Erdős-Ginzburg-Ziv theorem for finite commutative semigroups

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Abstract

Let $\mathcal{S}$ be a finite commutative semigroup written additively, and let $\exp(\mathcal{S})$ be its exponent which is defined as the least common multiple of all periods of the elements in $\mathcal{S}$. For every sequence T of elements in $\mathcal{S}$ (repetition allowed), let $\sigma(T) \in \mathcal{S}$ denote the sum of all terms of T. Define the Davenport constant $\mathsf{D}(\mathcal{S})$ of $\mathcal{S}$ to be the least positive integer d such that every sequence T over $\mathcal{S}$ of length at least d contains a proper subsequence T' with $\sigma(T') = \sigma(T)$, and define $\mathsf{E}(\mathcal{S})$ to be the least positive integer ℓ such that every sequence T over $\mathcal{S}$ of length at least ℓ contains a subsequence T' with $|T| - |T'| = \left\lceil \frac{|\mathcal{S}|}{\exp(\mathcal{S})} \right\rceil \exp(\mathcal{S})$ and $\sigma(T') = \sigma(T)$. When $\mathcal{S}$ is a finite abelian group, it is well known that $\left\lceil \frac{|\mathcal{S}|}{\exp(\mathcal{S})} \right\rceil \exp(\mathcal{S}) = |\mathcal{S}|$ and $\mathsf{E}(\mathcal{S}) = \mathsf{D}(\mathcal{S}) + |\mathcal{S}| - 1$. In this paper we investigate whether $\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \left\lceil \frac{|\mathcal{S}|}{\exp(\mathcal{S})} \right\rceil \exp(\mathcal{S}) - 1$ holds true for all finite commutative semigroups $\mathcal{S}$. We provide a positive answer to the question above for some classes of finite commutative semigroups, including group-free semigroups, elementary semigroups, and archimedean semigroups with certain constraints.

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1 Introduction

Zero-Sum Theory is a rapidly growing subfield of Combinatorial and Additive Number Theory. The main objects of study are sequences of terms from an abelian group (see [8] for a survey, or [16] for a recent monograph). Pushed forward by a variety of applications the last years have seen (at least) three substantially new directions:

(a) The study of weighted zero-sum problems in abelian groups.

(b) The study of zero-sum problems in (not necessarily cancellative) commutative semigroups.

(c) The study of zero-sum problems (product-one problems) in non-commutative groups (see [4, 9, 10, 15]).

In this paper we shall focus on direction (b). Let G be an additive finite abelian group. A sequence T of elements in G is called a *zero-sum sequence* if the sum of all terms of T equals to zero, the identify element of G. Investigations on zero-sum problems were initiated by pioneering research on two themes, one of which is the following result obtained in 1961 by P. Erdős, A. Ginzburg and A. Ziv.

Throrem A. [5] (Erdős-Ginzburg-Ziv) *Every sequence of $2n-1$ elements in an additive finite abelian group of order n contains a zero-sum subsequence of length n.*

Another starting point is the study on Davenport constant $\mathsf{D}(G)$ (named after H. Davenport) of a finite abelian group G, which is defined as the smallest integer d such that, every sequence T of d elements in G contains a nonempty zero-sum subsequence. Though attributed to H. Davenport who proposed the study of this constant in 1965, K. Rogers [20] had first studied it in 1962 and this reference was somehow missed out by most of the authors in this area.

Let G be a finite abelian group. For every integer ℓ with $\exp(G) \mid \ell$, let $\mathsf{s}_\ell(G)$ denote the least integer d such that every sequence T over G of length $|T| \geq d$ contains a zero-sum subsequence of length ℓ. For $\ell = \exp(G)$, we abbreviate $\mathsf{s}(G) = \mathsf{s}_{\exp(G)}(G)$ which is called EGZ constant, and for $\ell = |G|$ we abbreviate $\mathsf{E}(G) = \mathsf{s}_{|G|}(G)$, which is sometimes called Gao-constant (see [15, page 193]). In 1996, the second author of this paper established a connection between Erdős-Ginzburg-Ziv Theorem and Davenport constant.

Throrem B. (Gao, [6, 7]) *If G is a finite abelian group, then $\mathsf{s}_\ell(G) = \mathsf{D}(G) + \ell - 1$ holds for all positive integer ℓ providing that $\ell \geq |G|$ and $\exp(G) \mid \ell$.*

From Theorem B we know that

$$\mathsf{E}(G) = \mathsf{D}(G) + |G| - 1. \quad (*)$$

This formula has stimulated a lot of further research (see [1–3, 11, 14, 16, 18, 25] for example). Among others, the full Chapter 16 in the recent monograph [16] is devoted to this result. Indeed, the final Corollary in this chapter, (Corollary 16.1, page 260), provides a far-reaching generalization of the initial formula (*), which is called the Ψ-weighted Gao Theorem. The formula $\mathsf{E}(G) = \mathsf{D}(G) + |G| - 1$ has also been generalized to some finite groups (not necessarily commutative, for example see [4] and [9]).

In this paper, we aim to generalize $\mathsf{E}(G) = \mathsf{D}(G) + |G| - 1$ to some abstract finite commutative semigroups. To proceed, we shall need some preliminaries. The notations on zero-sum theory used in this paper are consistent with [8] and notations on commutative semigroups are consistent with [13]. For sake of completeness, we introduce some necessary ones.

• Throughout this paper, we shall always denote by $\mathcal{S}$ a finite commutative semigroup and by G a finite commutative group. We abbreviate "finite commutative semigroup" into "**f.c.s.**".

Let $\mathcal{F}(\mathcal{S})$ be the (multiplicatively written) free commutative monoid with basis $\mathcal{S}$. Then any $A \in \mathcal{F}(\mathcal{S})$, say $A = a_1 a_2 \cdot \ldots \cdot a_n$, is a sequence of elements in the semigroup $\mathcal{S}$. The identify element of $\mathcal{F}(\mathcal{S})$ is denoted by $1 \in \mathcal{F}(\mathcal{S})$ (traditionally, the identity element is also called the empty sequence). For any subset $S_0 \subset \mathcal{S}$, let $A(S_0)$ denote the subsequence of A consisting of all the terms from S_0.

The operation of the semigroup $\mathcal{S}$ is denoted by "+". The identity element of $\mathcal{S}$, denoted $0_{\mathcal{S}}$ (if exists), is the unique element e of $\mathcal{S}$ such that $e + a = a$ for every $a \in \mathcal{S}$. The zero element of $\mathcal{S}$, denoted $\infty_{\mathcal{S}}$ (if exists), is the unique element z of $\mathcal{S}$ such that $z + a = z$ for every $a \in \mathcal{S}$. Let

$$\sigma(A) = a_1 + a_2 + \cdots + a_n$$

be the sum of all terms in the sequence A. If S has an identity element $0_{\mathcal{S}}$, we allow $A = 1$, the empty sequence and adopt the convention that $\sigma(1) = 0_{\mathcal{S}}$. Denote

$$\mathcal{S}^0 = \begin{cases} \mathcal{S}, & \text{if } \ \mathcal{S} \text{ has an identity element;} \\ \mathcal{S} \cup \{0\}, & \text{if } \ \mathcal{S} \text{ does not have an identity element.} \end{cases}$$

For any element $a \in \mathcal{S}$, the **period** of a is the least positive integer t such that $ra = (r + t)a$ for some integer $r > 0$. We define $\exp(\mathcal{S})$ to be the period of $\mathcal{S}$, which is the least common multiple of all the periods of the elements of $\mathcal{S}$. Let $A, B \in \mathcal{F}(\mathcal{S})$ be sequences on $\mathcal{S}$. We call B a *proper subsequence* of A if $B \mid A$ and $B \neq A$. We say that A is *reducible* if $\sigma(B) = \sigma(A)$ for some proper subsequence B of A (note that, B is probably the empty sequence 1 if $\mathcal{S}$ has the identity element $0_{\mathcal{S}}$ and $\sigma(A) = 0_{\mathcal{S}}$). Otherwise, we call S *irreducible*.

Definition 1.1. *Let $\mathcal{S}$ be an additively written commutative semigroup.*

1. *Let $\mathsf{d}(\mathcal{S})$ denote the smallest $\ell \in \mathbb{N}_0 \cup \{\infty\}$ with the property:*

   For any $m \in \mathbb{N}$ and $c_1, \ldots, c_m \in \mathcal{S}$ there exists a subset $J \subset [1, m]$ such that $|J| \leq \ell$ and

$$\sum_{j=1}^{m} c_j = \sum_{j \in J} c_j.$$

2. *Let $\mathsf{D}(\mathcal{S})$ denote the smallest $\ell \in \mathbb{N} \cup \{\infty\}$ such that every sequence $A \in \mathcal{F}(\mathcal{S})$ of length $|A| \geq \ell$ is reducible.*

3. *We call $\mathsf{d}(\mathcal{S})$ the small Davenport constant of $\mathcal{S}$, and $\mathsf{D}(\mathcal{S})$ the (large) Davenport constant of $\mathcal{S}$.*

The small Davenport constant was introduced in [11, Definition 2.8.12], and the large Davenport constant was first studied in [24]. For convenience of the reader, we state the following (well known) conclusion.

Proposition 1.2. *Let $\mathcal{S}$ be a finite commutative semigroup. Then,*

1. $\mathsf{d}(\mathcal{S}) < \infty$.

2. $\mathsf{D}(\mathcal{S}) = \mathsf{d}(\mathcal{S}) + 1$.

Proof. 1. See [11, Proposition 2.8.13].

2. Take an arbitrary sequence $A \in \mathcal{F}(\mathcal{S})$ of length at least $\mathsf{d}(\mathcal{S}) + 1$. By the definition of $\mathsf{d}(\mathcal{S})$, there exists a subsequence A' of A such that $|A'| \leq \mathsf{d}(\mathcal{S}) < |A|$ and $\sigma(A') = \sigma(A)$. This proves

$$\mathsf{D}(\mathcal{S}) \leq \mathsf{d}(\mathcal{S}) + 1.$$

Now it remains to prove $\mathsf{d}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) - 1$. Take an arbitrary sequence $T \in \mathcal{F}(\mathcal{S})$. Let T' be the minimal (in length) subsequence of T such that $\sigma(T') = \sigma(T)$. It follows that T' is irreducible and $|T'| \leq \mathsf{D}(\mathcal{S}) - 1$. By the arbitrariness of T, we have

$$\mathsf{d}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) - 1.$$

$\square$

Definition 1.3. *Define $\mathsf{E}(\mathcal{S})$ of any f.c.s. $\mathcal{S}$ as the smallest positive integer ℓ such that, every sequence $A \in \mathcal{F}(\mathcal{S})$ of length ℓ contains a subsequence B with $\sigma(B) = \sigma(A)$ and $|A|-|B| = \kappa(\mathcal{S})$, where*

$$\kappa(\mathcal{S}) = \left\lceil \frac{|\mathcal{S}|}{\exp(\mathcal{S})} \right\rceil \exp(\mathcal{S}). \tag{1}$$

Note that if $\mathcal{S} = G$ is a finite abelian group, the invariants $\mathsf{E}(\mathcal{S})$ and $\mathsf{D}(\mathcal{S})$ are consistent with the classical invariants $\mathsf{D}(G)$ and $\mathsf{E}(G)$, respectively. We suggest the following generalization of $\mathsf{E}(G) = \mathsf{D}(G) + |G| - 1$.

Conjecture 1.4. *For any f.c.s. $\mathcal{S}$,*

$$\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1.$$

We remark that $\mathsf{E}(\mathcal{S}) \geq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1$, the converse of the above inequality, holds trivially when $\mathcal{S}$ contains an identity element $0_{\mathcal{S}}$, i.e., when $\mathcal{S}$ is a finite commutative monoid. The extremal sequence can be obtained by any irreducible sequence T of length $\mathsf{D}(\mathcal{S}) - 1$ adjoined with a sequence of length $\kappa(\mathcal{S}) - 1$ of all terms equaling $0_{\mathcal{S}}$. So, Conjecture 1.4, if true, would imply the following.

Conjecture 1.5. *For any finite commutative monoid $\mathcal{S}$,*

$$\mathsf{E}(\mathcal{S}) = \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1.$$

Nevertheless, to make the study more general, the existence of the identity element is not necessary. We shall verify Conjecture 1.4 holds for some important f.c.s., including group-free f.c.s, elementary f.c.s, and archimedean f.c.s with certain constraints.

2 On group-free semigroups

We begin this section with some definitions.

On a commutative semigroup $\mathcal{S}$ the Green's preorder, denoted $\leqq_{\mathcal{H}}$, is defined by

$$a \leqq_{\mathcal{H}} b \Leftrightarrow a = b + t$$

for some $t \in \mathcal{S}^0$. Green's congruence, denoted $\mathcal{H}$, is a basic relation introduced by Green for semigroups which is defined by:

$$a \, \mathcal{H} \, b \Leftrightarrow a \ \leqq_{\mathcal{H}} \ b \text{ and } b \ \leqq_{\mathcal{H}} \ a.$$

For an element a of $\mathcal{S}$, let H_a be the congruence class by $\mathcal{H}$ containing a. We call a f.c.s. $\mathcal{S}$ **group-free**, provided that all its subgroups are trivial, equivalently, $\exp(\mathcal{S}) = 1$. The group-free f.c.s is fundamental for Semigroup Theory due to the following property.

Property C. (see [13], Proposition 2.4 of Chapter V) *For any f.c.s. $\mathcal{S}$, the quotient semigroup $\mathcal{S}/\mathcal{H}$ of $\mathcal{S}$ by $\mathcal{H}$ is group-free.*

We first show that Conjecture 1.4 holds true for any group-free f.c.s., for which the following lemma will be necessary.

Lemma 2.1. *(See [13], Proposition 2.3 of Chapter V) For any group-free f.c.s. $\mathcal{S}$, the Green's congruence $\mathcal{H}$ is the equality on $\mathcal{S}$.*

Now we are ready to give the following.

Theorem 2.2. *For any group-free f.c.s. $\mathcal{S}$,*

$$\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1.$$

Proof. Take an arbitrary sequence $T \in \mathcal{F}(S)$ of length at least $\mathsf{D}(\mathcal{S})+\kappa(\mathcal{S})-1$. By the definition of $\mathsf{D}(\mathcal{S})$, there exists a subsequence T' of T with $|T'| \leq \mathsf{D}(\mathcal{S}) - 1$ and $\sigma(T') = \sigma(T)$. Let T'' be a subsequence of T containing T', i.e.,

$$T' \mid T'',$$

with length

$$|T''| = \mathsf{D}(\mathcal{S}) - 1. \tag{2}$$

Note that T'' is perhaps equal to T' for example when $|T'| = \mathsf{D}(\mathcal{S}) - 1$. We see that

$$\sigma(T') = \sigma(T) \ \leqq_{\mathcal{H}} \ \sigma(T'') \ \leqq_{\mathcal{H}} \ \sigma(T'),$$

and thus $\sigma(T'') \ \mathcal{H} \ \sigma(T)$. By Lemma 2.1, we have

$$\sigma(T'') = \sigma(T).$$

Combined with (2), we have the theorem proved. □

Definition 2.3. *A commutative nilsemigroup $\mathcal{S}$ is a commutative semigroup with a zero element $\infty_{\mathcal{S}}$ in which every element x is nilpotent, i.e., $nx = \infty_{\mathcal{S}}$ for some $n > 0$.*

Since any finite commutative nilsemigroup is group-free, we have the following immediate corollary of Theorem 2.2.

Corollary 2.4. *Let $\mathcal{S}$ be a finite commutative nilsemigroup. Then* $\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1$.

In the rest of this paper, we need only to consider the case of $\exp(\mathcal{S}) > 1$. Two classes of important f.c.s., finite elementary semigroups and finite archimedean semigroups, will be our emphasis as both these semigroups are basic components in two kinds of decompositions of semigroups, namely, subdirect decompositions and semilattice decompositions, correspondingly. Both decompositions have been the mainstay of Commutative Semigroup Theory for many years (see [13]).

3 On elementary semigroups

With respect to the subdirect decompositions, Birkhoff in 1944 proved the following.

Theorem D. ([13], Theorem 1.4 of Chapter IV) *Every commutative semigroup is a subdirect product of subdirectly irreducible commutative semigroups.*

Hence, we shall give the following result with respect to the subdirect decomposition.

Theorem 3.1. *For any subdirectly irreducible f.c.s. $\mathcal{S}$,*

$$\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1.$$

To prove Theorem 3.1, several preliminaries will be necessary.

Lemma 3.2. *[12] Any subdirectly irreducible f.c.s. is either a nilsemigroup, or an abelian group, or an elementary semigroup. In particular, any f.c.s. is a subdirect product of a commutative nilsemigroup, an abelian group, and several elementary semigroups.*

Definition 3.3. *A commutative semigroup $\mathcal{S}$ is elementary in case it is the disjoint union $\mathcal{S} = G \cup N$ of a group G and a nilsemigroup N, in which N is an ideal of $\mathcal{S}$, the zero element ∞_N of N is the zero element $\infty_{\mathcal{S}}$ of $\mathcal{S}$ and the identity element of G is the identity element of S.*

Lemma 3.4. *([13], Proposition 3.2 of Chapter IV) On any commutative nilsemigroup N, the relation $\mathcal{P}_N$ on N given by $a \, \mathcal{P}_N \, b \Leftrightarrow \infty : a = \infty : b$ is a congruence on N with $\{\infty_N\}$ being a $\mathcal{P}_N$ class, where $\infty : a = \{x \in N^0 : x + a = \infty_N\}$.*

Lemma 3.5. *([13], Proposition 5.1 of Chapter IV) In an elementary semigroup $\mathcal{S} = G \cup N$, the action of every $g \in G$ on N permutes every $\mathcal{P}_N$-class.*

Lemma 3.6. *Let N be a finite commutative nilsemigroup, and let a, b be two elements in N. If $a + b = a$ then $a = \infty_N$.*

Proof. It is easy to see that $a = a + b = a + 2b = \cdots = a + nb = \infty_N$ for some $n \in \mathbb{N}$, done. □

Lemma 3.7. *Let N be a finite commutative nilsemigroup, and let a, b be two elements of N with $a <_{\mathcal{H}} b$. Then $\infty : b \subsetneq \infty : a$.*

Proof. The conclusion $\infty : b \subseteq \infty : a$ is clear. Hence, we need only to show that $\infty : b \neq \infty : a$. If $a = \infty_N$, then $\infty : a = N^0 \neq N \supseteq \infty : b$, done. Hence, we assume $\infty_N <_{\mathcal{H}} a$. It follows that there exists some **minimal** element c of N such that

$$\infty_N <_{\mathcal{H}} c \leqq_{\mathcal{H}} a.$$

Then there exists some $x \in N^0$ such that

$$a + x = c.$$

Since $a <_{\mathcal{H}} b$, then there exists some $y \in N$ such that

$$b + y = a.$$

Since $c \neq \infty_N$, by Lemma 3.6 we have $c + y <_{\mathcal{H}} c$ and hence combined with the minimality of c, we have $c + y = \infty_N$. Therefore, it follows that $a + (x + y) = (a + x) + y = c + y = \infty_N$ and $b + (x + y) = (b + y) + x = a + x = c$, and thus, $x + y \in \infty : a$ and $x + y \notin \infty : b$, and we are done. □

Proof of Theorem 3.1. By Lemma 3.2, Theorem B and Corollary 2.4, it suffices to consider the case of $\mathcal{S} = G \cup N$ is an elementary semigroup.

Take an arbitrary sequence $T \in \mathcal{F}(\mathcal{S})$ of length $\mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1$. Let

$$T_1 = T(G)$$

be the subsequence of T consisting of all the terms from G, and let

$$T_2 = T(N)$$

be the subsequence of T consisting of all the terms from N. Then T_1 and T_2 are disjoint and $T_1 \cdot T_2 = T$.

If $T_2 = 1$, the empty sequence, then $T = T_1$ is a sequence of elements in the subsemigroup G (group). Since $\mathsf{D}(\mathcal{S}) \geq \mathsf{D}(G)$ and $\kappa(\mathcal{S}) \geq |\mathcal{S}| \geq |G|$ is a multiple of $\exp(G) = \exp(\mathcal{S})$, it follows from Theorem B that there exists a subsequence T' of T with

$$|T'| = \kappa(\mathcal{S})$$

and

$$\sigma(T') = 0_G = 0_{\mathcal{S}}.$$

Let

$$T'' = T \cdot T'^{-1}.$$

We see that $\sigma(T'') = \sigma(T'') + 0_{\mathcal{S}} = \sigma(T'') + \sigma(T') = \sigma(T)$, and we are done. Hence, we need only to consider the case that

$$T_2 \neq 1. \tag{3}$$

Assume $\sigma(T) = \infty_{\mathcal{S}}$. Then there exists a subsequence U of T with

$$\sigma(U) = \sigma(T)$$

and $|U| \leq \mathsf{D}(\mathcal{S}) - 1$. Take a subsequence U' of T with

$$U \mid U'$$

and

$$|U'| = \mathsf{D}(\mathcal{S}) - 1.$$

We check that $\sigma(U') = \sigma(U) + \sigma(U'U^{-1}) = \infty_{\mathcal{S}} + \sigma(U'U^{-1}) = \infty_{\mathcal{S}} = \sigma(T)$, and we are done. Hence, we need only to consider the case that

$$\sigma(T) \neq \infty_{\mathcal{S}}. \tag{4}$$

Define the subgroup

$$K = \{g \in G : g + \sigma(T_2) = \sigma(T_2)\}$$

of G, i.e., K is the stabilizer of $\sigma(T_2)$ in G when considering the action of G on N. We claim that

$$\mathsf{D}(\mathcal{S}) \geq |T_2| + \mathsf{D}(G/K). \tag{5}$$

Take a sequence $W \in \mathcal{F}(G) \subset \mathcal{F}(\mathcal{S})$ such that $\varphi_{G/K}(W)$ is zero-sum free in the quotient group G/K with

$$|W| = \mathsf{D}(G/K) - 1,$$

where $\varphi_{G/K}$ denotes the canonical epimorphism of G onto G/K. To prove (5), it suffices to verify that $W \cdot T_2$ is irreducible in $\mathcal{S}$. Suppose to the contrary that $W \cdot T_2$ contains a proper subsequence V with

$$\sigma(V) = \sigma(W \cdot T_2). \tag{6}$$

Let

$$V = V_1 \cdot V_2$$

with

$$V_1 \mid W$$

and

$$V_2 \mid T_2. \tag{7}$$

By (3), we have $\sigma(W \cdot T_2) \in N$, which implies

$$V_2 \neq 1$$

By Lemma 3.5, we have that $\sigma(V_2)\,\mathcal{P}_N\,\sigma(V)$ and $\sigma(T_2)\,\mathcal{P}_N\,\sigma(W \cdot T_2)$. Combined with (6), we have that

$$\sigma(V_2)\,\mathcal{P}_N\,\sigma(T_2). \tag{8}$$

By (4), we have

$$\sigma(T_2) \neq \infty_N. \tag{9}$$

By (7), we have

$$\sigma(T_2) \leqq_{\mathcal{H}} \sigma(V_2),$$

where $\leqq_{\mathcal{H}}$ denotes the Green's preorder in the nilsemigroup N. Combined with (8) and Lemma 3.7, we derive that $\sigma(T_2)\ \mathcal{H}\ \sigma(V_2)$. It follows from Lemma 2.1 that

$$\sigma(T_2) = \sigma(V_2).$$

Combined with (7), (9) and Lemma 3.6, we conclude that

$$V_2 = T_2.$$

Recalling that V is a proper subsequence of $W \cdot T_2$, we have $V_1 \neq W$. Since $\varphi_{G/K}(W)$ is zero-sum free in the group G/K, we derive that $\sigma(V_1) - \sigma(W) \notin K$, and thus, $\sigma(V) = \sigma(V_1) + \sigma(V_2) = \sigma(V_1) + \sigma(T_2) \neq \sigma(W) + \sigma(T_2) = \sigma(W \cdot T_2)$, a contradiction with (6). This proves (5).

By (5), we have that

$$\begin{array}{rcl} |T_1| & = & |T| - |T_2| \\ & \geq & \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1 - (\mathsf{D}(\mathcal{S}) - \mathsf{D}(G/K)) \\ & = & \mathsf{D}(G/K) + \kappa(\mathcal{S}) - 1. \end{array}$$

Applying Theorem B, we derive that T_1 contains a subsequence T_1' with

$$|T_1'| = \kappa(\mathcal{S})$$

such that $\varphi_{G/K}(\sigma(T_1')) = 0_{G/K}$, i.e.,

$$\sigma(T_1') \in K. \tag{10}$$

Let $T_2' = T \cdot T_1'^{-1}$. Observe $T_2 \mid T_2'$. By (10), we check that

$$\begin{array}{rcl} \sigma(T) & = & \sigma(T_1') + \sigma(T_2') \\ & = & \sigma(T_1') + (\sigma(T_2) + \sigma(T_2' \cdot T_2^{-1})) \\ & = & (\sigma(T_1') + \sigma(T_2)) + \sigma(T_2' \cdot T_2^{-1}) \\ & = & \sigma(T_2) + \sigma(T_2' \cdot T_2^{-1}) \\ & = & \sigma(T_2'). \end{array}$$

This completes the proof of the theorem. □

We remark that since the elementary semigroup $\mathcal{S} = G \cup N$ has an identity element $0_{\mathcal{S}} = 0_G$, the equality in the above theorem holds as noted in the introductory section, i.e., $\mathsf{E}(\mathcal{S}) = \mathsf{D}(\mathcal{S}) + \kappa(S) - 1$.

4 On archimedean semigroups

In this section, we shall deal with the class of semigroups associated to semilattice decomposition of semigroups. The semilattice decompositions were obtained for f.c.s by Schwarz [21] and Thierrin [23], and then were extended to all commutative semigroups by Tamura and Kimura [22], in which they proved the following.

Theorem E. *Every commutative semigroup is a semilattice of commutative archimedean semigroups.*

Definition 4.1. *A commutative semigroup $\mathcal{S}$ is called archimedean provided that for any two elements $a, b \in \mathcal{S}$, there exist $m, n > 0$ and $x, y \in \mathcal{S}$ with $ma = b + x$ and $nb = a + y$.*

To be precise, for any commutative semigroup $\mathcal{S}$ there exists a semilattice Y and a partition $\mathcal{S} = \bigcup_{a \in Y} \mathcal{S}_a$ into subsemigroups $\mathcal{S}_a$ (one for every $a \in Y$) with $\mathcal{S}_a + \mathcal{S}_b \subseteq \mathcal{S}_{a \wedge b}$ for all $a, b \in Y$, and moreover, each component $\mathcal{S}_a$ is archimedean. Hence, we shall consider Conjecture 1.4 on archimedean semigroups in what follows. To proceed with it, several preliminaries will be necessary.

Definition 4.2. *We call a commutative semigroup $\mathcal{S}$ nilpotent if $|\underbrace{\mathcal{S} + \cdots + \mathcal{S}}_{t}| = 1$ for some $t > 0$. For any commutative nilpotent semigroup $\mathcal{S}$, the least such positive integer t is called the nilpotency index and is denoted by $\mathcal{L}(\mathcal{S})$.*

Note that, when the commutative semigroup $\mathcal{S}$ is finite, $\mathcal{S}$ is nilpotent if and only if $\mathcal{S}$ is a nilsemigroup. With respect to finite semigroups, the famous Kleitman-Rothschild-Spencer conjecture (see [17]) states that, on a statistical basis, almost all finite semigroups are nilpotent of index at most three, for which there is considerable evidence, but gaps in the original proof have remained unfilled. For the commutative version of this conjecture, there is also some evidence.

We need to give some important notions, namely the Rees congruence and the Rees quotient. Let $\mathcal{I}$ be an ideal of a commutative semigroup $\mathcal{S}$. The relation $\mathcal{J}$ defined by

$$a \, \mathcal{J} \, b \Leftrightarrow a = b \text{ or } a, b \in \mathcal{I}$$

is a congruence on $\mathcal{S}$, the **Rees congruence** of the ideal $\mathcal{I}$. Let $\mathcal{S}/\mathcal{I}$ denote the quotient semigroup $\mathcal{S}/\mathcal{J}$, which is called the **Rees quotient semigroup** of $\mathcal{S}$ by $\mathcal{I}$. The Rees congruence and the resulting Rees quotient semigroup introduced by Rees [19] in 1940 have been among the basic notions in Semigroup Theory. In some sense, the Rees quotient semigroup is obtained by squeezing $\mathcal{I}$ to a zero element (if $\mathcal{I} \neq \emptyset$) and leaving $S \setminus I$ untouched. Hence, it is not hard to obtain the following lemma.

Lemma 4.3. *For any ideal $\mathcal{I}$ of a f.c.s. $\mathcal{S}$,*

$$\mathsf{D}(\mathcal{S}) \geq \mathsf{D}(\mathcal{S}/\mathcal{I}).$$

Lemma 4.4. *([13], Chapter III, Proposition 3.1) A commutative semigroup S which contains an idempotent e (for instance a f.c.s.) is archimedean if and only if it is an ideal extension of an abelian group G by a commutative nilsemigroup N; then S has a kernel $K = H_e = e + S$ and S/K is a commutative nilsemigroup.*

Lemma 4.5. *For any finite commutative nilsemigroup N,*

$$\mathcal{L}(N) \leq \mathsf{D}(N) \leq \mathcal{L}(N) + 1.$$

Proof. By the definition of $\mathcal{L}(N)$, there exists a sequence $T \in \mathcal{F}(N)$ with $|T| = \mathcal{L}(N) - 1$ and $\sigma(T) \neq \infty_N$. By Lemma 3.6, we have that T is irreducible, which implies $\mathsf{D}(N) \geq |T| + 1 = \mathcal{L}(N)$. On the other hand, since any sequence in $\mathcal{F}(N)$ of length $\mathcal{L}(N)$ has a sum ∞_N, we have $\mathsf{D}(N) \leq \mathcal{L}(N) + 1$, and we are through. $\square$

Now we are in a position to put out our result on archimedean semigroup as follows.

Theorem 4.6. *For any finite archimedean semigroup $\mathcal{S}$, $\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S})$. Moreover, if the nilsemigroup $\mathcal{S}/K$ has a nilpotency index at most three, then $\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1$, where K denotes the kernel of $\mathcal{S}$.*

Proof. Let e be the unique idempotent of $\mathcal{S}$. By Lemma 4.4, we have that the kernel $K = H_e = e + S$ and the Rees quotient semigroup

$$N = \mathcal{S}/K \tag{11}$$

is a nilsemigroup. By Theorem B and Corollary 2.4, we need only to consider the case that both K and N are nontrivial. We claim that

$$\mathsf{D}(\mathcal{S}) \geq \max(\mathsf{D}(N),\ \mathsf{D}(K)+1). \tag{12}$$

$\mathsf{D}(\mathcal{S}) \geq \mathsf{D}(N)$ follows from (11) and Lemma 4.3. Now take a minimal zero-sum sequence U of elements in the group K (the kernel of $\mathcal{S}$) with length $|U| = \mathsf{D}(K)$. Since N is nontrivial, the semigroup $\mathcal{S}$ has no identity element, which implies that U is irreducible in $\mathcal{S}$, and thus, $\mathsf{D}(\mathcal{S}) \geq |U| + 1 = \mathsf{D}(K) + 1$. This proves (12).

Now we take a sequence $T \in \mathcal{F}(\mathcal{S})$ with $|T| = \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - \epsilon$, where $\epsilon = 0$ or $\epsilon = 1$ according to the conclusions to prove in what follows. Since $|T| = \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - \epsilon \geq \mathsf{D}(N) + \kappa(G) - \epsilon \geq \mathsf{D}(N)$, it follows from Lemma 4.5 that $\sigma(T)$ belongs to the kernel of $\mathcal{S}$, i.e.,

$$\sigma(T) \in K. \tag{13}$$

Let $\psi_K : \mathcal{S} \to K$ be the canonical retraction of $\mathcal{S}$ onto K, i.e.,

$$\psi_K(a) = e + a$$

for every $a \in \mathcal{S}$. Notice that $\psi_K(T)$ is a sequence of elements in the kernel K with length

$$|\psi_K(T)| = |T| = \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - \epsilon \geq \mathsf{D}(K) + \kappa(\mathcal{S}).$$

Since $|\kappa(\mathcal{S})| \geq |K|$ and $\exp(K) = \exp(\mathcal{S}) \mid \kappa(\mathcal{S})$, it follows from Theorem B that there exists a subsequence T' of T with

$$|T'| = \kappa(\mathcal{S})$$

such that $\psi_K(T')$ is a zero-sum sequence in the kernel K, i.e.,

$$\psi_K(\sigma(T')) = \sigma(\psi_K(T')) = e. \tag{14}$$

Now we assert the following.

Claim. If $\mathsf{D}(\mathcal{S}) - \epsilon \geq \mathcal{L}(N)$ then $\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - \epsilon$.

Since $|TT'^{-1}| = \mathsf{D}(\mathcal{S}) - \epsilon \geq \mathcal{L}(N)$, it follows that

$$\sigma(TT'^{-1}) \in K.$$

Combined with (13) and (14), we have that

$$\begin{aligned}\sigma(TT'^{-1}) &= \sigma(TT'^{-1}) + e \\ &= \sigma(TT'^{-1}) + \psi_K(\sigma(T')) \\ &= \sigma(TT'^{-1}) + (e + \sigma(T')) \\ &= (\sigma(TT'^{-1}) + \sigma(T')) + e \\ &= \sigma(T) + e \\ &= \sigma(T).\end{aligned}$$

Recall $|T'| = \kappa(\mathcal{S})$. This proves the claim.

By (12), Lemma 4.5, and applying the above claim with $\epsilon = 0$, we conclude that

$$\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}).$$

It remains to show $\mathsf{E}(\mathcal{S}) \leq \mathsf{D}(\mathcal{S}) + \kappa(\mathcal{S}) - 1$ when $\mathcal{S}/K$ has a nilpotency index at most three. Take $\epsilon = 1$. By the above claim, we may assume without loss of generality that $\mathsf{D}(\mathcal{S}) \leq \mathcal{L}(N)$. Since K is nontrivial, we have $\mathsf{D}(K) \geq 2$. Combined with $\mathcal{L}(N) \leq 3$ and (12) and Lemma 4.5, we conclude that

$$\mathsf{D}(\mathcal{S}) = \mathcal{L}(N) = 3,$$

and $\mathsf{D}(K) = 2$ which implies

$$K = C_2,$$

the group of two elements. Take a subsequence T'' of T with length

$$|T''| \leq \mathsf{D}(\mathcal{S}) - 1$$

and

$$\sigma(T'') = \sigma(T).$$

If $|T''| = \mathsf{D}(\mathcal{S}) - 1$, we are done. Now assume $|T''| < \mathsf{D}(\mathcal{S}) - 1$, equivalently,

$$|TT''^{-1}| > \kappa(\mathcal{S}).$$

By (13), we have

$$\sigma(T'') \in K, \tag{15}$$

and thus, $\sigma(\psi_K(TT''^{-1})) = \psi_K(\sigma(TT''^{-1})) = e + \sigma(TT''^{-1}) = e$. Since $\exp(\mathcal{S}) = \exp(K) = 2$, we can find a subsequence U of TT''^{-1} of length exactly $\kappa(\mathcal{S})$ such that

$$\sigma(\psi_K(U)) = e.$$

Since $T'' \mid TU^{-1}$, it follows from (15) that $\sigma(TU^{-1}) \in K$, and thus,

$$\begin{aligned} \sigma(TU^{-1}) &= \sigma(TU^{-1}) + e \\ &= \sigma(TU^{-1}) + \sigma(\psi_K(U)) \\ &= \sigma(TU^{-1}) + \psi_K(\sigma(U)) \\ &= \sigma(TU^{-1}) + e + \sigma(U) \\ &= \sigma(T) + e \\ &= \sigma(T). \end{aligned}$$

This completes the proof of the theorem. □

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