Tight upper bound of the rainbow vertex-connection number for 2-connected graphs*

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Abstract

The *rainbow vertex-connection number*, $rvc(G)$, of a connected graph G is the minimum number of colors needed to color its vertices such that every pair of vertices is connected by at least one path whose internal vertices have distinct colors. In this paper we prove that for a 2-connected graph G of order n,

$$rvc(G) \leq \begin{cases} \lceil n/2 \rceil - 2 & \text{if } n = 3, 5, 9 \\ \lceil n/2 \rceil - 1 & \text{if } n = 4, 6, 7, 8, 10, 11, 12, 13 \text{ or } 15 \\ \lceil n/2 \rceil & \text{if } n \geq 16 \text{ or } n = 14. \end{cases}$$

The upper bound is tight since the cycle C_n on n vertices has its $rvc(C_n)$ equal to this bound.

Keywords: rainbow vertex coloring, rainbow vertex-connection number, ear decomposition, 2-connected graph.

AMS subject classification 2010: 05C40, 05C15.

1 Introduction

All graphs in this paper are finite, undirected and simple. We follow the terminology and notation of Bondy and Murty [1]. An *edge coloring* of a graph is a function from its edge set to the set of natural numbers. A path in an edge-colored graph with no two edges sharing the same color is called a *rainbow path*. An edge-colored graph is said to

*Supported by NSFC No. 11371205 and 11071130, and the "973" program.

be *rainbow connected* if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a *rainbow coloring* of the graph. The minimum number of colors required to rainbow color a connected graph is called its *rainbow connection number*, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, and that of a tree is the number of edges in the tree. For a basic introduction to the topic, see Chapter 11 in [4]. For more results, see [2, 3, 7, 9, 12, 13, 15, 16, 17, 18] and [11, 14].

The above is an edge-version of the rainbow connection of a graph. A vertex-version of it was introduced by Krivelevich and Yuster in [7]. Let G be a vertex-colored connected graph. A path of G is a *rainbow path* if its *internal* vertices have distinct colors. The vertex-colored graph G is called *rainbow vertex-connected* if any two vertices are connected by at least one rainbow path and the vertex coloring is called *a rainbow vertex coloring* of G. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. If F is a subgraph of a graph with a vertex coloring c, we denote the set of all colors appearing on F by $c(F)$. Note that an uncolored graph is also thought as a special vertex-colored graph with 0 colors.

Some easy observations about the rainbow vertex-connection number include that if G is a connected graph of order n, then $diam(G) - 1 \leq rvc(G) \leq n - 2$; $rvc(G) = 0$ if and only if G is a complete graph; $rvc(G) = 1$ if and only if $diam(G) = 2$ and if G' is a connected spanning subgraph of G, then $rvc(G) \leq rvc(G')$. Note that the parameters $rc(G)$ and $rvc(G)$ are independent of each other. Indeed, $rvc(G)$ may be much smaller than $rc(G)$ for some graphs G. For example, $rvc(K_{1,n-1}) = 1$ while $rc(K_{1,n-1}) = n - 1$. Moreover, $rvc(G)$ may also be much larger than $rc(G)$ for some graphs G. For example, take n vertex-disjoint triangles and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph has n cut-vertices and hence $rvc(G) \geq n$. In fact, $rvc(G) = n$ by coloring only the cut-vertices with distinct colors. On the other hand, it is not difficult to see that $rc(G) \leq 4$. Just color the edges of the K_n with, say color 1, and color the edges of each triangles with the colors 2, 3, 4. In fact, one can show that $rc(G) = 4$ when $n \geq 4$.

Krivelevich and Yuster [7] showed that if a connected graph G has n vertices and minimum degree δ, then $rvc(G) \leq 11n/\delta$. In [10], Li and Shi improved the bound. In [6, 5], Chen et al. studied the computational complexity of rainbow vertex-connection and proved that computing $rvc(G)$ is NP-hard.

In [8], we obtained a tight upper bound of the rainbow connection number for 2-connected graphs. This paper is to investigate the upper bound of its rainbow vertex-connection number. The following notation and terminology are needed in the sequel.

Let F be a subgraph of a graph G. An *ear* of F in G is a nontrivial path whose two ends are in F but whose internal vertices are not. A nested sequence of graphs is a sequence $G_0, G_1, \cdots, G_k$ of graphs such that $G_i \subset G_{i+1}, 0 \leq i \leq k-1$. An *ear decomposition* of a 2-connected graph G is a nested sequence $G_0, G_1, \cdots, G_k$ of 2-connected subgraphs of G satisfying the following conditions: (1) G_0 is a cycle; (2) $G_i = G_{i-1} \bigcup P_i$, where P_i is an ear of G_{i-1} in $G, 1 \leq i \leq k$; (3) $G_k = G$. Note that the two end vertices of P_i $(1 \leq i \leq k)$ are distinct and that if G is minimal 2-connected then its ear decompositions do not contain any ears of length 1.

A maximal connected subgraph of a graph G without any cut vertex is called a *block* of G. Thus, every block of a nontrivial connected graph is either a maximal 2-connected subgraph or a K_2. All the blocks of a graph G form a *block decomposition* of G. Given a graph G, a set $D \subseteq V(G)$ is called a *k-step dominating set* of G, if every vertex in G is at a distance at most k from D. For two vertices v_i and v_j on a walk W, v_iWv_j denotes the segment of W from v_i to v_j. Let $W_1 = u_0u_1 \cdots u_k$ and $W_2 = v_0v_1 \cdots v_\ell$ be two walks such that $u_k = v_0$, and v_i is a vertex on W_2. Then $W_1(v_0W_2v_i)$ denotes a walk obtained by concatenating W_1 and the segment $v_0W_2v_i$ of W_2.

Since every 2-connected graph can be constructed from a cycle by adding ears inductively, we first determine the rainbow vertex-connection number $rvc(C_n)$ of a cycle C_n $(n \geq 3)$. Based on it, we then prove that for any 2-connected graph G of order $n \geq 3$, $rvc(G) \leq rvc(C_n)$. Thus the bound is tight since C_n is 2-connected.

2 Main results

As an inductive basis, we first determine the rainbow vertex-connection number of a cycle.

Theorem 2.1. *Let C_n be a cycle of order $n \geq 3$. Then*

$$rvc(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 2 & \text{if } n = 3, 5, 9 \\ \lceil \frac{n}{2} \rceil - 1 & \text{if } n = 4, 6, 7, 8, 10, 11, 12, 13 \text{ or } 15 \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 16 \text{ or } n = 14. \end{cases}$$

Proof. Assume that $C_n = v_1v_2 \cdots v_nv_{n+1}(= v_1)$ $(n \geq 3)$. It is obvious that $rvc(C_3) = 0$. Since $rvc(G) = 1$ if and only if $diam(G) = 2$, we have $rvc(C_4) = rvc(C_5) = 1$.

It is easy to check that the vertex colorings of C_n shown in Figure 1 are rainbow vertex colorings. So $rvc(C_n) \leq \lceil \frac{n}{2} \rceil - 1$ for $6 \leq n \leq 13$ and $n = 15$ and $rvc(C_9) \leq 3$. Since $rvc(C_n) \geq diam(C_n) - 1$, $rvc(C_n) = \lceil \frac{n}{2} \rceil - 1$ for $n = 6, 8, 10, 12$ and $rvc(C_9) = 3$. For any vertex coloring c of C_7 using 2 colors, there exist two adjacent vertices (say v_1, v_2) having

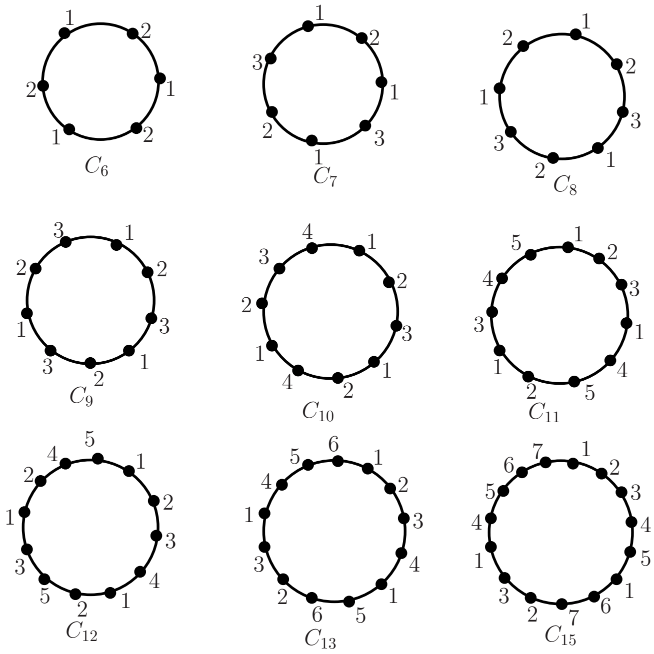

Figure 1. Rainbow vertex colorings for small cycles.

the same color. Then neither $P_1 = v_7v_1v_2v_3$ nor $P_2 = v_7v_6v_5v_4v_3$ on C_n is a rainbow path, i.e., there is no rainbow path between v_7 and v_3. So c is not a rainbow vertex coloring of G. Hence, $rvc(C_7) = 3$.

Assume, to the contrary, that C_n ($n = 11, 13, 15$) has a rainbow vertex coloring c with $\lceil \frac{n}{2} \rceil - 2$ colors. Then some three vertices (say $v_1, v_i, v_j \in V(C_n), 1 < i < j \leq n$) have the same color and one pair of vertices among them (say v_1, v_i) has distance no more than $\lfloor \frac{n}{3} \rfloor$, i.e., $d_{C_n}(v_1, v_i) \leq \lfloor \frac{n}{3} \rfloor$. Suppose that $P = v_1v_2 \cdots v_i$ is the path on C_n from v_1 to v_i with length $d_{C_n}(v_1, v_i)$. Since $c(v_1) = c(v_i)$, $P_1 = v_nv_1 \cdots v_iv_{i+1}$ is not a rainbow path. So $C_n - P = v_nv_{n-1} \cdots v_{i+1}$ is the rainbow path on C_n from v_n to v_{i+1}. Since $\ell(C_n - P) = n - (\ell(P) + 2) \geq n - \lfloor \frac{n}{3} \rfloor - 2 = \lceil \frac{n}{2} \rceil$ for $n = 11, 13, 15$, $C_n - P$ has $\lceil \frac{n}{2} \rceil - 1$ internal vertices. So $C_n - P$ is not a rainbow path, a contradiction. Hence, $rvc(C_n) = \lceil \frac{n}{2} \rceil - 1$ for $n = 11, 13, 15$.

In the following, we consider the rainbow vertex-connection number of C_n for $n \geq 16$ or $n = 14$. Define a vertex coloring c of C_n by $c(v_i) = x_i$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and $c(v_i) = x_{i - \lceil \frac{n}{2} \rceil}$ if $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$. Since for any two vertices u, v of C_n, the path on C_n with length $d_{C_n}(u, v)$ is a rainbow path, c is a rainbow vertex coloring of C_n. Hence, $rvc(C_n) \leq \lceil \frac{n}{2} \rceil$ for $n \geq 16$ or $n = 14$.

Next, we show that $rvc(C_n) \geq \lceil \frac{n}{2} \rceil$ for $n \geq 16$ or $n = 14$. Assume, to the contrary, that $rvc(C_n) \leq \lceil \frac{n}{2} \rceil - 1$. Then there exists a rainbow vertex coloring c of C_n with $\lceil \frac{n}{2} \rceil - 1$ colors. Obviously, there are three vertices (say $v_1, v_i, v_j, 1 < i < j \leq n$) of C_n having the same color. And one pair of vertices among $\{v_1, v_i, v_j\}$ (say v_1, v_i) satisfy that $d_{C_n}(v_1, v_i) \leq \lfloor \frac{n}{3} \rfloor$. Without loss of generality, assume that $P = v_1 v_2 \cdots v_i$ is the path on C_n with length $d_{C_n}(v_1, v_i)$. Now consider the vertices v_n and v_{i+1}. Since v_1 and v_i have the same color, the rainbow path between v_n and v_{i+1} on C_n must be $C_n - P = v_n v_{n-1} \cdots v_{i+2} v_{i+1}$. Since $\ell(C_n - P) = n - (\ell(P) + 2) \geq n - \lfloor \frac{n}{3} \rfloor - 2$, the number of internal vertices of $C_n - P$ is at least $n - \lfloor \frac{n}{3} \rfloor - 3$. For $n \geq 16$ or $n = 14$, $n - \lfloor \frac{n}{3} \rfloor - 3 > \lceil \frac{n}{2} \rceil - 1$ which contradicts that $C_n - P$ is a rainbow path. Hence, $rvc(C_n) \geq \lceil \frac{n}{2} \rceil$ for $n \geq 16$ or $n = 14$. Therefore, $rvc(C_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 16$ or $n = 14$. □

Now, we need to introduce the concept of strict rainbow vertex coloring which will be used in the following proofs.

Definition 2.1. Let G be a connected graph with a vertex coloring c. A path P of G is called a *strict rainbow path* if it is a rainbow path and the colors of its end vertices do not appear on its internal vertices. The vertex coloring c of G is called a *strict rainbow vertex coloring* if any two vertices of G are connected by at least one strict rainbow path. The *strict rainbow vertex-connection number* of a connected graph G, denoted by $rvc^*(G)$, is the smallest number of colors that are needed in order to make G strict rainbow vertex-connected.

Since a strict rainbow path is also a rainbow path, $rvc(G) \leq rvc^*(G)$ for any connected graph G.

Let G be a connected graph and a, b be two nonadjacent vertices of G. Assume that c is a vertex coloring of G and x is a color of c satisfying that $c(a) \neq x$ and $c(b) \neq x$. We say that c has *the property* $P(x, a, b)$, if for any vertex u of G, there exists a rainbow path P from u to one of a and b such that all vertices of P have distinct colors and $x \notin c(P)$ if $c(u) \neq x$. The order of a graph G is denoted by $|G|$ in the following. A rainbow vertex coloring or a strict rainbow vertex coloring of a graph G with k colors is called *equitable* if each color occurs on $\lfloor |G|/k \rfloor$ or $\lceil |G|/k \rceil$ vertices. In particular, in an equitable rainbow vertex coloring with $\lceil |G|/2 \rceil$ colors each color appears at most twice.

Lemma 2.1. Let H be a connected graph and $P = v_1 v_2 \cdots v_s$ ($s \geq 6$) be an ear of H such that $V(H) \bigcap V(P) = \{v_1, v_s\}$. Suppose that H has an equitable strict rainbow vertex coloring c_H with $\lceil |H|/2 \rceil$ colors, and moreover, when $|H|$ is odd, for the color x_0 that appears only once on H, $c_H(v_1) \neq x_0$ and $c_H(v_s) \neq x_0$, and c_H has the property $P(x_0, v_1, v_s)$. Then for any two nonadjacent vertices $a, b \in V(G)$, we have a vertex coloring c_G of $G := H \bigcup P$ satisfying the following conditions:

(a) c_G is an equitable strict rainbow vertex coloring with $\lceil |G|/2 \rceil$ colors.

(b) When $|G|$ is odd, for the color x that appears only once on G, $c_G(a) \neq x$ and $c_G(b) \neq x$, and c_G has the property $P(x, a, b)$.

Proof. We will prove the result by demonstrating a vertex coloring c_G of G satisfying the required conditions. Let $x_1, x_2, \cdots$ be new colors. We distinguish the following cases according to the parities of $|H|$ and s.

Case 1. $|H|$ and s are even.

In this case, $|G|$ is even. Define a vertex coloring c_G of G as follows. $c_G(v) = c_H(v)$ for $v \in V(H)\backslash\{v_1, v_s\}$ and the last $s/2$ vertices of P, i.e., $v_{s/2+1}, \cdots, v_s$ are colored by $c_H(v_s), x_1, \cdots, x_{s/2-1}$ in order. If $c_H(v_1) \neq c_H(v_s)$, the first $s/2$ vertices of P, i.e., $v_1, \cdots, v_{s/2}$ are colored by $x_1, \cdots, x_{s/2-1}, c_H(v_1)$ in order; otherwise, by $c_H(v_1), x_1, \cdots,$ $x_{s/2-1}$ in order. From the definition, the obtained vertex coloring c_G of G uses $|G|/2$ colors such that every color appears twice.

Now we prove that G is strict rainbow vertex-connected. Let v', v'' be any two vertices of G. If $v', v'' \in V(H)\backslash\{v_1, v_s\}$, there exists a strict rainbow path P_0 between v' and v'' in H with respect to c_H. From the definition of c_G and $x_1 \neq x_{s/2-1}$, P_0 is also a strict rainbow path between v' and v'' with respect to c_G. Let P' be a strict rainbow path in H from v_1 to v_s with respect to c_H. Then $P' \bigcup P$ is a cycle. If $v', v'' \in V(P)$, then there exists a strict rainbow path from v' to v'' on $P' \bigcup P$. Assume that $v' \in V(H)\backslash\{v_1, v_s\}$ and $v'' \in V(P)$. Let P_1 (resp. P_2) be a strict rainbow path in H from v' to v_1 (resp. v_s). If $v'' \in V(v_1 P v_{s/2})$, then $P_1(v_1 P v'')$ is a strict rainbow path from v' to v''. If $v'' \in V(v_s P v_{s/2+1})$, then $P_2(v_s P v'')$ is a strict rainbow path from v' to v''. Therefore, c_G is a required strict rainbow vertex coloring of G.

Case 2. $|H|$ and s are odd.

In this case, $|G|$ is even. For the color x_0 that appears only once on H, $c_H(v_1) \neq x_0$ and $c_H(v_s) \neq x_0$, and c_H has the property $P(x_0, v_1, v_s)$. Define a vertex coloring c_G of G as follows. $c_G(v) = c_H(v)$ for $v \in V(H)\backslash\{v_1, v_s\}$ and the last $\lceil s/2 \rceil - 1$ vertices of P, i.e., $v_{\lceil s/2 \rceil+1}, \cdots, v_s$ are colored by $c_H(v_s), x_1, \cdots, x_{\lceil s/2 \rceil-2}$ in order. If $c_H(v_1) \neq c_H(v_s)$, the first $\lceil s/2 \rceil$ vertices of P, i.e., $v_1, v_2, \cdots, v_{\lceil s/2 \rceil}$ are colored by $x_1, \cdots, x_{\lceil s/2 \rceil-2}$, $c_H(v_1)$, x_0 in order; otherwise, by $c_H(v_1), x_1, \cdots, x_{\lceil s/2 \rceil-2}, x_0$ in order. From the definition, the vertex coloring c_G of G uses $|G|/2$ colors such that every color appears twice on G.

Now we show that c_G is a strict rainbow vertex coloring of G. Since other cases can be proved similar to Case 1, we just need to prove that $v_{\lceil s/2 \rceil}$ has a strict rainbow path to any vertex v in $V(H)\backslash\{v_1, v_s\}$. Since c_H has the property $P(x_0, v_1, v_s)$, there exists a rainbow path P_0 in H from v to one of v_1, v_s (say v_1) with respect to c_H such that $x_0 \notin c_H(P_0)$ if $c_H(v) \neq x_0$. Hence, $P_0(v_1 P v_{\lceil s/2 \rceil})$ is a strict rainbow path from v to $v_{\lceil s/2 \rceil}$. Therefore, G is strict rainbow vertex-connected.

Case 3. $|H|$ is even and s is odd.

In this case, $|G|$ is odd. We consider the following cases.

Subcase 3.1. $a, b \in V(H)\backslash\{v_1, v_s\}$.

Define a vertex coloring c_G of G as follows. $c_G(v) = c_H(v)$ for $v \in V(H)\backslash\{v_1, v_s\}$ and the last $\lceil s/2\rceil - 1$ vertices of P are colored by $c_H(v_s), x_1, \cdots, x_{\lceil s/2\rceil-2}$ in order. If $c_H(v_1) \neq c_H(v_s)$, the first $\lceil s/2\rceil$ vertices of P are colored by $x_1, \cdots, x_{\lceil s/2\rceil-2}, c_H(v_1), x_{\lceil s/2\rceil-1}$ in order; otherwise, by $c_H(v_1), x_1, \cdots, x_{\lceil s/2\rceil-1}$ in order. Obviously, the vertex coloring c_G of G uses $\lceil|G|/2\rceil$ colors such that $x_{\lceil s/2\rceil-1}$ appears once and every other color appears twice on G. Similar to Case 1, G is strict rainbow vertex-connected. From the definition of c_G, it is obvious that $c_G(a) \neq x_{\lceil s/2\rceil-1}$ and $c_G(b) \neq x_{\lceil s/2\rceil-1}$.

Now we prove that c_G has the property $P(x_{\lceil s/2\rceil-1}, a, b)$. Let u be any vertex of G. If $u = a$ or $u = b$, there exists a trivial rainbow path P_u from u to one of a and b such that all vertices of P have distinct colors and $x_{\lceil s/2\rceil-1} \notin c_G(P_u)$. Assume that $u \neq a$ and $u \neq b$. We distinguish the following three cases. (1) Assume that $u \in V(H)\backslash\{v_1, v_s\}$. Since every color appears at most twice, we have $c_G(u) \neq c_G(a)$ or $c_G(u) \neq c_G(b)$. Hence, there exists a rainbow path P_u in H from u to one of a, b such that all its vertices have distinct colors and $x_{\lceil s/2\rceil-1} \notin c_G(P_u)$. (2) Assume that $u \in V(v_1Pv_{\lceil s/2\rceil})$. Since $c_H(v_1) \neq c_H(a)$ or $c_H(v_1) \neq c_H(b)$, without loss of generality, assume $c_H(v_1) \neq c_H(a)$. There exists a rainbow path P_a from a to v_1 in H whose vertices have distinct colors with respect to c_H. Hence, $P_a(v_1Pu)$ is a rainbow path from a to u such that all vertices of $P_a(v_1Pu)$ have distinct colors and $x_{\lceil s/2\rceil-1} \notin c_G(P_a(v_1Pu))$ if $c_G(u) \neq x_{\lceil s/2\rceil-1}$. (3) If $u \in V(v_sPv_{\lceil s/2\rceil})$, we can prove the result similarly.

Subcase 3.2. Exactly one of a, b belongs to $V(H)\backslash\{v_1, v_s\}$.

Without loss of generality, assume that $a \in V(H)\backslash\{v_1, v_s\}$ and $b \in V(v_1Pv_{\lceil s/2\rceil})$. Define a vertex coloring c_G of G as follows. $c_G(v) = c_H(v)$ for $v \in V(H)\backslash\{v_1, v_s\}$ and the last $\lceil s/2\rceil - 1$ vertices of P are colored by $x_{\lceil s/2\rceil-1}, x_1, \cdots, x_{\lceil s/2\rceil-2}$ in order. If $c_H(v_1) \neq c_H(v_s)$, the first $\lceil s/2\rceil$ vertices of P are colored by $x_1, \cdots, x_{\lceil s/2\rceil-2}, c_H(v_1), c_H(v_s)$ in order; otherwise, by $c_H(v_1), x_1, \cdots, x_{\lceil s/2\rceil-2}, c_H(v_s)$ in order. Obviously, the vertex coloring c_G of G uses $\lceil|G|/2\rceil$ colors such that $x_{\lceil s/2\rceil-1}$ appears once and every other color appears twice on G. Similar to Case 1, c_G is a strict rainbow vertex coloring of G. From the definition of c_G, it is obvious that $c_G(a) \neq x_{\lceil s/2\rceil-1}$ and $c_G(b) \neq x_{\lceil s/2\rceil-1}$.

Now we show that c_G has the property $P(x_{\lceil s/2\rceil-1}, a, b)$. Let u be any vertex of G. If $u = v_1$, then uPb is a rainbow path on P from u to b such that all its vertices have distinct colors and $x_{\lceil s/2\rceil-1} \notin c_G(v_1Pb)$. If $u = a$, it holds trivially. Assume that $u \neq a$ and $u \neq v_1$. We distinguish the following three cases.

(1) Assume that $u \in V(H)\backslash\{v_1, v_s\}$. Since every color of c_H appears at most twice on H,

we have $c_H(u) \neq c_H(a)$ or $c_H(u) \neq c_H(v_1)$. If $c_H(u) \neq c_H(a)$, there exists a rainbow path P_u from u to a in H such that all vertices of P_u have distinct colors and $x_{\lceil s/2 \rceil -1} \notin c_G(P_u)$. If $c_H(u) \neq c_H(v_1)$, there exists a rainbow path P_u from u to v_1 in H whose vertices have distinct colors with respect to c_H. So $P_u(v_1Pb)$ is a rainbow path from u to b such that all vertices of $P_u(v_1Pb)$ have distinct colors and $x_{\lceil s/2 \rceil -1} \notin c_G(P_u(v_1Pb))$.

(2) If $u \in V(v_1Pv_{\lceil s/2 \rceil +1})$, then uPb is a rainbow path from u to b such that all vertices of P have distinct colors and $x_{\lceil s/2 \rceil -1} \notin c_G(uPb)$ if $c_G(u) \neq x_{\lceil s/2 \rceil -1}$.

(3) Assume that $u \in V(v_sPv_{\lceil s/2 \rceil +2})$. Let P_a be a strict rainbow path from a to v_s in H. Then $P_a(v_sPu)$ is a rainbow path from a to u such that all vertices of $P_a(v_sPu)$ have distinct colors and $x_{\lceil s/2 \rceil -1} \notin c_G(P_a(v_sPu))$.

Subcase 3.3. One of a, b belongs to $V(v_1Pv_{\lceil s/2 \rceil -1})$ and the other belongs to $V(v_sP v_{\lceil s/2 \rceil +1})$.

Without loss of generality, assume that $a \in V(v_1Pv_{\lceil s/2 \rceil -1})$ and $b \in V(v_sPv_{\lceil s/2 \rceil +1})$. Define a vertex coloring c_G of G as follows. $c_G(v) = c_H(v)$ for $v \in V(H) \backslash \{v_1, v_s\}$ and the last $\lceil s/2 \rceil - 1$ vertices of P are colored by $c_H(v_s), x_1, \cdots, x_{\lceil s/2 \rceil -2}$ in order. If $c_H(v_1) \neq c_H(v_s)$, the first $\lceil s/2 \rceil$ vertices of P are colored by $x_1, \cdots, x_{\lceil s/2 \rceil -2}, c_H(v_1), x_{\lceil s/2 \rceil -1}$ in order; otherwise, by $c_H(v_1), x_1, \cdots, x_{\lceil s/2 \rceil -1}$ in order. Similarly, c_G is a strict rainbow vertex coloring of G with $\lceil |G|/2 \rceil$ colors such that $x_{\lceil s/2 \rceil -1}$ appears once and every other color appears twice on G. Obviously, $c_G(a) \neq x_{\lceil s/2 \rceil -1}$ and $c_G(b) \neq x_{\lceil s/2 \rceil -1}$.

Now we show that c_G has the property $P(x_{\lceil s/2 \rceil -1}, a, b)$. Let u be any vertex of G. If $u \in V(v_1Pv_{\lceil s/2 \rceil})$, then uPa is a rainbow path from u to a such that all vertices of uPa have distinct colors and $x_{\lceil s/2 \rceil -1} \notin c_G(uPa)$ if $c_G(u) \neq x_{\lceil s/2 \rceil -1}$. If $u \in V(v_sPv_{\lceil s/2 \rceil +1})$, then uPb is a required rainbow path from u to b. Assume that $u \in V(H) \backslash \{v_1, v_s\}$. Since every color of c_H appears at most twice in H, we have $c_H(u) \neq c_H(v_1)$ or $c_H(u) \neq c_H(v_s)$. Let P_u be a rainbow path from u to v_1 in H whose vertices have distinct colors. If $c_H(u) \neq c_H(v_1)$, then $P_u(v_1Pa)$ is a rainbow path from u to a such that all its vertices have distinct colors and $x_{\lceil s/2 \rceil -1} \notin c_G(P_u(v_1Pa))$. If $c_H(u) \neq c_H(v_s)$, then the required rainbow path exists similarly.

Subcase 3.4. $a, b \in V(v_1Pv_{\lceil s/2 \rceil})$ or $a, b \in V(v_sPv_{\lceil s/2 \rceil})$.

Without loss of generality, assume that $a, b \in V(v_1Pv_{\lceil s/2 \rceil})$ and $a = v_i, b = v_j$ ($1 \leq i < j \leq \lceil s/2 \rceil$). Since a, b are nonadjacent, we have that $i \leq j + 2 \leq \lceil s/2 \rceil$, i.e., $i \leq \lceil s/2 \rceil - 2$. Define a vertex coloring c_G of G as follows. $c_G(v) = c_H(v)$ for $v \in V(H) \backslash \{v_1, v_s\}$. If $c_H(v_1) \neq c_H(v_s)$, the first $\lceil s/2 \rceil - 1$ vertices of P are colored by $x_1, \cdots, x_{\lceil s/2 \rceil -2}, c_H(v_1)$ in order; otherwise, by $c_H(v_1), x_1, \cdots, x_{\lceil s/2 \rceil -2}$ in order. If $b = v_j$ with $j \leq \lceil s/2 \rceil - 1$ and $c_G(b) = x_k$, then color the last $\lceil s/2 \rceil$ vertices of P by $c_H(v_s), x_1, \cdots, x_{k-1}, x_{\lceil s/2 \rceil -1}, x_k, \cdots, x_{\lceil s/2 \rceil -2}$ in order; otherwise, by $c_H(v_s), x_1, \cdots, x_{\lceil s/2 \rceil -1}$

in order. It can be checked that c_G is a strict rainbow vertex coloring of G with $\lceil |G|/2 \rceil$ colors such that $x_{\lceil s/2 \rceil -1}$ appears once and every other color appears twice on G. Obviously, $c_G(a) \neq x_{\lceil s/2 \rceil -1}$ and $c_G(b) \neq x_{\lceil s/2 \rceil -1}$.

Now we show that c_G has the property $P(x_{\lceil s/2 \rceil -1}, a, b)$. Let u be any vertex of G. Let P' be a strict rainbow path from v_1 to v_s in H. If $u \in V(P)$, then there exists a rainbow path P_u on $P' \bigcup P$ from u to one of a, b such that all vertices of P_u have distinct colors and $x_{\lceil s/2 \rceil -1} \notin c_G(P_u)$ if $c_G(u) \neq x_{\lceil s/2 \rceil -1}$. If $u \in V(H) \backslash \{v_1, v_s\}$, there exists a rainbow path P_u through v_1 from u to a such that all its vertices have distinct colors and $x_{\lceil s/2 \rceil -1} \notin c_G(P_u)$.

Case 4. $|H|$ is odd and s is even.

In this case, $|G|$ is odd. For the color x_0 that appears only once on H, $c_H(v_1) \neq x_0$ and $c_H(v_s) \neq x_0$, and c_H has the property $P(x_0, v_1, v_s)$. We consider the following cases.

Subcase 4.1. $a, b \in V(H) \backslash \{v_1, v_s\}$.

Since $c_H(a) \neq x_0$ or $c_H(b) \neq x_0$, without loss of generality, assume that $c_H(a) \neq x_0$. From the property $P(x_0, v_1, v_s)$ of c_H, there exists a rainbow path P_a in H from a to one of v_1, v_s, say v_1, such that $x_0 \notin c_H(P_a)$. Define a vertex coloring c_G of G as follows. $c_G(v) = c_H(v)$ for $v \in V(H) \backslash \{v_1, v_s\}$ and the last $s/2$ vertices of P are colored by $x_{s/2-1}, c_H(v_s), x_1, \cdots, x_{s/2-2}$ in order. If $c_H(v_1) \neq c_H(v_s)$, the first $s/2$ vertices of P are colored by $x_1, \cdots, x_{s/2-2}, c_H(v_1), x_0$ in order; otherwise, by $c_H(v_1), x_1, \cdots, x_{s/2-2}, x_0$ in order.

Similarly, c_G is a strict rainbow vertex coloring of G with $\lceil |G|/2 \rceil$ colors such that $x_{s/2-1}$ appears once and every other color appears twice on G. Note that $P_a(v_1 P v_{s/2})$ is a rainbow path from a to $v_{s/2}$ such that all its vertices have distinct colors and $x_{s/2-1} \notin c_G(P_a(v_1 P v_{s/2}))$. It can be checked that $c_G(a) \neq x_{s/2-1}$ and $c_G(b) \neq x_{s/2-1}$, and c_G has the property $P(x_{s/2-1}, a, b)$.

Subcase 4.2. Exactly one of a, b belongs to $V(H) \backslash \{v_1, v_s\}$.

Without loss of generality, assume that $a \in V(H) \backslash \{v_1, v_s\}$ and $b \in V(v_1 P v_{s/2})$. Define a vertex coloring c_G of G as follows. $c_G(v) = c_H(v)$ for $v \in V(H) \backslash \{v_1, v_s\}$ and the last $s/2$ vertices of P are colored by $c_H(v_s), x_{s/2-1}, x_1, \cdots, x_{s/2-2}$ in order. If $c_H(v_1) \neq c_H(v_s)$, the first $s/2$ vertices of P are colored by $x_1, \cdots, x_{s/2-2}, c_H(v_1), x_0$ in order; otherwise, by $c_H(v_1), x_1, \cdots, x_{s/2-2}, x_0$ in order. Similar to the above cases, it can be checked that the vertex coloring c_G of G satisfies the conditions (a) and (b).

If $a, b \in V(P)$, we can prove the results similar to the above cases. □

An ear decomposition $G_0, G_1, \cdots, G_k$ of a 2-connected graph G is a *non-increasing ear decomposition* if each P_i $(1 \leq i \leq k)$ is a longest ear of G_{i-1} in G, and $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$. In the following, suppose that $G_{i-1} \bigcap P_i = \{a_i, b_i\}$ $(1 \leq i \leq k)$ and

$|G_i| = n_i$ $(0 \leq i \leq k)$. The following lemma shows that we may restrict our attention to non-increasing ear decompositions.

Lemma 2.2. If G is a 2-connected graph, then G admits a non-increasing ear decomposition $G_0, G_1, \cdots, G_k$.

Proof. Since G is 2-connected, there is a cycle G_0 in G, and there are ears of G_0 in G if $G_0 \neq G$. We can chose a longest ear P_1 of G_0 in G, then $G_1 = G_0 \bigcup P_1$ is a 2-connected subgraph of G. If $G_1 \neq G$, we can also chose a longest ear P_2 of G_1 in G, and $G_2 = G_1 \bigcup P_2$ is a 2-connected subgraph of G. Continuing the process until $G_k = G$, we can get an ear decomposition $G_0, G_1, \cdots, G_k$ of G. We claim that $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$. Suppose, on the contrary, there is an i $(1 \leq i \leq k-1)$ such that $\ell(P_i) < \ell(P_{i+1})$. Since P_{i+1} is an ear, all of its internal vertices are outside G_i. If the two end vertices of P_{i+1} are not internal vertices of P_i, then P_{i+1} is also an ear of G_{i-1}. So P_i is not a longest ear of G_{i-1}, a contradiction. If at least one of the end vertices of P_{i+1} is an internal vertex of P_i, then we can find another ear P' of G_{i-1} that consists of the entire P_{i+1} and some segments of P_i. Obviously, $\ell(P') > \ell(P_i)$, a contradiction. Hence, $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$, i.e., $G_0, G_1, \cdots, G_k$ is a non-increasing ear decomposition of G. □

Next, we give a property of the ear decomposition of minimal 2-connected graphs, which will be used in the sequel.

Lemma 2.3. If G is a minimal 2-connected graph, then in any of its ear decompositions the two ends of any ear are non-adjacent.

Proof. Suppose for a contradiction that the assertion is false. Let P_i be the first ear in the decomposition whose two ends a_i, b_i are adjacent, and suppose that the edge $e = a_i b_i$ belongs to ear P_j. Replacing P_i with e and P_j with $(P_i \cup P_j) - e$ we obtain an ear decomposition in which e is an ear of length 1. This implies that $G - e$ is also 2-connected, contradicting the assumption that G is minimal. □

Lemma 2.4. Let G be a minimal 2-connected graph of order n $(n \geq 16)$. If a non-increasing ear decomposition $G_0, G_1, \cdots, G_t$ of G satisfies that $\ell(P_1) \geq \cdots \geq \ell(P_t) \geq 5$, then $rvc(G) \leq rvc^*(G) \leq \lceil n/2 \rceil$, moreover, G has an equitable rainbow vertex coloring with $\lceil n/2 \rceil$ colors.

Proof. From Lemma 2.3, the end vertices a_i, b_i of P_i $(1 \leq i \leq t)$ are nonadjacent. We will apply induction on t to show that each G_i $(0 \leq i \leq t)$ has a vertex coloring c_i satisfying the following conditions: (a) c_i is an equitable strict rainbow vertex coloring of G_i with $\lceil n_i/2 \rceil$ colors; (b) when n_i is odd and $i < t$, for the color x_i that appears only once on G_i, $c_i(a_{i+1}) \neq x_i$ and $c_i(b_{i+1}) \neq x_i$, and c_i has the property $P(x_i, a_{i+1}, b_{i+1})$.

Consider the case $i = 0$, i.e., $G_0 = C_{n_0} = v_1 v_2 \cdots v_{n_0} v_{n_0+1} (= v_1)$. If $t > 0$, without loss of generality, assume that $a_1 = v_1$. Define a vertex coloring c_0 of G_0 by $c_0(v_j) = y_j$ for j with $1 \leq j \leq \lceil n_0/2 \rceil$ and $c(v_j) = y_{j - \lceil n_0/2 \rceil}$ for j with $\lceil n_0/2 \rceil + 1 \leq j \leq n_0$. Note that if n_0 is odd, $y_{\lceil n_0/2 \rceil}$ appears only once on G_0. It can be checked that c_0 satisfies the conditions (a) and (b).

Assume that every graph G_i $(0 \leq i \leq t-1)$ has a vertex coloring c_i satisfying conditions (a) and (b). Consider the graph G_{i+1}. It is obvious that c_i satisfies the conditions of Lemma 2.1. From Lemma 2.1, G_{i+1} has a vertex coloring c_{i+1} satisfying conditions (a) and (b).

Hence, c_t is an equitable strict rainbow vertex coloring of G with $\lceil n/2 \rceil$ colors. □

Theorem 2.2. *Let G be a 2-connected graph of order n $(n \geq 3)$. Then*

$$rvc(G) \leq \begin{cases} \lceil n/2 \rceil - 2 & \text{if } n = 3, 5, 9 \\ \lceil n/2 \rceil - 1 & \text{if } n = 4, 6, 7, 8, 10, 11, 12, 13 \text{ or } 15 \\ \lceil n/2 \rceil & \text{if } n \geq 16 \text{ or } n = 14, \end{cases}$$

and the upper bound is tight, which is achieved by the cycle C_n.

Proof. Without loss of generality, assume that G is a minimal 2-connected graph. So there exists a non-increasing ear decomposition $G_0, G_1, \cdots, G_k$ of G satisfying that $\ell(P_1) \geq \cdots \geq \ell(P_k) \geq 2$. First, we show that $rvc(G) \leq \lceil n/2 \rceil$ for all $n \geq 3$. If $k = 0$ or $\ell(P_1) \geq \cdots \geq \ell(P_k) \geq 5$, then G has a strict rainbow vertex coloring with $\lceil n/2 \rceil$ colors from Lemma 2.4. Hence, $rvc(G) \leq rvc^*(G) \leq \lceil n/2 \rceil$.

Now assume that $5 \leq \ell(P_t) \leq \cdots \leq \ell(P_1)$ and $2 \leq \ell(P_k) \leq \cdots \leq \ell(P_{t+1}) \leq 4$ with $0 \leq t < k$. From Lemma 2.4, G_t has an equitable strict rainbow vertex coloring c_t with $\lceil n_t/2 \rceil$ colors. Let x be a color of c_t and $y, x_{t+1}, \cdots, x_k$ be new colors.

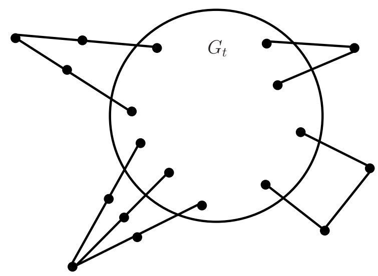

Figure 2. The structure of the graph G.

Figure 2 shows the structure of G, where G_t possibly has ears with lengths 2, 3 or 4. Note that the end vertices of P_i $(t+1 \leq i \leq k)$ with length 3 or 4 must belong to $V(G_t)$ and one end vertex of an ear with length 2 possible is the center vertex of some ear with length

4. Define a vertex coloring c of G from c_t as follows. For any $v \in V(G_t)\backslash\{a_{t+1}, \cdots, a_k\}$, $c(v) = c_t(v)$. If there exists only one ear, say $P_j = a_j v_{j_1} v_{j_2} v_{j_3} b_j$ $(j = t+1)$ with length 4, then $c(a_j) = c(v_{j_3}) = x_j$, $c(v_{j_1}) = c_t(a_j)$ and $c(v_{j_2}) = x$. If there exist at least two ears with length 4 and $P_j = a_j v_{j_1} v_{j_2} v_{j_3} b_j$ $(t+1 \leq j \leq k)$ is such an ear with length 4, then $c(a_j) = c(v_{j_3}) = x_j$, $c(v_{j_1}) = c_t(a_j)$ and $c(v_{j_2}) = y$. Note that the center vertices of all ears with length 4 are colored by the new color y. If $P_j = a_j v_{j_1} v_{j_2} b_j$ $(t+1 \leq j \leq k)$ with length 3, then $c(a_j) = c(v_{j_2}) = x_j$ and $c(v_{j_1}) = c_t(a_j)$. If $P_j = a_j v_{j_1} b_j$ $(t+1 \leq j \leq k)$ with length 2, then $c(a_j) = c_t(a_j)$ and $c(v_{j_1}) = x$. Note that if $\ell(P_j) = 2$ with $t+1 \leq j \leq k$, then the color x_j is not used in c. Hence, we obtain a vertex coloring c of G with at most $\lceil n/2 \rceil$ colors.

Now we show that G is rainbow vertex-connected. Let v', v'' be any two vertices of G. We distinguish the following three cases. (1) Assume that $v', v'' \in V(G_t)$. Since c_t is a strict rainbow vertex coloring of G_t, there exists a rainbow path from v' to v'' in G_t with respect to c_t. From the definition of c, this path is also a rainbow path with respect to c. (2) Assume that $v' \in V(G)\backslash V(G_t)$ and $v'' \in V(G_t)$, i.e., $v' \in V(P_j)\backslash V(G_t)$ with $t+1 \leq j \leq k$. Let P' (resp. P'') be a strict rainbow path from a_j (resp. b_j) to v'' in G_t with respect to c_t. Then one of $(v'P_j a_j)P'$ and $(v'P_j b_j)P''$ is a rainbow path from v' to v''. (3) Assume that $v', v'' \in V(G)\backslash V(G_t)$. If $d_G(v', v'') \leq 2$, then there is a rainbow path from v' to v'' trivially. If $d_G(v', v'') \geq 3$, without loss of generality, assume that $v' \in V(P_{j_1})$ and $v'' \in V(P_{j_2})$ $(t+1 \leq j_1 < j_2 \leq k)$. Since $\ell(P_{j_2}) \leq 4$, one of $a_{j_2} P_{j_2} v''$ and $b_{j_2} P_{j_2} v''$ (say $a_{j_2} P_{j_2} v''$) has length no more than 2. Let P'_{j_1} (resp. P''_{j_1}) be a strict rainbow path from a_{j_1}(resp. b_{j_1}) to a_{j_2} in G_t with respect to c_t. Then one of $(v'P_{j_1} a_{j_1})P'_{j_1}(a_{j_2} P_{j_2} v'')$ and $(v'P_{j_1} b_{j_1})P''_{j_1}(a_{j_2} P_{j_2} v'')$ is a rainbow path from v' to v''. Hence, c is a rainbow vertex coloring of G, i.e., $rvc(G) \leq \lceil n/2 \rceil$ for $n \geq 3$. Therefore, the result holds for $n \geq 16$ or $n = 14$. The upper bound is tight from Theorem 2.1.

In the following, we prove that $rvc(G) \leq rvc(C_n)$ for $3 \leq n \leq 13$ or $n = 15$. It can be checked that the result holds for $n = 3, 4, 5$. If $G = C_n$ $(6 \leq n \leq 13$ or $n = 15)$, the result holds obviously. Now assume that G is a 2-connected graph with order n $(6 \leq n \leq 13$ or $n = 15)$ and $G \neq C_n$. Let $G_0, G_1, \cdots, G_k$ be an ear decomposition of G such that $G_0 = C_{n_0}$ is a longest cycle of G. Note that $4 \leq n_0 \leq 14$ and the length of ears of G_0 is at most $\lfloor n_0/2 \rfloor$. Define *a standard vertex coloring* c_0 of $G_0 = C_{n_0} = v_1 \cdots v_{n_0} v_{n_0+1} (= v_1)$ by $c_0(v_i) = x_i$ for i with $1 \leq i \leq \lceil n_0/2 \rceil$ and $c_0(v_i) = x_{i - \lceil n_0/2 \rceil}$ for i with $\lceil n_0/2 \rceil + 1 \leq i \leq n_0$. It is obvious that c_0 is a strict rainbow vertex coloring of G_0 with $\lceil n_0/2 \rceil$ colors. There are two simple claims.

Claim 1. If $n \geq n_0 + 2$ and $V(G_0)$ is a 1-step dominating set of G, then $rvc(G) \leq \lceil n/2 \rceil - 1$. In fact, define a standard vertex coloring c_0 of G_0 with $\lceil n_0/2 \rceil$ colors and color the vertices in $V(G)\backslash V(G_0)$ by colors already used properly. Then we can get a rainbow vertex coloring of G with at most $\lceil n/2\rceil-1$ colors.

Claim 2. If $n=n_0+1$ with $6\leq n_0\leq 12$, then $rvc(G)\leq rvc(C_n)$. In fact, define a vertex coloring c_0 of G_0 as shown in Figure 1 and color the vertex in $V(G)\backslash V(G_0)$ by color already used on G_0. It can be checked that G is rainbow vertex-connected.

If $n_0=4$ or 5, $V(G_0)$ must be a 1-step dominating set of G. Hence, from Claim 1 the result holds for $n_0=4$ or $n_0=5$ and $n\geq 7$. If $n_0=5$ and $n=6$, define a vertex coloring c of G by $c(v_i)=x_1$ if i is odd, $c(v_i)=x_2$ if i is even and for $v\in V(G)\backslash V(G_0)$, $c(v)=x_1$. The vertex coloring c of G is a rainbow vertex coloring with 2 colors, i.e., $rvc(G)\leq 2$. Therefore, the result holds for $n_0=4$ or 5.

For $n_0=7$ and $n=9$, color G_0 as shown in Figure 1 and color vertices in $V(G)\backslash V(G_0)$ by colors already used such that adjacent vertices of G are colored different. The obtained vertex coloring of G is a rainbow vertex coloring with 3 colors, i.e., $rvc(G)\leq 3$. For $n_0=6$ or 7, $V(G_0)$ must be a 1-step dominating set of G. From Claims 1 and 2 and the above case that $n_0=7$ and $n=9$, the result holds for $n_0=6$ or 7.

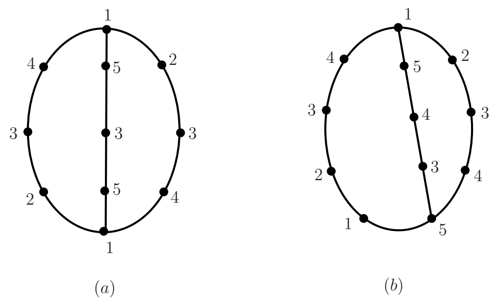

Figure 3. The vertex colorings for $n_0=8$ and 9.

Consider the cases that $n_0=8$ or 9. If $V(G_0)$ is a 1-step dominating set of G, the result holds from Claims 1 and 2. Assume that $V(G_0)$ is a 2-step dominating of G. If $n_0=8$ (resp. 9), G_1 is shown in Figure 3 (a) (resp. Figure 3 (b)) and the vertex coloring is a rainbow vertex coloring of G_1. If $n_0=8$ and $V(G_1)$ is a 1-step dominating set of G, then color all vertices in $V(G)\backslash V(G_1)$ by color 1. If $n_0=9$ and $V(G_1)$ is a 1-step dominating set of G, then color all vertices in $V(G)\backslash V(G_1)$ by a new color 6. G is rainbow vertex-connected and the result holds. If G_1 has an ear $P_2=v_1'v_2'\cdots v_5'$ such that $V(G_1)\bigcap V(P_2)=\{v_1',v_5'\}$, define a vertex coloring of G as follows. Color the vertices in $V(G_1)$ as shown in Figure 3, $c(v_2')=c(v_4')=6$ and $c(v_3')=7$. For $v\in V(G)\backslash(V(G_1)\bigcup V(P_2))$, color v by color 7. It can be checked that G is rainbow vertex-connected. Hence, the result holds for $n_0=8$ or 9.

Consider the case that $n_0=10$. If $V(G_0)$ is a 1-step dominating set of G, the result holds from Claims 1 and 2. If $V(G_0)$ is a 2-step dominating set of G, then $n=13$ or 15. First, we give G_0 a standard vertex coloring c_0 with colors $x_1,\cdots,x_5$. If $P_1=v_1'v_2'\cdots v_5'$

such that $V(G_0)\bigcap V(P_1) = \{v'_1, v'_5\}$, define the colors of vertices in $V(G)\backslash V(G_0)$ as follows. $c(v'_2) = c(v'_4) = 6$, $c(v'_3) = 1$ and color the other uncolored vertices by color 1. If $P_1 = v'_1v'_2\cdots v'_6$ such that $V(G_0)\bigcap V(P_1) = \{v'_1, v'_6\}$, then $n = 15$ and define the colors of vertices in $V(G)\backslash V(G_0)$ as follows. $c(v'_2) = c(v'_4) = 6$, $c(v'_3) = c(v'_5) = 7$ and color the other uncolored vertices by color 1. The obtained vertex coloring of G is a rainbow vertex coloring with at most $\lceil\frac{n}{2}\rceil - 1$ colors. Therefore, the result holds for $n_0 = 10$.

Consider the case that $n_0 = 11$. If $V(G_0)$ is a 1-step dominating set of G, the result holds from Claims 1 and 2. If $V(G_0)$ is a 2-step dominating set of G, then $n = 15$. First, we give G_0 a standard vertex coloring c_0 with colors $x_1, \cdots, x_6$. If $P_1 = v'_1v'_2\cdots v'_5$ such that $V(G_0)\bigcap V(P_1) = \{v'_1, v'_5\}$, then define the colors of vertices in $V(G)\backslash V(G_0)$ as follows. $c(v'_2) = c(v'_4) = 7$, $c(v'_3) = 1$ and color the other uncolored vertices by color 1. If $P_1 = v'_1v'_2\cdots v'_6$ such that $V(G_0)\bigcap V(P_1) = \{v'_1, v'_6\}$, define the colors of vertices in $V(G)\backslash V(G_0)$ by $c(v'_2) = c(v'_5) = 7$, $c(v'_3) = 1$ and $c(v'_4) = 2$. It can be checked that the obtained vertex coloring of G is a rainbow vertex coloring with 7 colors. Therefore, the result holds for $n_0 = 11$.

Consider the case that $n_0 = 12$. From Claims 1 and 2, the result holds if $V(G_0)$ is a 1-step dominating set of G. If $V(G_0)$ is a 2-step dominating set of G, then $n = 15$ and $G = G_0\bigcup P_1$, where $P_1 = v'_1v'_2\cdots v'_5$ such that $V(G_1)\bigcap V(P_1) = \{v'_1, v'_5\}$. We give G_0 a standard vertex coloring with colors $x_1, \cdots, x_6$. Define the colors of the vertices in $V(G)\backslash V(G_0)$ as $c(v'_2) = c(v'_4) = 7$ and $c(v'_3) = 1$. The vertex coloring c of G is a rainbow vertex coloring with 7 colors. Therefore, the result holds for $n_0 = 12$.

Consider the cases that $n_0 = 13, 14$. We give G_0 a standard vertex coloring with colors $x_1, \cdots, x_7$ and the other vertices are colored by color 1. Then G is rainbow vertex-connected. Therefore, the result holds for $n_0 = 13$ or 14.

The proof is now complete. □

Because the proof methods above are constructive, one can obtain a concrete rainbow vertex coloring from the proofs for any given 2-connected graph, using at most $\lceil\frac{n}{2}\rceil$ colors.

Acknowledgement: The authors are very grateful to the referees and the editor for their useful comments and suggestions, which helped to improve the presentation of our paper.

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