L}_t(a(n,0)) \geq 0$. It remains to prove that, for any $n \geq 2$ and $0 \leq t \leq n$, there exists $k'$ such that $\mathcal{L}_t(a(n,k)) \geq 0$ for $1 \leq k \leq k'$ and $\mathcal{L}_t(a(n,k)) \leq 0$ for $k' < k \leq t/2$. In Lemma 3.4, we have considered the case $0 \leq t \leq n-1$, whereas the case $t = n$ has been dealt with in Lemma 3.5, and hence the proof is complete. ∎

Combining Theorems 2.1, 3.1 and 3.2, we reach the following conclusion.

**Theorem 3.6.** *The polynomial sequence $\{S_n(q)\}_{n\geq 0}$ is q-log-convex.*

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# References

[1] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, in: Contemp. Math., 178 (1994), 71–89.

[2] L.M. Butler, The $q$-log-concavity of $q$-binomial coefficients, J. Combin. Theory Ser. A 54 (1990), 54–63.

[3] L.M. Butler and W.P. Flanigan, A note on log-convexity of $q$-Catalan numbers, Ann. Combin. 11 (2007), 369–373.

[4] W.Y.C. Chen, R.L. Tang, L.X.W. Wang and A.L.B. Yang, The $q$-log-convexity of the Narayana polynomials of type B, Adv. Appl. Math. 44 (2010), 85–110.

[5] W.Y.C. Chen, L.X.W. Wang and A.L.B. Yang, Schur positivity and the $q$-log-convexity of the Narayana polynomials, J. Alegbraic Combin. 32 (2010), 303–338.

[6] W.Y.C. Chen, L.X.W. Wang and A.L.B. Yang, Recurrence relations for strongly $q$-log-convex polynomials, Canad. Math. Bull. 54 (2011), 217–229.

[7] C. Krattenthaler, On the $q$-log-concavity of Gaussian binomial coefficients, Monatsh. Math. 107 (1989), 333–339.

[8] P. Leroux, Reduced matrices and $q$-log-concavity properties of $q$-Stirling numbers, J. Combin. Theory Ser. A 54 (1990), 64–84.

[9] L.L. Liu and Y. Wang, On the log-convexity of combinatorial sequences, Adv. Appl. Math. 39 (2007), 453–476.

[10] B.E. Sagan, Inductive proofs of $q$-log concavity, Discrete Math. 99 (1992), 289–306.

[11] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Ann. New York Acad. Sci. 576 (1989), 500–535.

[12] Z.W. Sun, List of conjectural series for powers of $\pi$ and other constants, arXiv:1102.5649.