Graphs with 3-rainbow index $n-1$ and $n-2$*

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Abstract

Let $G=(V(G),E(G))$ be a nontrivial connected graph of order n with an edge-coloring $c: E(G)\rightarrow\{1,2,\ldots,q\}$, $q\in\mathbb{N}$, where adjacent edges may be colored the same. A tree T in G is a *rainbow tree* if no two edges of T receive the same color. For a vertex set $S\subseteq V(G)$, a tree connecting S in G is called an S-tree. The minimum number of colors that are needed in an edge-coloring of G such that there is a rainbow S-tree for each k-subset S of $V(G)$ is called the k-rainbow index of G, denoted by $rx_k(G)$, where k is an integer such that $2\leq k\leq n$. Chartrand et al. got that the k-rainbow index of a tree is $n-1$ and the k-rainbow index of a unicyclic graph is $n-1$ or $n-2$. So there is an intriguing problem: Characterize graphs with the k-rainbow index $n-1$ and $n-2$. In this paper, we focus on $k=3$, and characterize the graphs whose 3-rainbow index is $n-1$ and $n-2$, respectively.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. Let $G = (V(G), E(G))$ be a nontrivial connected graph of order n with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *rainbow path* if every two edges of the path have distinct colors. The graph G is *rainbow connected* if for every two vertices u and v of G, there is a rainbow path connecting u and v. The minimum number of colors for which there is an edge coloring of G such that G is rainbow connected is called the *rainbow connection number*, denoted by $rc(G)$. Results on the rainbow connectivity can be found in [2, 5, 6, 7, 8, 9]. In the sequel we will simply denote the order of a graph G by $|G|$, i.e., $|G| = |V(G)|$.

These concepts were introduced by Chartrand et al. in [2]. In [3], they generalized the concept of rainbow path to rainbow tree. A tree T in G is a *rainbow tree* if no two edges of T receive the same color. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree connecting S. Given a fixed integer k with $2 \leq k \leq n$, the edge-coloring c of G is called a *k-rainbow coloring* of G if for every k-subset S of $V(G)$, there exists a rainbow S-tree. In this case, G is called *k-rainbow connected*. The minimum number of colors that are needed in a *k-rainbow coloring* of G is called the *k-rainbow index* of G, denoted by $rx_k(G)$. Clearly, when $k = 2$, $rx_2(G)$ is the rainbow connection number $rc(G)$ of G. For every connected graph G of order n, it is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$.

The *Steiner distance* $d_G(S)$ of a set S of vertices in G is the minimum size of a tree in G connecting S. The *k-Steiner diameter* $sdiam_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G. Then there is a simple upper bound and lower bound for $rx_k(G)$.

Observation 1 ([3])**.** For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$.

They showed that the k-rainbow index of trees attains the upper bound.

Proposition 1 ([3])**.** Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$, $rx_k(T) = n - 1$.

They also showed that the k-rainbow index of a unicyclic graph is $n - 1$ or $n - 2$.

Theorem 1 ([3])**.** *If G is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then*

$$rx_k(G) = \begin{cases} n-2, & k = 3 \text{ and } g \geq 4; \\ n-1, & g = 3 \text{ or } 4 \leq k \leq n. \end{cases} \quad (1)$$

A natural thought is that which graph of order n has the k-rainbow index $n - 1$ except for a tree and a unicyclic graph of girth 3? Furthermore, which graph of order n has the

k-rainbow index $n-2$ except for a unicyclic graph of girth at least 4? In this paper, we focus on $k=3$. In addition, some known results are mentioned.

Observation 2 ([3]). Let G be a connected graph of order n containing two bridges e and f. For each integer k with $2 \leq k \leq n$, every k-rainbow coloring of G must assign distinct colors to e and f.

Lemma 1 ([4]). *Let G be a connected graph and $\{V_1, V_2, \cdots, V_k\}$ a partition of $V(G)$. If each V_i induces a connected subgraph H_i of G, then $rx_3(G) \leq k-1+\sum_{i=1}^{k} rx_3(H_i)$.*

Theorem 2 ([4]). *Let G be a connected graph of order n. Then $rx_3(G)=2$ if and only if $G=K_5$ or G is a 2-connected graph of order 4 or G is of order 3.*

Observation 3 ([4]). Let G be a connected graph of order n, and H be a connected spanning subgraph of G. Then $rx_3(G) \leq rx_3(H)$.

Let G be a connected graph with n vertices and m edges. Define the *cyclomatic number* of G as $c(G)=m-n+1$. A graph G with $c(G)=k$ is called *k-cyclic*. According to this definition, if a graph G meets $c(G)=0$, 1, 2 or 3, then the graph G is called acyclic (or a tree), unicyclic, bicyclic, or tricyclic, respectively.

This paper is organized as follows. In Section 2, some basic results and notation are presented. In Section 3, we characterize the graphs whose 3-rainbow index is $n-1$ and $n-2$, respectively. For the latter case, we take two steps to finish our proof: we deal with it for bicyclic graphs first, and then for tricyclic graphs.

2 Some basic results

First of all, we need some more terminology and notation.

Definition 1. For a subgraph H of G and $v \in V(G)$, let $d(v,H)=min\{d_G(v,x): x \in V(H)\}$.

Next we define some new notations.

Definition 2. Let G be a connected graph of order n with $V(G)=\{v_1, v_2, \cdots, v_n\}$. For any edge e of G with end-vertices v_k and v_r, if $k<r$ then we will write $e=v_kv_r$. It is clear that in this way any edge has a unique expression. Then, we define a *lexicographic ordering* between any two edges of G by $v_iv_j < v_sv_t$ if and only if $i<s$ or $i=s$, $j<t$.

Note that, the lexicographic ordering of a connected graph is unique. Given a coloring c of a connected graph G, denote by $c_\ell(G)$ a sequence of colors of the edges which are ordered by the lexicographic ordering.

For a connected graph G, to *contract* an edge $e=xy$ is to delete e and replace its ends by a single vertex incident to all the edges which were incident to either x or y. Let G'

be the graph obtained by contracting some edges of G. Given a rainbow coloring of G', when it comes back to G, we keep the colors of corresponding edges of G' in G and assign a new color to a new edge, which makes G 3-rainbow connected. Hence, the following lemma holds.

Lemma 2. *Let G be a connected graph, and G' be a connected graph by contracting some edges of G. Then $rx_3(G) \leq rx_3(G') + |G| - |G'|$.*

Definition 3. Let G_0 be the graph obtained by contracting all the cut edges of G. Then G_0 is called the *basic graph* of G.

3 Main results

3.1 Characterize the graphs with $rx_3(G)$= n − 1

Theorem 3. *Let G be a connected graph of order n. Then $rx_3(G) = n - 1$ if and only if G is a tree or G is a unicyclic graph with girth 3.*

Proof. If G is a tree or a unicyclic graph with girth 3, by Proposition 1 and Theorem 1, $rx_3(G) = n - 1$. Conversely, suppose G is a graph with $rx_3(G) = n - 1$ but not a tree. Then G must contain cycles. Let $\{H_1, H_2, \cdots, H_k\}$ be a partition of $V(G)$ into connected subgraphs. If G contains a cycle of length r at least 4, then let H_1 be the r-cycle, and each other subgraph a single vertex. We color H_1 with $r - 2$ dedicated colors, then by Lemma 1, $rx_3(G) \leq n - r + rx_3(H_1) = n - 2$. Suppose then G contains at least two triangles C_1 and C_2. If C_1 and C_2 have a vertex in common, then let H_1 be the union of C_1 and C_2, and each other subgraph a single vertex. We color both C_1 and C_2 with the same three dedicated colors, thus $rx_3(G) \leq n - 5 + rx_3(H_1) = n - 2$. If C_1 and C_2 are vertex disjoint, then let $H_1 = C_1$, $H_2 = C_2$, and each other subgraph a single vertex. We color H_1 with three new colors and H_2 with the same three colors of H_1, thus $rx_3(G) \leq n - 5 + rx_3(H_1) + rx_3(H_2) = n - 2$. Combining the above two cases, G is a unicyclic graph with girth 3. Therefore, the result holds. □

3.2 Characterize the graphs with $rx_3(G)$= n − 2

Next, we characterize the graphs whose 3-rainbow index is $n - 2$. We begin with a useful theorem from [4].

A 3-*sun* is a graph constructed from a cycle $C_6 = v_1v_2 \cdots v_6v_1$ by adding three edges v_2v_4, v_2v_6 and v_4v_6.

Theorem 4 ([4])**.** *Let G be a 2-edge-connected graph of order n ($n \geq 4$). Then $rx_3(G) \leq n - 2$, with equality if and only if $G = C_n$ or G is a spanning subgraph of one of the*

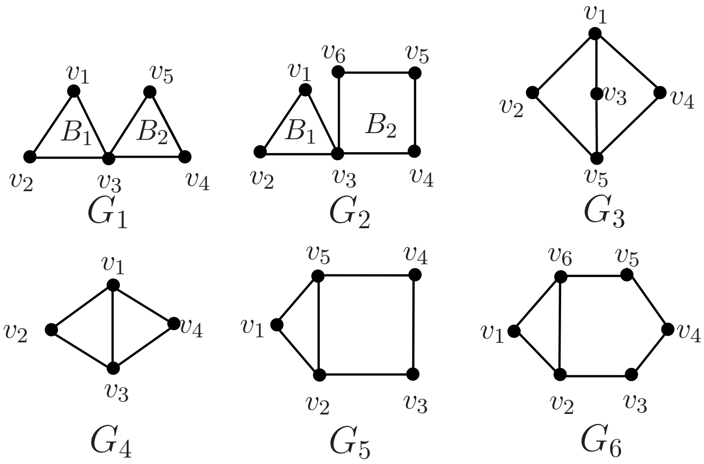

Figure 1. The basic graphs for Lemma 3.

following graphs: a 3-sun, $K_5 - e$, K_4, G_1, G_2, H_1, H_2, H_3, *where* G_1, G_2 *are defined in Figure 1 and* H_1, H_2, H_3 *are defined in Figure 2.*

Since all the 2-edge-connected graphs with the 3-rainbow index $n-2$ have been characterized in Theorem 4, it remains to characterize the graphs with 3-rainbow index $n-2$ which have cut edges. Notice that the cut edges of a graph must be assigned with distinct colors, our main purpose is to check out how the addition of cut edges to G affect the 3-rainbow index of a 2-connedted graph G when $rx_3(G) = n-2$. In other words, share the colors of cut edges with the colors of the non-cut edges as many as possible.

Given a connected graph G of order n, and a coloring c of G, we always let A_1 be the set of colors assigned to the non-cut edges of G and A_2 the set of colors assigned to the cut edges of G. For each positive integer k, let $N_k = \{1, 2, \cdots, k\}$. We always set that $A_2 = N_s$, where s is the number of cut edges of G. Note that, A_1 and A_2 may intersect and suppose $|A_1 \cap A_2| = p$. We can interchange the colors of cut edges suitably such that $A_1 \cap A_2 = \{1, 2, \cdots, p\}$. Set $A_1 \setminus A_2 = \{a_1, \cdots, a_t\}$, $t \leq m - s$ and $a_j \in N_{|c|}$.

For a connected graph G, a *block* is a maximal 2-connected subgraph. In this paper, we regard K_2 other than a block. An *internal cut edge* is a cut edge which is on the unique path joining some two blocks. Denote the cut edges of G by $e_1 = x_1y_1, \cdots, e_p = x_py_p$ and the colors of these cut edges by $1, \cdots, p$, respectively. Moreover, if x_iy_i is not an internal cut edge, we always set $d(x_i, B) \leq d(y_i, B)$ where B is an arbitrary block.

Let H be a connected subgraph of G. Denote by $i \in H$ if the color i appears in H. Given a graph G, let G_0 be its basic graph. Deleting the corresponding edges of G_0 in G, we obtain a forest. Each component corresponds to a vertex v in G_0, denoted by $T(v)$. Denote by $U(v)$ the number of leaves of $T(v)$ in G and $U(G) = \sum_{v\in V(G)} U(v)$. Let $W(v)$ be the number of edges of $T(v)$ whose colors are appeared in A_1, that is, $W(v) = |c(T(v)) \cap A_1|$.

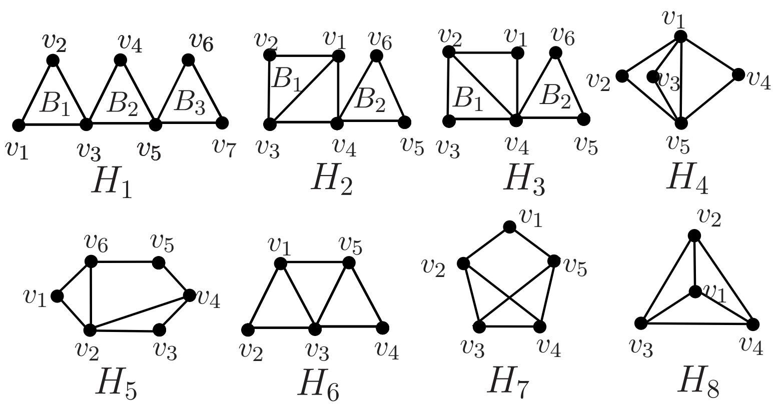

Figure 2. The basic graphs for Lemma 4.

3.2.1 Bicyclic graphs with $rx_3(G)$= $\mathbf{n-2}$

First, we introduce some graph classes. Let G_i be the graphs shown in Figure 1. Define by $\mathcal{G}_i^*$ the set of graphs whose basic graph is G_i, where $1 \leq i \leq 6$. Set
$\mathcal{G}_1 = \{G \in \mathcal{G}_1^* | U(v_3) \leq 1\}$,
$\mathcal{G}_2 = \{G \in \mathcal{G}_2^* | U(v_3) + U(v_i) \leq 1,\ i = 4, 6\}$,
$\mathcal{G}_3 = \{G \in \mathcal{G}_3^* | U(v_i) + U(v_j) \leq 2,\ v_iv_j \in E(G_3)\}$,
$\mathcal{G}_4 = \{G \in \mathcal{G}_4^* | U(v_i) \leq 2,\ i = 1, 3\}$,
$\mathcal{G}_5 = \{G \in \mathcal{G}_5^* | U(v_2) + U(v_3) \leq 2,\ U(v_4) + U(v_5) \leq 2\}$,
$\mathcal{G}_6 = \{G \in \mathcal{G}_6^* | U(v_2) = U(v_6) = 0,\ U(v_4) \leq 1,\ U(v_4) + U(v_i) \leq 2,\ i = 3, 5\}$
and set $\mathcal{G} = \mathcal{G}_1 \bigcup \mathcal{G}_2 \bigcup \cdots \bigcup \mathcal{G}_6$.

Lemma 3. *Let G be a connected bicyclic graph of order n. Then $rx_3(G) = n - 2$ if and only if $G \in \mathcal{G}$.*

Proof. Suppose that G is a graph with $rx_3(G) = n - 2$ but $G \notin \mathcal{G}$. Let G_0 be the basic graph of G. Then G_0 is a 2-edge-connected bicyclic graph. If $G_0 \neq G_i$, by Theorem 4, $rx_3(G_0) \leq |G_0| - 3$. Moreover, by Lemma 2, we have $rx_3(G) \leq rx_3(G_0) + |G| - |G_0| \leq n-3$. Hence $G_0 = G_i$. Next we show that if $G \in \mathcal{G}_i^* \setminus \mathcal{G}_i$, then $rx_3(G) \leq n - 3$, where $1 \leq i \leq 6$. As pointed out before, all the cut edges of G are colored with $1, 2, \cdots$. We only provide a coloring c_ℓ of G_0, namely, color the corresponding edges of G, with parts of colors used in cut edges, and the position of cut edges will be determined as following: $\{1, 2, \cdots, q\} \subseteq T(v)$ means to assign colors $\{1, 2, \cdots, q\}$ to q leaves of $T(v)$ arbitrarily. If $G \in \mathcal{G}_1^* \setminus \mathcal{G}_1$, then $U(v_3) \geq 2$, set $c_\ell(G_1) = 1a_1a_2a_2a_12$ and $\{1, 2\} \subseteq T(v_3)$. If $G \in \mathcal{G}_2^* \setminus \mathcal{G}_2$, then $U(v_3) + U(v_4) \geq 2$ or $U(v_3) + U(v_6) \geq 2$. By contracting v_3v_4 or v_3v_6, we obtain a graph G' belonging to $\mathcal{G}_1^* \setminus \mathcal{G}_1$. Then the coloring of G can be obtained easily from G' by Lemma 2. If $G \in \mathcal{G}_3^* \setminus \mathcal{G}_3$, then there is an edge $v_iv_j \in E(G_3)$ such that $U(v_i) + U(v_j) \geq 3$. By symmetry, there exist four cases for G: (1) $U(v_1) \geq 3$; (2) $U(v_1) \geq 2$, $U(v_2) \geq 1$; (3) $U(v_1) \geq 1$, $U(v_2) \geq 2$; (4) $U(v_2) \geq 3$. Set $c_\ell(G_3) = a_1a_2a_2123$ and set $\{1, 2, 3\} \subseteq T(v_1)$

for (1); $\{1\} \subseteq T(v_2)$, $\{2,3\} \subseteq T(v_1)$ for (2); $\{1\} \subseteq T(v_1)$, $\{2,3\} \subseteq T(v_2)$ for (3); $\{1,2,3\} \subseteq T(v_2)$ for (4). If $G \in \mathcal{G}_4^* \setminus \mathcal{G}_4$, then $U(v_1) \geq 3$ or $U(v_3) \geq 3$. By symmetry, suppose $U(v_1) \geq 3$ and set $c_\ell(G_4) = 123a_1a_1$ and $\{1,2,3\} \subseteq T(v_1)$. If $G \in \mathcal{G}_5^* \setminus \mathcal{G}_5$, then by contracting v_2v_3 or v_4v_5, we obtain a graph G' belonging to $\mathcal{G}_4^* \setminus \mathcal{G}_4$. Now consider $G \in \mathcal{G}_6^* \setminus \mathcal{G}_6$. Then $U(v_2) \geq 1$, or $U(v_6) \geq 1$, or $U(v_4) \geq 2$, or $U(v_4) + U(v_3) \geq 3$, or $U(v_4) + U(v_5) \geq 3$. For the last two cases, it belongs to $\mathcal{G}_5^* \setminus \mathcal{G}_5$ by contracting v_3v_4 or v_4v_5. If $U(v_2) \geq 1$, set $c_\ell(G_6) = a_3a_2a_4a_4a_2a_31$ and $\{1\} \subseteq T(v_2)$. If $U(v_4) \geq 2$, set $c_\ell(G_6) = a_31a_22a_1a_2a_1$ and $\{1,2\} \subseteq T(v_4)$. It is not hard to check that the colorings above make G rainbow connected with $n-3$ colors, thus $rx_3(G) \leq n-3$.

Conversely, let G be a bicyclic graph such that $G \in \mathcal{G}$. Assume, to the contrary, that $rx_3(G) \leq n-3$. Then there exists a rainbow coloring c such that $A_1 \cup A_2 = N_{n-3}$. By Theorem 4, we focus on the graphs with cut edges and $|A_1 \cap A_2| \geq 1$. We write $d_{G_i}(u,v,w)$ to mean that the number of edges of a $\{u,v,w\}$-tree in G which correspond to the edges of G_i, the basic graph of G. We divide into three cases.

Case 1. $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.

First assume that $G \in \mathcal{G}_2$ and we give the following claims. If there is a nontrivial path P_ℓ connecting B_1 and B_2 in G, then denote its ends by $v_3'(\in B_1)$ and $v_3''(\in B_2)$.

Claim 1. Each block B_i has at most one edge use the color from A_2, where $i \in \{1,2\}$. Moreover, if a color of A_2 appears in B_i, then the other edges of B_i must be assigned with different colors in $A_1 \setminus A_2$.

Proof. Suppose two edges of B_1 are colored with 1, 2, respectively. We also set $d(x_i, B_1) \leq d(y_i, B_1)$, where x_iy_i belongs to P_ℓ. Since the cut edges colored with 1 and 2 should be contained in the rainbow tree whose vertices contain y_1 and y_2, by deleting the edges assigned with 1 and 2 in B_1, G is disconnected. Let w be a vertex in the component that does not contain y_1, then there is no rainbow tree connecting $\{y_1, y_2, w\}$, a contradiction. We can take the similar argument for the other cases when two edges of B_1 (B_2) are colored with 1 or two edges of B_2 are colored with 1, 2, respectively.

Now suppose $1 \in B_i \cap A_2$ and two edges of B_i have the same color a_1. Let w_1, w_2 be the end vertices of the edge assigned with 1, then $\{y_1, w_1, w_2\}$ has no rainbow tree. □

Claim 2 The colors of the path P_ℓ can not appear in A_1.

Proof. Assume e is the edge of P_ℓ colored with 1. The color 1 can not appear in B_1. Otherwise suppose the three edges of B_1 are assigned with 1, a_1 and a_2, respectively. Consider $\{v_1, v_2, v_5\}$, then $c(v_3''v_4), c(v_4v_5) \in \{2, a_3\}$ or $c(v_3''v_6), c(v_5v_6) \in \{2, a_3\}$. Without loss of generality, suppose $c(v_3''v_4), c(v_4v_5) \in \{2, a_3\}$, then by Claim 1, $c(v_3''v_6), c(v_5v_6) \in \{a_1, a_2\}$, thus $\{v_1, v_2, v_6\}$ has no rainbow tree. On the other hand, 1 can not be in B_2. It is easy to see that neither $c(v_3''v_4)$ nor $c(v_3''v_6)$ can be 1 by considering $\{v_1, v_2, v_6\}$. If $c(v_5v_6) = 1$, consider $\{v_1, v_5, v_6\}$, $\{v_2, v_5, v_6\}$, then $c(v_1v_3'), c(v_2v_3') \in A_2$, a contradiction to Claim 1. □

By Claim 1, we have $1 \leq |A_1 \cap A_2| \leq 2$ and only color 1 and 2 can exist in A_1.

We should discuss all the situations according to which cut edges are colored with 1, 2. By the definition of G, $U(v_3) = 1$ or $U(v_3) = 0$. By similarity, we only deal with the former case. First assume $|A_1 \cap A_2| = 1$, then $A_1 = \{1, a_1, a_2, a_3\}$. We consider the subcase when $1 \in T(v_3)$. In this case we claim that the color 1 appears in neither B_1 nor B_2. Indeed, if $c(v_3''v_6) = 1$, since every tree whose vertices contain y_1 must contain the cut edge colored with 1, $d_{G_2}(y_1, v_1, v_6) = 4$. Thus $\{y_1, v_1, v_6\}$ has no rainbow tree. If now $c(v_5v_6) = 1$, then consider $\{y_1, v_5, v_6\}$, $\{y_1, v_5, v_1\}$, $\{y_1, v_5, v_2\}$ successively, we have $c(v_1v_3') = c(v_2v_3') = c(v_3''v_6)$, leading to a contradiction when considering $\{v_1, v_2, v_6\}$. Else if $c(v_1v_3') = 1$, then $\{y_1, v_1, v_5\}$ has no rainbow tree. The last possibility is that $c(v_1v_2) = 1$, we may set $c(v_1v_3) = a_1$, $c(v_2v_3) = a_2$. Consider $\{y_1, v_1, v_4\}$, $\{y_1, v_2, v_4\}$, $\{y_1, v_1, v_6\}$, $\{y_1, v_2, v_6\}$ successively, we have $c(v_3''v_4) = c(v_3''v_6) = a_3$ and 1 can not appear in B_2, hence $\{v_1, v_4, v_6\}$ has no rainbow tree. The other subcases are similar.

Thus $|A_1 \cap A_2| = 2$, $A_1 = \{1, 2, a_1, a_2, a_3\}$. By Claim 1, set $1 \in B_1$, $2 \in B_2$, and the other edges in each block have distinct colors. If $1, 2 \in T(v_3)$, assume that $d(y_1, T(v_3)) > d(y_2, T(v_3))$, there always exist two vertices which come from different blocks such that there is no rainbow tree connecting them and y_1. If $1 \in T(v_3)$, $2 \in T(v_1)$, the most difficult case is that $c(v_1v_2) = 1$, $c(v_5v_6) = 2$. In this case, consider $\{y_2, v_5, v_6\}$, forcing that one of v_1v_3', $v_3''v_4$, $v_3''v_6$, v_4v_5 is colored with 1, contradicting Claim 1. With an analogous argument, we would get a contradiction if 1, 2 are in other cut edges of G.

For $G \in \mathcal{G}_1$, it can be obtained by contracting an edge of a graph in $\mathcal{G}_2$. Then by Lemma 2, $rx_3(G) \geq n - 2$.

Case 2. $G \in \mathcal{G}_3$.

First note that each path from v_1 to v_5 in G_3 can have at most one color in A_2. Thus $|A_1 \cap A_2| \leq 3$. On the other hand, noticing that $d_{G_3}(v_2, v_3, v_4) = 3 > 2$, all the cases satisfying $W(v_1) = W(v_5) = 0$ and $W(v_2)$, $W(v_3)$, $W(v_4) \leq 1$ are easy to get a contradiction, so we omit them here.

First assume $|A_1 \cap A_2| = 1$, then $A_1 = \{1, a_1, a_2\}$. If $1 \in T(v_1)$, consider $\{y_1, v_2, v_3\}$, $\{y_1, v_2, v_4\}$ and $\{y_1, v_3, v_4\}$ successively, v_1v_2, v_1v_3, v_1v_4 must be colored with distinct colors from $A_1 \setminus \{1\}$, which is impossible.

Assume now $|A_1 \cap A_2| = 2$, then $A_1 = \{1, 2, a_1, a_2\}$. If $1, 2 \in T(v_1)$, then consider $\{y_1, y_2, v_5\}$, without loss of generality, set $c(v_1v_2) = a_1$, $c(v_2v_5) = a_2$. Thus $c(v_1v_3)$ can be neither 1 nor 2, otherwise there is no rainbow $\{y_1, y_2, v_3\}$-tree. On the other hand, $c(v_1v_3)$ cannot be a_1, otherwise $c(v_3v_5) = i (i = 1, 2)$, then $\{y_i, v_2, v_3\}$ has no rainbow tree. Meanwhile, v_1v_3 cannot be colored with a_2, otherwise $c(v_3v_5) = i (i = 1, 2)$, then $\{y_i, v_3, v_5\}$ has no rainbow tree. If $1 \in T(v_1)$, $2 \in T(v_5)$, then every path from v_1 to v_5 must color $\{a_1, a_2\}$, a contradiction to $|A_1 \cap A_2| = 3$. If $1, 2 \in T(v_2)$, then by the same reason, we conclude that $c(v_1v_2)$, $c(v_2v_5) \notin \{1, 2\}$ and we may set $c(v_1v_3) = 1$. But now $\{y_1, v_1, v_3\}$ has no rainbow tree. If $1 \in T(v_1)$, $2 \in T(v_2)$. By considering $\{y_1, y_2, v_3\}$, $\{y_1, y_2, v_4\}$, $\{y_1, y_2, v_5\}$, we may set $c(v_1v_3) = c(v_1v_4) = c(v_2v_5) = a_1$, this force $c(v_3v_5) = i (i = 1, 2)$. However, there is no rainbow tree connecting $\{y_i, v_3, v_5\}$.

Thus $|A_1 \cap A_2| = 3$, $A_1 = \{1, 2, 3, a_1, a_2\}$. If $1, 2, 3 \in T(v_1)$, since $U(v_1) \leq 2$, we may assume that y_2 is on the unique path from y_1 to v_1. Thus one path from v_1 to v_5 must be colored with $\{a_1, a_2\}$, a contradiction to $|A_1 \cap A_2| = 3$. If $1, 2 \in T(v_2)$, $3 \in T(v_5)$, and without loss of generality, y_2 is on the unique path from y_1 to v_2. Considering $\{y_1, v_3, y_3\}$ and $\{y_1, v_4, y_3\}$, we may set $c(v_1v_2) = a_1$, $c(v_1v_3) = c(v_1v_4) = a_2$. But there is no rainbow $\{y_1, v_3, v_4\}$-tree. Each other case is similar or easier.

Case 3. $G \in \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6$.

First let $G \in \mathcal{G}_6$. Similarly, each path from v_2 to v_6 in G_6 can have at most one color in A_2. Thus we have $1 \leq |A_1 \cap A_2| \leq 3$. Assume first $|A_1 \cap A_2| = 1$, $A_1 = \{1, a_1, a_2, a_3\}$.

We only focus on the case that $1 \in T(v_4)$. To make sure there are rainbow trees connecting $\{y_1, v_1, v_3\}$ and $\{y_1, v_1, v_5\}$, only $c(v_2v_6)$ can be 1, but now $\{y_1, v_2, v_6\}$ has no rainbow tree.

Assume then $|A_1 \cap A_2| = 2$, $A_1 = \{1, 2, a_1, a_2, a_3\}$. If $1, 2 \in T(v_1)$, we may set $c(v_1v_2) = a_1$, $c(v_2v_3) = a_2$, $c(v_3v_4) = a_3$ by considering $\{y_1, y_2, v_4\}$. $c(v_5v_6)$ can be neither 1 nor 2, otherwise $\{y_1, y_2, v_5\}$ has no rainbow tree. Moreover, $c(v_4v_5)$ can be neither 1 nor 2, otherwise when $c(v_4v_5) = i (i = 1, 2)$, there is no rainbow $\{y_i, v_4, v_5\}$-tree. Thus v_1v_6, v_2v_6 must use colors $\{1, 2\}$, but now $\{y_1, y_2, v_6\}$ has no rainbow tree. If $1, 2 \in T(v_3)$, first we claim that at most one edge of the triangle $v_1v_2v_6$ uses a color from $\{1, 2\}$. Otherwise if $c(v_1v_2)$, $c(v_2v_6) \in \{1, 2\}$, then $\{y_1, y_2, v_1\}$ has no rainbow tree. If $c(v_1v_6)$, $c(v_2v_6) \in \{1, 2\}$, the rest non-cut edges must color $\{a_1, a_2, a_3\}$. It is easy to verify that either $\{y_1, v_1, v_5\}$ or $\{y_1, v_1, v_4\}$ has no rainbow tree. So the longest path from v_2 to v_6 has an edge colored with 1 or 2. However, we will show that it is impossible. It is easy to check that $c(v_2v_3), c(v_3v_4) \notin \{1, 2\}$. If $c(v_5v_6) \in \{1, 2\}$, then we may set $c(v_5v_6) = 1$. Consider $\{y_1, v_1, v_5\}$ and $\{y_1, v_5, v_6\}$, then $c(v_1v_2) = c(v_2v_6) = 2$, a contradiction. It is similar to check that $c(v_4v_5)$ can not be 1 or 2, a contradiction.

Now assume that $|A_1 \cap A_2| = 3$, $A_1 = \{1, 2, 3, a_1, a_2, a_3\}$. If $1 \in T(v_1)$, $2, 3 \in T(v_3)$. Again, we may set $c(v_1v_2) = a_1$, $c(v_2v_3) = a_2$. Thus $c(v_1v_6), c(v_2v_6) \in \{1, 2, 3\}$. If $c(v_1v_6), c(v_2v_6) \in \{1, i\}$, then there is a contradiction by considering $\{y_1, y_i, v_6\}$ $(i = 2, 3)$. Thus we may set that $c(v_1v_6) = 2$, $c(v_2v_6) = 3$. By considering $\{y_1, y_3, v_4\}$ and $\{y_1, y_3, v_5\}$, we get that $c(v_3v_4) = c(v_5v_6) = a_3$, $c(v_4v_5) = 1$, but now $\{y_1, v_4, v_5\}$ has no rainbow tree. If $1 \in T(v_3)$, $2, 3 \in T(v_5)$, then we set $v_3v_4 = a_1$, $v_4v_5 = a_2$. If $c(v_2v_6) = i$, $c(v_5v_6) = j$, $i, j \in \{1, 2, 3\}$, then $\{y_i, y_j, v_6\}$ has no rainbow tree. The only possibility is $c(v_2v_3) = 2$, $c(v_2v_6) = 3$, $c(v_5v_6) = a_3$. However, $\{y_1, y_2, v_1\}$ has no rainbow tree.

For $G \in \mathcal{G}_5$, notice that $|A_1 \cap A_2| \leq 3$. If $U(v_2) = 0$ or $U(v_5) = 0$, then G can be obtained by contracting an edge of a graph in $\mathcal{G}_6$. Then by Lemma 2, $rx_3(G) \geq n - 2$. Thus we need to consider the case when $W(v_2) \geq 1$ and $W(v_5) \geq 1$. If $|A_1 \cap A_2| = 2$, then suppose $1 \in T(v_2)$, $2 \in T(v_5)$. Consider $\{y_1, y_2, v_3\}$, $\{y_1, y_2, v_4\}$, we have $c(v_2v_3)$, $c(v_2v_5)$, $c(v_4v_5) \in \{a_1, a_2\}$ and $c(v_2v_3) = c(v_4v_5)$. But now $c(v_3v_4) = i$ $(i = 1, 2)$, then there is no rainbow $\{y_i, v_3, v_4\}$-tree. If $|A_1 \cap A_2| = 3$, then $A_1 = \{1, 2, 3, a_1, a_2\}$. If $1 \in T(v_1)$, $2 \in T(v_2)$, $3 \in T(v_5)$, then consider $\{y_1, y_2, y_3\}$, we have that two of v_1v_2, v_1v_5, v_2v_5 have colors outside A_2, contradicting to $|A_1 \cap A_2| = 3$. If $1 \in T(v_2)$, $2 \in T(v_5)$, $3 \in T(v_2)$ and we may assume that y_3 is on the unique path from y_1 to v_2. Then consider $\{y_1, v_3, y_3\}$ and $\{y_1, v_4, y_3\}$, we have $c(v_2v_3) = c(v_4v_5)$, thus $c(v_3v_4)$ can not be in A_2, contradicting to $|A_1 \cap A_2| = 3$. If $1 \in T(v_2)$, $2 \in T(v_5)$, $3 \in T(v_i)$ $(i = 3, 4)$, then consider $\{y_1, y_2, y_3\}$, we have that $c(v_2v_5)$ is in $A_1 \setminus A_2$, contradicting to $|A_1 \cap A_2| = 3$.

Finally, for a graph G belonging to $\mathcal{G}_4$, it can be obtained by contracting an edge of a graph in $\mathcal{G}_3 \cup \mathcal{G}_6$. Then by Lemma 2, $rx_3(G) \geq n - 2$.

Combining all the cases above, we have $rx_3(G) \geq n - 2$ for $G \in \mathcal{G}$. By Theorem 3, it follows that $rx_3(G) = n - 2$. □

3.2.2 Tricyclic graphs with $rx_3(G)$= n − 2

Define by $\mathcal{H}_i^*$ the set of graphs whose basic graph is H_i, where H_i is shown in Figure 2 and $1 \leq i \leq 8$.

Now, we introduce another graph class $\mathcal{H}$. Set $\mathcal{H} = \mathcal{H}_1 \bigcup \mathcal{H}_2 \bigcup \cdots \bigcup \mathcal{H}_8$, where

$\mathcal{H}_1 = \{G \in \mathcal{H}_1^* | U(G) = 0\}$,
$\mathcal{H}_2 = \{G \in \mathcal{H}_2^* | U(v_i) \leq 1,\ U(v_j) = 0,\ i = 5, 6,\ j = 1, 3, 4\}$,
$\mathcal{H}_3 = \{G \in \mathcal{H}_3^* | U(v_2) \leq 1,\ U(v_5) + U(v_6) \leq 1,\ U(v_i) = 0,\ i = 1, 3, 4\}$,
$\mathcal{H}_4 = \{G \in \mathcal{H}_4^* | U(v_i) \leq 1,\ U(v_j) \leq 2,\ U(v_i) + U(v_j) \leq 1,\ U(v_j) + U(v_k) \leq 3,$
$i = 1, 5,\ j, k = 2, 3, 4\}$,
$\mathcal{H}_5 = \{G \in \mathcal{H}_5^* | U(v_i) \leq 1,\ U(v_j) = 0,\ i = 1, 3, 5,\ j = 2, 4, 6\}$,
$\mathcal{H}_6 = \{G \in \mathcal{H}_6^* | U(v_3) = 0,\ U(v_i) \leq 1,\ U(v_1) + U(v_5) \leq 1,\ i = 1, 2, 4, 5\}$,
$\mathcal{H}_7 = \{G \in \mathcal{H}_2^* | U(v_2) + U(v_4) \leq 1,\ U(v_3) + U(v_5) \leq 1,\ U(v_5) + U(v_1) \leq 1,$
$U(v_j) + U(v_{j+1}) \leq 1,\ j = 1, 2, 4\}$,
$\mathcal{H}_8 = \{G \in \mathcal{H}_8^* | U(v_i) \leq 2,\ U(v_i) + U(v_j) + U(v_k) \leq 3,\ i, j, k = 1, 2, 3, 4\}$.

Lemma 4. *Let G be a connected tricyclic graph of order n. Then $rx_3(G) = n - 2$ if and only if $G \in \mathcal{H}$.*

Proof. Suppose that $rx_3(G) = n - 2$ but $G \notin \mathcal{H}$. Let G_0 be the basic graph of G. Similar to Lemma 3, we have $G_0 = H_i$ and we can rainbow color G with $n - 3$ colors for $G \in \mathcal{H}_i^* \setminus \mathcal{H}_i$, $i = 1, \cdots, 8$.

If $G \in \mathcal{H}_1^* \setminus \mathcal{H}_1$, then if $U(v_2) \geq 1$, set $c_\ell(H_1) = a_4 1 a_1 a_2 a_2 a_4 a_3 a_3 a_4$ and $\{1\} \subseteq T(v_2)$; if $U(v_3) \geq 1$, set $c_\ell(H_1) = a_4 a_1 a_1 1 a_3 a_4 a_2 a_2 a_4$ and $\{1\} \subseteq T(v_3)$; if $U(v_4) \geq 1$, set $c_\ell(H_1) = a_3 a_2 a_2 a_1 1 a_3 a_4 a_4 a_1$ and $\{1\} \subseteq T(v_4)$.

If $G \in \mathcal{H}_2^* \setminus \mathcal{H}_2$, then if $U(v_3) \geq 1$ ($U(v_1) \geq 1$ is similar), set $c_\ell(H_2) = a_3 a_2 1 a_2 a_3 a_1 a_1 a_2$ and $\{1\} \subseteq T(v_3)$; if $U(v_4) \geq 1$, set $c_\ell(H_2) = a_1 a_3 a_2 1 a_2 a_3 a_3 a_1$ and $\{1\} \subseteq T(v_4)$; if $U(v_6) \geq 2$ ($U(v_5) \geq 2$ is similar), set $c_\ell(H_2) = a_3 1 a_2 2 a_2 a_3 a_1 1$ and $\{1, 2\} \subseteq T(v_3)$.

If $G \in \mathcal{H}_3^* \setminus \mathcal{H}_3$, then if $U(v_2) \geq 2$, set $c_\ell(H_3) = a_1 2 a_2 a_3 2 a_2 a_1 1$ and $\{1, 2\} \subseteq T(v_2)$; if $U(v_5) \geq 2$, set $c_\ell(H_3) = 2 a_2 a_3 1 a_2 a_1 1 a_3$ and $\{1, 2\} \subseteq T(v_5)$; if $U(v_5) + U(v_6) \geq 2$, set $c_\ell(H_3) = 1 a_1 2 a_2 a_3 a_1 a_2 a_3$ and $\{1\} \subseteq T(v_6)$, $\{2\} \subseteq T(v_5)$; if $U(v_3) \geq 1$, set $c_\ell(H_3) =$

$a_1a_2a_21a_1a_3a_3a_2$ and $\{1\} \subseteq T(v_3)$; if $U(v_4) \geq 1$, set $c_\ell(H_3) = a_2a_11a_1a_2a_3a_3a_2$ and $\{1\} \subseteq T(v_4)$.

If $G \in \mathcal{H}_4^* \setminus \mathcal{H}_4$, then there are four cases for the graph G: (1) $U(v_i) \geq 3$ $(i = 2, 3, 4)$; (2) $U(v_i) \geq 2$ $(i = 1, 5)$; (3) $U(v_i) + U(v_j) \geq 2$ $(i \in \{1, 5\}, j \in \{2, 3, 4\})$; (4) $U(v_i) \geq 2$ and $U(v_j) \geq 2$, $i, j \in \{2, 3, 4\}$. If G is a graph in case (1), then there exists a graph in $\mathcal{G}_3^* \setminus \mathcal{G}_3$ which is a subgraph of G. Thus the result is obvious. If $U(v_1) \geq 2$, set $c_\ell(H_4) = a_2a_2a_1a_121a_2$ and $\{1, 2\} \subseteq T(v_1)$; if $U(v_1)+U(v_2) \geq 2$, set $c_\ell(H_4) = a_1a_2a_22a_21a_1$ and $\{1\} \subseteq T(v_1)$, $\{2\} \subseteq T(v_2)$; if $U(v_2) \geq 2$ and $U(v_4) \geq 2$, set $c_\ell(H_4) = 3421a_2a_2a_1$ and $\{1, 2\} \subseteq T(v_2)$, $\{3, 4\} \subseteq T(v_4)$.

If $G \in \mathcal{H}_5^* \setminus \mathcal{H}_5$, then $U(v_i) \geq 1$ $(i = 2, 4, 6)$ or $U(v_i) \geq 2$ $(i = 1, 3, 5)$. If G is a graph in the former case, there exists a graph in $\mathcal{G}_6^* \setminus \mathcal{G}_6$ which is a subgraph of G. If $U(v_3) \geq 2$, set $c_\ell(H_5) = a_22a_1a_2a_31a_3a_2$ and $\{1, 2\} \subseteq T(v_3)$; if $U(v_5) \geq 2$, set $c_\ell(H_5) = 1a_22a_1a_3a_2a_3a_1$ and $\{1, 2\} \subseteq T(v_5)$;

If $G \in \mathcal{H}_6^* \setminus \mathcal{H}_6$, then $U(v_3) \geq 1$ or $U(v_i) \geq 2$ $(i = 1, 2, 4, 5)$ or $U(v_1) + U(v_5) \geq 2$. If $U(v_3) \geq 1$, set $c_\ell(H_6) = a_2a_11a_1a_2a_2a_1$ and $\{1\} \subseteq T(v_3)$; if $U(v_1) \geq 2$, set $c_\ell(H_6) = a_2a_1a_11a_212$ and $\{1, 2\} \subseteq T(v_1)$; if $U(v_2) \geq 2$, set $c_\ell(H_6) = a_21a_1a_1a_212$ and $\{1, 2\} \subseteq T(v_2)$; if $U(v_1) + U(v_5) \geq 2$, set $c_\ell(H_6) = a_1a_1a_2a_221a_1$ and $\{1\} \subseteq T(v_1)$, $\{2\} \subseteq T(v_5)$;

If $G \in \mathcal{H}_7^* \setminus \mathcal{H}_7$, then $U(v_i) \geq 2$ $(i = 1, 2, 3, 4, 5)$ or $U(v_1)+U(v_2) \geq 2$ or $U(v_2)+U(v_3) \geq 2$ or $U(v_4) + U(v_5) \geq 2$ or $U(v_2) + U(v_4) \geq 2$ or $U(v_3) + U(v_5) \geq 2$ or $U(v_1) + U(v_5) \geq 2$. If $U(v_1) \geq 2$, set $c_\ell(H_7) = a_1a_2a_212a_1a_1$ and $\{1, 2\} \subseteq T(v_3)$; if $U(v_2) \geq 2$, set $c_\ell(H_7) = a_2a_1a_1a_1a_212$ and $\{1, 2\} \subseteq T(v_2)$; if $U(v_3) \geq 2$, set $c_\ell(H_7) = a_2a_1a_12a_1a_21$ and $\{1, 2\} \subseteq T(v_3)$; if $U(v_1) + U(v_2) \geq 2$, set $c_\ell(H_7) = a_2a_1a_1a_1a_221$ and $\{1\} \subseteq T(v_1)$, $\{2\} \subseteq T(v_2)$; if $U(v_2) + U(v_3) \geq 2$, set $c_\ell(H_7) = a_2a_1a_12a_2a_21$ and $\{1\} \subseteq T(v_2)$, $\{2\} \subseteq T(v_3)$; if $U(v_2) + U(v_4) \geq 2$, set $c_\ell(H_7) = a_212a_1a_2a_1a_2$ and $\{1\} \subseteq T(v_2)$, $\{2\} \subseteq T(v_4)$;

If $G \in \mathcal{H}_8^* \setminus \mathcal{H}_8$, then $U(v_i) \geq 3$ $(i = 1, 2, 3, 4)$ or $U(v_i) + U(v_j) + U(v_k) \geq 4$, $i, j, k = 1, 2, 3, 4$. If G is a graph in the former case, then a graph belonging to $\mathcal{G}_4^* \setminus \mathcal{G}_4$ is a subgraph of G. If $U(v_1) + U(v_2) + U(v_4) \geq 4$, set $c_\ell(H_8) = 1a_1a_1423$ and $\{1, 2\} \subseteq T(v_1)$, $\{3\} \subseteq T(v_2)$, $\{4\} \subseteq T(v_4)$; if $U(v_2)+U(v_3) \geq 4$, set $c_\ell(H_8) = 12a_1a_134$ and $\{1, 2\} \subseteq T(v_2)$, $\{3, 4\} \subseteq T(v_3)$.

It is not hard to check that the colorings above make G rainbow connected with $n - 3$ colors, thus $rx_3(G) \leq n - 3$.

Conversely, let G be a tricyclic graph such that $G \in \mathcal{H}$. Similar to Lemma 3, we only need to consider the case that G has cut edges and $|A_1 \cap A_2| \geq 1$. Assume, to the contrary, that $rx_3(G) \leq n - 3$. Then there exists a rainbow coloring c of G using colors in N_{n-3}.

For $G \in \mathcal{H}_1$, if there is a nontrivial path P' connecting B_1 and B_2 in G, then denote its ends by $v_3'(\in B_1)$ and $v_3''(\in B_2)$ and if there is a nontrivial path P'' connecting B_2 and B_3 in G, then denote its ends by $v_5'(\in B_2)$ and $v_5''(\in B_3)$. Similar to Claim 2 in Lemma 3, the colors in the path $P^{'}$ and $P^{''}$ can not appear in A_1, which implies $|A_1 \cap A_2| = 0$, contradicting to $|A_1 \cap A_2| \geq 1$. For $G \in \mathcal{H}_5$, notice that $d_{H_5}(v_1, v_3, v_5) = 4$

and $|A_1 \setminus A_2| = 3$, the result holds. The same argument applies to the case when $G \in \mathcal{H}_6$. Thus, we mainly discuss the rest cases for G as follows.

Case 1. $G \in \mathcal{H}_2$. we have $1 \leq |A_1 \cap A_2| \leq 4$ and $|A_1 \setminus A_2| = 3$. If there is a nontrivial path P' connecting B_1 and B_2 in G, then denote its ends by $v_4'(\in B_1)$ and $v_4''(\in B_2)$. We can also claim that $c(P') \cap A_1 = \emptyset$. Noticing that $d_{H_2}(v_2, v_5, v_6) = 4 > 3$, we only check the case when $W(v_2) \geq 2$. Since the case of $|A_1 \cap A_2| = 1$ or $|A_1 \cap A_2| = 4$ is easy to check, we consider the remaining two cases. Assume $|A_1 \cap A_2| = 2$, $A_1 = \{1, 2, a_1, a_2, a_3\}$ and $1, 2 \in T(v_2)$, consider $\{y_1, y_2, v_5\}$ and we may set $c(v_2v_3) = a_1$, $c(v_3v_4') = a_2$, $c(v_4''v_5) = a_3$. If 1 and 2 are in B_1, and 1 appears in v_1v_2 or v_1v_4', then we have $c(v_4''v_6) = a_3$ by considering $\{y_1, y_2, v_6\}$, and thus $c(v_5v_6) \notin A_2$, but now $\{y_1, v_5, v_6\}$ has no rainbow tree. So one of 1, 2, say 1, is in B_2 and $c(v_5v_6) = 1$. Now we have $c(v_4''v_6) \neq a_3$ and $c(v_1v_2), c(v_1v_4'), c(v_4''v_6) \in \{a_1, a_2, a_3\}$ by considering $\{y_1, y_2, v_6\}$. Then every $\{y_1, v_5, v_6\}$-tree of size 5 can not have the color 2. Thus there is no rainbow $\{y_1, v_5, v_6\}$-tree.

Assume then $|A_1 \cap A_2| = 3$ and $A_1 = \{1, 2, a_1, a_2, a_3\}$. If $1, 2, 3 \in T(v_2)$, first we claim that v_2v_3, v_3v_4' can not use colors from A_2 both. Otherwise assume $c(v_2v_3) = 1$, $c(v_3v_4) = 2$, $c(v_1v_2) = a_1$, $c(v_1v_4') = a_2$, and by considering $\{y_1, y_2, v_5\}$ and $\{y_1, y_2, v_6\}$, we have $c(v_4''v_5) = c(v_4''v_6) = a_3$, and $c(v_5v_6)$ can be a_1 or a_2. However, there is no rainbow $\{y_1, v_5, v_6\}$-tree or $\{y_2, v_5, v_6\}$-tree. With the same reason, we conclude that exactly one edge of the unique 4-cycle of H_2 can be colored with a color from A_2. Thus there are two cases by symmetry. If $c(v_1v_2) = 1$, $c(v_1v_3) = 2$, $c(v_5v_6) = 3$. Consider y_1 and y_2, together with v_5, v_6 respectively, we have $c(v_4''v_5) = c(v_4''v_6)$, which is impossible. If $c(v_1v_4') = 1$, $c(v_1v_3) = 2$, $c(v_5v_6) = 3$. Consider y_1 and y_3, together with v_5, v_6 respectively. suppose $c(v_4''v_5) = a_1$, $c(v_4''v_6) = a_2$ and $c(v_3v_4') = a_3$, $c(v_1v_2), c(v_2v_3) \in \{a_1, a_2\}$, but now there is no rainbow $\{y_3, v_5, v_6\}$-tree. If $1, 2 \in T(v_2)$, $3 \in T(v_6)$, similarly set $c(v_1v_2) = a_1$, $c(v_1v_4') = a_2$, $c(v_4''v_6) = a_3$. First we can easily claim that the color 3 can not appear in B_2. Thus there are three possibilities for the color 3. If $c(v_3v_4') = 3$, then consider $\{y_1, y_3, v_3\}$ and $\{y_2, y_3, v_3\}$, we have $c(v_1v_3), c(v_2v_3) \in \{1, 2\}$. Consider $\{y_1, y_2, v_5\}$, one of $c(v_4''v_5)$ and $c(v_5v_6)$ is a_3, but now there is no rainbow $\{y_2, y_3, v_5\}$-tree. The case when $c(v_1v_3) = 3$ is similar to the case of $c(v_3v_4') = 3$. If $c(v_2v_3) = 3$, then similarly we get $c(v_1v_3), c(v_2v_3) \in \{1, 2\}$ and one of $c(v_4''v_5)$ and $c(v_5v_6)$ is a_3. Consider $\{y_1, y_3, v_5\}$, this forces one of $c(v_4''v_5)$ and $c(v_5v_6)$ is 2, it is impossible.

Case 2. $G \in \mathcal{H}_3$. Then $1 \leq |A_1 \cap A_2| \leq 4$ and $|A_1 \setminus A_2| = 3$. If there is a nontrivial path P' connecting B_1 and B_2 in G, then denote its ends by $v_4'(\in B_1)$ and $v_4''(\in B_2)$. Similarly, it is easy to check that $c(P') \cap A_1 = \emptyset$. We only focus on the case that $|A_1 \cap A_2| = 1$, where $A_1 = \{1, a_1, a_2, a_3\}$. If $1 \in T(v_6)$, consider $\{y_1, v_1, v_3\}$, set $c(v_4''v_6) = a_1$, $c(v_1v_4') = a_2$, $c(v_3v_4') = a_3$. Then $c(v_5v_6) \neq 1$, otherwise there is no rainbow tree connecting $\{y_1, v_1, v_5\}$ or $\{y_1, v_3, v_5\}$ depending on $c(v_4''v_5)$. Similarly, $c(v_4''v_5) \neq 1$. Next $c(v_2v_4') \neq 1$ by considering $\{y_1, v_2, v_5\}$. Suppose $c(v_1v_2) = 1$, to make sure there is a rainbow $\{y_1, v_1, v_2\}$-tree and $\{y_1, v_2, v_3\}$-tree, we have $c(v_2v_4') = a_3$ and $c(v_2v_3) = a_2$. But now $\{v_2, v_3, v_5\}$ has no rainbow tree. If $1 \in T(v_2)$, then consider $\{y_1, v_5, v_6\}$, $\{y_1, v_1, v_5\}$,

$\{y_1, v_1, v_6\}$, $\{y_1, v_3, v_5\}$, $\{y_1, v_3, v_6\}$ successively. Set $c(v_2v_4') = a_1$, then $c(v_1v_2)$, $c(v_1v_4')$, $c(v_2v_3)$, $c(v_3v_4')$, $c(v_4''v_5)$ and $c(v_4''v_6)$ can only be a_2 or a_3. It is easy to check that $\{v_2, v_3, v_5\}$ has no rainbow tree.

Case 3. $G \in \mathcal{H}_4$. $1 \leq |A_1 \cap A_2| \leq 4$ and $|A_1 \setminus A_2| = 2$. First notice that $d_{H_4}(v_2, v_3, v_4) = 3$, the case that $W(v_1) = W(v_5) = 0$, $W(v_2)$, $W(v_3)$, $W(v_4) \leq 1$ is evident. Assume $|A_1 \cap A_2| = 1$, then $1 \in T(v_1)$, the case is similar with the case that $G \in \mathcal{G}_3$ in Lemma 3. Assume now $|A_1 \cap A_2| = 2$, if $1 \in T(v_1)$, $2 \in T(v_5)$, by considering all the trees containing y_1 and y_2, without loss of generality, set $c(v_1v_5) = a_1$, $c(v_1v_2) = 1(2)$, $c(v_2v_5) = a_2$. Moreover, by considering $\{y_1(y_2), v_2, v_3\}$ and $\{y_1(y_2), v_2, v_4\}$, the remaining two paths of length 2 from v_1 to v_5 must be colored with $2(1)$, a_2, respectively. However, there is no rainbow $\{y_2(y_1), v_3, v_4\}$-tree. If $1, 2 \in T(v_2)$, by considering $\{y_1, y_2, v_4\}$, set $c(v_2v_5) = a_1$, $c(v_4v_5) = a_2$. Since the two possible rainbow trees connecting $\{y_1, v_3, v_4\}$ and $\{y_2, v_3, v_4\}$ are the same, we may set $c(v_3v_5) = 1$. It is easy to see that $c(v_1v_2)$, $c(v_1v_3)$ cannot use colors from A_2 by considering $\{y_1, y_2, v_3\}$, and $c(v_1v_4) = 2$ by considering $\{y_1, v_3, v_4\}$. But now if $c(v_1v_5) = 1$ or $c(v_1v_5) = 2$, there is no rainbow $\{y_1, v_2, v_5\}$-tree or $\{y_2, v_1, v_4\}$-tree, respectively.

Assume $|A_1 \cap A_2| = 3$, then $1, 2 \in T(v_2)$, $3 \in T(v_4)$. Similarly as above, we may set $c(v_2v_5) = a_1$, $c(v_4v_5) = a_2$, $c(v_3v_5) = 1$, $c(v_1v_5) = 3$, one of $c(v_1v_2)$, $c(v_1v_4)$ is 2. However, there is no rainbow $\{y_2, y_3, v_1\}$-tree.

Finally assume $|A_1 \cap A_2| = 4$ and $1, 2 \in T(v_2)$, $3 \in T(v_3)$, $4 \in T(v_4)$, consider $\{y_1, y_3, y_4\}$ and $\{y_2, y_3, y_4\}$, at least four of the non-cut edges must be colored with $\{a_1, a_2\}$. This contradicts to $|A_1 \cap A_2| = 4$.

Case 4. $G \in \mathcal{H}_7$. Since $d_{H_7}(v_1, v_3, v_4) = 3$, we only focus on the case $1 \in T(v_2)$. Consider all the three vertices containing y_1, it is not hard to obtain a contradiction.

Case 5. $G \in \mathcal{H}_8$. First notice $d_{H_8}(v_1, v_2, v_3) = 2$, the case that $W(v_1)$, $W(v_2)$, $W(v_3)$, $W(v_4) \leq 1$ is evident. Assume $|A_1 \cap A_2| = 2$, then $1, 2 \in T(v_1)$. Consider $\{y_1, y_2, v_2\}$, $\{y_1, y_2, v_3\}$, $\{y_1, y_2, v_4\}$ successively, we have $c(v_1v_2) = c(v_1v_3) = c(v_1v_4) = a_1$. However, there is no rainbow tree connecting $\{y_1, v_2, v_3\}$ or $\{y_2, v_2, v_3\}$, a contradiction. Now focus on $|A_1 \cap A_2| = 3$, then $1, 2 \in T(v_1)$, $3 \in T(v_2)$. Consider $\{y_1, y_3, v_3\}$, $\{y_1, y_3, v_4\}$, $\{y_2, y_3, v_3\}$ and $\{y_2, y_3, v_4\}$ successively, $c(v_1v_3)$, $c(v_1v_4)$, $c(v_2v_3)$, $c(v_2v_4)$ must be 1 or 2. Again, there is no rainbow $\{y_1, y_2, v_3\}$-tree.

By the detailed analysis above, we have $rx_3(G) \geq n - 2$ for $G \in \mathcal{H}$. By Theorem 3, it follows that $rx_3(G) = n - 2$. □

3.2.3 Characterize the graphs with $rx_3(G) = \mathbf{n-2}$

We begin with a lemma about a connected 5-cyclic graph.

Lemma 5. *Let G be a connected 5-cyclic graph of order n. Then $rx_3(G) = n - 2$ if and only if $G = K_5 - e$.*

Proof. Let $G \neq K_5 - e$ and $rx_3(G) = n-2$, by Lemma 2 and Theorem 4, $rx_3(G) \leq n-3$, a contradiction. Conversely, suppose $G = K_5 - e$, by Theorem 2, $rx_3(G) \geq 3$, on the other hand, $rx_3(G) \leq rx_3(C_5) = 3$. Thus $rx_3(G) = n-2$. □

For $n \geq 3$, the *wheel* W_n is a graph constructed by joining a vertex v_0 to every vertex of a cycle $C_n : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$.

A third graph class is defined as follows. Let $\mathcal{J}_1$ be a class of graphs such that every graph is obtained from a graph in $\mathcal{H}_5$ by adding an edge v_4v_6. Let $\mathcal{J}_2$ be a class of graphs such that every graph is obtained from a graph in $\mathcal{H}_7$ where $U(v_2) = 0$ and $U(v_5) = 0$ by adding an edge v_2v_5. Set $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, W_4\}$.

Now we are ready to show our second main theorem of this paper.

Theorem 5. *Let G be a connected graph of order n ($n \geq 6$). Then $rx_3(G) = n-2$ if and only if G is unicyclic with the girth of G at least 4 or $G \in \mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$ or $G = K_5 - e$.*

Proof. Let G be a t-cyclic graph with $rx_3(G) = n-2$, but not a graph listed in the theorem. By Proposition 1, Theorem 1, Lemma 3 and Lemma 4, we need to consider the cases $t \geq 4$. If $t = 4$, by Theorem 4, the basic graph of G should be a 3-sun or the basic graph of $\mathcal{J}_2$ or W_4. If $G \notin \mathcal{J}_1$ or $G \notin \mathcal{J}_2$, then by the similar arguments with Lemma 3, we have $rx_3(G) \leq n-3$, a contradiction. If the basic graph of G is W_4 and there are some cut edges in G. If $U(v_0) \geq 1$, then a graph belonging to $\mathcal{G}_6^* \setminus \mathcal{G}_6$ and satisfying $U(v_3) \geq 1$ is a subgraph of G. If $U(v_1) \geq 1$(other cases are similar), then set $c_\ell(W_4) = a_2 1 a_1 a_1 a_1 a_2 a_2 a_1$ and $\{1\} \subseteq T(v_1)$. If $t \geq 5$, by Theorem 4, the basic graph of G should be $K_5 - e$, since $n \geq 6$, by the similar argument with $t = 4$, we have $rx_3(G) \leq n-3$, a contradiction.

Conversely, by Theorem 1, Theorem 2, Lemma 3, Lemma 4 and Lemma 5, suppose G is a graph such that $G \in \mathcal{J}_1$ or $G \in \mathcal{J}_2$. Assume, to the contrary, that $rx_3(G) \leq n-3$. Then there exists a rainbow coloring c of G using $n-3$ colors. Both cases can be considered similar to the case that $G \in \mathcal{H}_5$ or $G \in \mathcal{H}_7$ in Lemma 4, a contradiction. □

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References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[2] G. Chartrand, G. Johns, K. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133 (2008), 85-98.

[3] G. Chartrand, F. Okamoto, P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks 55 (2010), 360-367.

[4] L. Chen, X. Li, K. Yang, Y. Zhao, *The 3-rainbow index of a graph*, accepted for publication in Discuss. Math. Graph Theory.

[5] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, Electron. J. Combin. 15(1)(2008), R57.

[6] G. Chartrand, S. Kappor, L. Lesniak, D. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. 2(1984), 1-6.

[7] G. Chartrand, G. Johns, K. McKeon, P. Zhang, *The rainbow connectivity of a graph*, Networks 54(2) (2009), 75-81.

[8] X. Li, Y. Shi, Y. Sun, *Rainbow connections of graphs*: A survey, Graphs & Combin 29 (2013), 1-38.

[9] X. Li, Y. Sun, *Rainbow Connections of Graphs*, Springer Briefs in Math., Springer, 2012.