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Semisymmetric graphs admitting primitive groups of degree $9p$

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Abstract Let Γ be a connected regular bipartite graph of order $18p$, where p is a prime. Assume that Γ admits a group acting primitively on one of the bipartition subsets of Γ. Then, in this paper, it is shown that either Γ is arc-transitive, or Γ is isomorphic to one of 17 semisymmetric graphs which are constructed from primitive groups of degree $9p$.

Keywords edge-transitive graph, arc-transitive graph, semisymmetric graph, primitive permutation group, suborbit

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1 Introduction

All graphs in this paper are assumed to be finite, simple and undirected.

For a graph Γ, we use $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\Gamma$ to denote its vertex set, edge set and automorphism group, respectively. A graph Γ is said to be *vertex-transitive* or *edge-transitive* if $\mathsf{Aut}\Gamma$ acts transitively on $V\Gamma$ or $E\Gamma$, respectively. A regular edge-transitive graph is called *semisymmetric* if it is not vertex-transitive graph. An *arc* in a graph Γ is an ordered pair of adjacent vertices. A graph Γ is said to be *arc-transitive* if $\mathsf{Aut}\Gamma$ acts transitively on the set of arcs in Γ.

The class of semisymmetric graphs was first systematically studied by Folkman [10]. Afterwards, many authors have done much work on this topic, see [1,2,4,7–9,12,15,16,21,23–26] for references. In particular, lots of interesting examples of such graphs were found. For example, the Folkman graph on 20 vertices, the smallest semisymmetric graph, was constructed by Folkman [10]; the Gray graph, a cubic graph of order 54, was first observed to be semisymmetric by Bouwer [1] and proved to be the smallest cubic semisymmetric graph by Malnič et al [22]. In 1985, Iofinova and Ivanov [15] classified all bi-primitive cubic semisymmetric graphs, they proved that there are only five such graphs. Tutte's 12-cage is one of those graphs, which is the unique cubic semisymmetric graph on 126 vertices and is the fifth smallest cubic semisymmetric graph, see [4].

This paper is a starting point of the project devoted to characterizing edge-transitive regular bipartite graphs order $18p$, where p is a prime. Let Γ be a connected regular bipartite graph of order $18p$. In this paper, we deal with the case where Γ admits a group acting transitively on the edges and primitively on one of the bipartition subsets of Γ. Our main result is stated as follows.

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Theorem 1.1. *Let Γ be a connected regular graph of order $18p$, where p is a prime. Assume that a subgroup $G \leqslant \mathsf{Aut}\Gamma$ acts transitively on $E\Gamma$ but not on $V\Gamma$. If G acts primitively on one of G-orbits, then Γ is either arc-transitive or isomorphic to one of the semisymmetric graphs given in Section* 3.

2 Preliminaries

Let Γ be a graph and let $G \leqslant \mathsf{Aut}\Gamma$. The graph Γ is called *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive* if G acts transitively on its vertex set, edge set or arc set, respectively. The graph Γ is called a *G-semisymmetric* graph if it is regular, G-edge-transitive but not G-vertex-transitive.

Assume that Γ is a G-edge-transitive but not G-vertex-transitive graph, where $G \leqslant \mathsf{Aut}\Gamma$. Then Γ is a bipartite graph with two bipartition subsets being the G-orbits on $V\Gamma$. It follows that Γ is semiregular, that is, the vertices in a same bipartition subset have the same valency. For a given vertex $u \in V\Gamma$, denote by $\Gamma(u)$ the neighborhood of u, that is, the set of vertices adjacent to u in Γ. Then the vertex-stabilizer G_u acts transitively on $\Gamma(u)$. Take $w \in \Gamma(u)$. Then each vertex of Γ can be written as u^g or w^h for some $g, h \in G$. Then, for two arbitrary vertices u^g and w^h, they are adjacent in Γ if and only if u and $w^{hg^{-1}}$ are adjacent, i.e., $hg^{-1} \in G_wG_u$. Moreover, it is well-known and easily shown that Γ is connected if and only if $\langle G_u, G_w\rangle = G$.

Let Γ be a G-semisymmetric graph with two bipartition subsets U and W. Suppose that G has a subgroup R which is regular on both U and W. Take an edge $\{u, w\} \in E\Gamma$. Then each vertex in U (W, resp.) can be written uniquely as u^x (w^x, resp.) for some $x \in R$. Set $S = \{s \in R \mid w^s \in \Gamma(u)\}$. Then u^x and w^y are adjacent if and only if $yx^{-1} \in S$. If R is abelian, then it is easily shown that $u^x \mapsto w^{x^{-1}}, w^x \mapsto u^{x^{-1}}$, $\forall x \in R$ is an automorphism of Γ, which leads to the vertex-transitivity of Γ, refer to [8, 20].

Lemma 2.1. *Let Γ be a G-semisymmetric graph. Assume that G has an abelian subgroup which is regular on both bipartition subsets of Γ. Then Γ is arc-transitive.*

Let Γ be a G-semisymmetric graph. Suppose that G has a normal subgroup N which acts intransitively on at least one of the bipartition subsets of Γ. Then we define the *quotient graph Γ_N* to have vertices the N-orbits on $V\Gamma$, and two N-orbits B and B' are adjacent in Γ_N if and only if some $v \in B$ and some $v' \in B'$ are adjacent in Γ. It is easy to see that G induces an edge-transitive subgroup of $\mathsf{Aut}\Gamma_N$.

Let Γ be a connected G-semisymmetric graph with $G \leqslant \mathsf{Aut}\Gamma$. Denote by $\mathrm{soc}(G)$ the subgroup generated by all minimal normal subgroups of G, which is called the *socle* of G. Take an edge $\{u, w\} \in E\Gamma$ and let $U = w^G$ and $W = w^G$ be the G-orbits on $V\Gamma$. Denote respectively by G^U and G^W the restrictions of G on U and on W. Assume that G is unfaithful on U, and let K be the kernel of G acting on U. Then K is faithful on W. It follows that there are two distinct vertices in W which have the same neighborhood in Γ. Thus, as observed in [8], Γ is semisymmetric while any two distinct vertices in U have different neighborhoods in the quotient graph Γ_K. The next lemma is quoted from [13].

Lemma 2.2. *Let Γ be a connected G-semisymmetric graph with bipartition subsets U and W, where $G \leqslant \mathsf{Aut}\Gamma$. Assume that G^U is quasiprimitive, that is, each minimal normal subgroup of G^U is transitive on U. Then one of the following statements hold.*

(1) Γ is isomorphic to the complete bipartite graph $\mathsf{K}_{|U|,|U|}$;

(2) G is faithful on both U and W, and if G^U is of affine type then Γ is semisymmetric if and only if $\mathrm{soc}(G)$ is intransitive on W;

(3) G is faithful on W but not faithful on U, $G^U \cong G^{\bar{W}}$, and Γ is semisymmetric if further G^U is primitive, where K is the kernel of G acting on U and $\bar{W}$ is the set of K-orbits on W.

Let G be a finite transitive permutation group on a set Ω. The orbits of G on the cartesian product $\Omega \times \Omega$ are the *orbitals* of G, and the diagonal orbital $\{(\alpha, \alpha)^g \mid g \in G\}$ is said to be *trivial.* For a G-orbital Δ and $\alpha \in \Omega$, the set $\Delta(\alpha) = \{\beta \mid (\alpha, \beta) \in \Delta\}$ is a G_α-orbit on Ω and called a *suborbit* of G at α. The *rank* of G on Ω is the number of G-orbitals, which equals to the number of G_α-orbits on Ω for any given $\alpha \in \Omega$. A G-orbital Δ is called *self-paired* if $(\beta,\alpha) \in \Delta$ for some $(\alpha,\beta) \in \Delta$, while the suborbit $\Delta(\alpha)$ is said to be *self-paired*. For a G-orbital Δ, the *paired orbital* Δ^* is defined as $\{(\beta,\alpha) \mid (\alpha,\beta) \in \Delta\}$. Then a G-orbital Δ is self-paired if and only if $\Delta^* = \Delta$. For a non-trivial G-orbital Δ, the *orbital bipartite graph* $B(G,\Omega,\Delta)$ is the graph on two copies of Ω, say $\Omega\times\{1,2\}$, such that $\{(\alpha,1),(\beta,2)\}$ is an edge if and only if $(\alpha,\beta) \in \Delta$. Then $B(G,\Omega,\Delta)$ is G-semisymmetric, where G acts on $\Omega\times\{1,2\}$ as follows:

$$(\alpha,i)^g = (\alpha^g, i),\ g \in G,\ i = 1,\ 2.$$

If Δ is self-paired, then $(\alpha,1) \leftrightarrow (\alpha,2)$, $\alpha \in \Omega$ gives an automorphism of $B(G,\Omega,\Delta)$, which yields that $B(G,\Omega,\Delta)$ is G-arc-transitive. Moreover, the next lemma is easily shown, see also [11].

Lemma 2.3. *Assume that Γ is a connected G-semisymmetric graph of valency at least 2 with bipartition subsets U and W, and that, for an edge $\{u,w\} \in E\Gamma$, the two stabilizers G_u and G_w are conjugate in G. Then there is a bijection $\iota : U \leftrightarrow W$ such that $G_u = G_{\iota(u)}$ and $\{u,\iota(u)\} \notin E\Gamma$ for all $u \in U$. Moreover, $\Delta = \{(u,\iota^{-1}(w)) \mid \{u,w\} \in E\Gamma, u \in U, w \in W\}$ is a G-orbital on U. In particular, $\Gamma \cong B(G,U,\Delta)$, and ι extends to an automorphism of Γ if and only if Δ is self-paired.*

Remak on Lemma 2.3. Let Γ and $G \leqslant \mathsf{Aut}\Gamma$ be as in Lemma 2.3. Then $\{G_u \mid u \in U\} = \{G_w \mid w \in W\}$, and so $\cap_{u\in U} G_u = \cap_{w\in W} G_w = 1$ as $G \leqslant \mathsf{Aut}\Gamma$. Thus G is faithful on both parts of Γ. Take $u \in U$ and $w \in W$ with $G_u = G_w$. Then $u^g \leftrightarrow w^g$, $g \in G$ gives a bijection meeting the requirement of Lemma 2.3. Thus one can define l^2 bijections ι, where l is the number of the points in U fixed by a stabilizer G_u. By [6, Theorem 4.2A], $l = |\mathbf{N}_G(G_u) : G_u|$.

Let G be a finite transitive permutation group on Ω and Δ be a G-orbital. If Δ is self-paired, then $B(G,\Omega,\Delta)$ is arc-transitive. The next lemma indicates it is possible that $B(G,\Omega,\Delta)$ is arc-transitive even if Δ is not self-paired.

Lemma 2.4. *Let X be a permutation group on Ω and let G be a transitive subgroup of X with index $|X : G| = 2$. Let Δ be a G-orbital. If $\Delta \cup \Delta^*$ is an X-orbital, then $B(G,\Omega,\Delta)$ is arc-transitive.*

Proof. Assume that $\Delta \cup \Delta^*$ is an X-orbital. To shown $\Gamma := B(G,\Omega,\Delta)$ is arc-transitive, it suffices to find an automorphism of Γ which interchanges two bipartition subsets of Γ. Take $x \in X \setminus G$. It is easily shown that $\Delta^x = \Delta^*$ and $(\Delta^*)^x = \Delta$. Define $\hat{x} : \Omega\times\{0,1\} \to \Omega\times\{0,1\}$; $(\alpha,0) \mapsto (\alpha^x,1)$, $(\beta,1) \mapsto (\beta^x,0)$. It is easy to check that $\hat{x} \in \mathsf{Aut}\Gamma$, and so the lemma follows. □

The next result is a special version of [8, Lemma 2.6].

Lemma 2.5. *Let Γ be a G-semisymmetric graph with bipartition subsets U and W. Assume that G has an automorphism σ of order 2 such that $G_u^\sigma = G_w$ for some $u \in U$ and $w \in W$. If all G_u-orbits on W have distinct lengths, then Γ is arc-transitive.*

3 Some semisymmetric graphs of order $18p$

In this section we construct the semisymmetric graphs involved in Theorem 1.1.

For a prime power q and positive integer d, we denote respectively by $\mathbb{F}_q$ and $\mathbb{F}_q^d$ the field of order q and the d-dimensional vector space over $\mathbb{F}_q$.

Example 3.1. Let $U = \mathbb{F}_3^3$ and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ a basis of U. Let $W = \{y + \langle\mathbf{e}_i\rangle \mid y \in U, 1 \leqslant i \leqslant 3\}$, which consists of 27 1-dimensional affine subspaces of U. Set $V = U \cup W$ and take 5 subsets of W as follows:

$$\begin{aligned}
&\Delta_0 := \{\langle\mathbf{e}_i\rangle \mid 1 \leqslant i \leqslant 3\},\\
&\Delta_1 := \{\pm\mathbf{e}_2 + \langle\mathbf{e}_1\rangle, \pm\mathbf{e}_3 + \langle\mathbf{e}_2\rangle, \pm\mathbf{e}_1 + \langle\mathbf{e}_3\rangle\},\\
&\Delta_2 := \{\pm\mathbf{e}_3 + \langle\mathbf{e}_1\rangle, \pm\mathbf{e}_1 + \langle\mathbf{e}_2\rangle, \pm\mathbf{e}_2 + \langle\mathbf{e}_3\rangle\},\\
&\Delta_3 := \{\pm\mathbf{e}_i \pm \mathbf{e}_j + \langle\mathbf{e}_k\rangle \mid \{i,j,k\} = \{1,2,3\}\},\\
&\Delta_4 := \Delta_1 \cup \Delta_2.
\end{aligned}$$

For each s with $0 \leqslant s \leqslant 4$, define a bipartite graph $\mathit{\Gamma}_s$ on V such that $\mathbf{x} \in U$ and $\mathbf{y} + \langle \mathbf{e}_i \rangle \in W$ are adjacent in $\mathit{\Gamma}_s$ if and only if $\mathbf{y} - \mathbf{x} + \langle \mathbf{e}_i \rangle \in \Delta_s$. Clearly, these graph have valency 3, 6, 6, 12 and 12, respectively. Moreover, $\mathit{\Gamma}_4$ is the edge-disjoint union of $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$. □

Lemma 3.2. *The graphs given in Example* 3.1 *are all semisymmetric. Moreover,* $\mathit{\Gamma}_1 \cong \mathit{\Gamma}_2$*, and* $\mathit{\Gamma}_0$ *is isomorphic to the Gray graph.*

Proof. We continue the notation used in Example 3.1. Take $h_0, h_1, h_2 \in \mathrm{GL}(3,3)$ such that

$$\begin{array}{lll} \mathbf{e}_1^{h_0} = \mathbf{e}_2, & \mathbf{e}_2^{h_0} = \mathbf{e}_3, & \mathbf{e}_3^{h_0} = \mathbf{e}_1, \\ \mathbf{e}_1^{h_1} = -\mathbf{e}_1, & \mathbf{e}_2^{h_1} = \mathbf{e}_2, & \mathbf{e}_3^{h_1} = \mathbf{e}_3, \\ \mathbf{e}_1^{h_2} = \mathbf{e}_1, & \mathbf{e}_2^{h_2} = \mathbf{e}_3, & \mathbf{e}_3^{h_2} = \mathbf{e}_2. \end{array} \tag{3.1}$$

Set $H = \langle h_1, h_2, h_0 \rangle$ and $H_1 = \langle h_1, h_0 \rangle$. Then both H and H_1 are irreducible subgroups of $\mathrm{GL}(3,3)$. Let N be the group consists of all affine transformations of the form $\tau_{\mathbf{x}} : \mathbb{F}_3^3 \to \mathbb{F}_3^3$, $\mathbf{y} \mapsto \mathbf{y} + \mathbf{x}$. Then we get two primitive permutation groups $G = N{:}H$ and $G_1 = N{:}H_1$ (on U). Define an action of G on W by

$$(\mathbf{y} + \langle \mathbf{e}_i \rangle)^{\tau_{\mathbf{x}}} = \mathbf{y} + \mathbf{x} + \langle \mathbf{e}_i \rangle,\ (\mathbf{y} + \langle \mathbf{e}_i \rangle)^h = \mathbf{y}^h + \langle \mathbf{e}_i^h \rangle;\ \mathbf{x}, \mathbf{y} \in U, h \in H. \tag{3.2}$$

It is easily shown that G is transitive on $E(\mathit{\Gamma}_0)$, $E(\mathit{\Gamma}_3)$ and $E(\mathit{\Gamma}_4)$, and that G_1 is transitive $E(\mathit{\Gamma}_0)$, $E(\mathit{\Gamma}_1)$ and $E(\mathit{\Gamma}_2)$. Note that $\mathrm{soc}(G_1) = \mathrm{soc}(G) = N$ and N is intransitive on W. By Lemma 2.2 (2), each $\mathit{\Gamma}_s$ is semisymmetric. Moreover, it is easily shown that h_2 gives an isomorphism from $\mathit{\Gamma}_1$ to $\mathit{\Gamma}_2$.

It is known that, up to isomorphism, the Gray graph is the unique cubic semisymmetirc graph of order 54 (refer to [4]). Thus $\mathit{\Gamma}_0$ is isomorphic to the Gray graph. This completes the proof. □

It is well-know that Tutte's 12-cage is a cubic semisymmetric graph with automorphism group isomorphic to $\mathrm{P\Gamma U}(3,3)$. In the next example we give a construction for Tutte's 12-cage based on the argument in [3, pp.383, 12.4].

Example 3.3. Equip $V = \mathbb{F}_9^3$ with the standard unitary inner product

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1^3 + x_2 y_2^3 + x_3 y_3^3,\ \mathbf{x}, \mathbf{y} \in V.$$

A non-zero vector $\mathbf{x} \in V$ is called non-isotropic if $(\mathbf{x}, \mathbf{x}) \neq 0$. Then V has 504 non-isotropic vectors. These vectors span 63 1-dimensional subspaces (non-isotropic points in $\mathrm{PG}(2,3)$). Let U be the set of these subspaces. Define a graph Φ on U such that $\langle \mathbf{x} \rangle, \langle \mathbf{y} \rangle \in U$ are adjacent if and only if $(\mathbf{x}, \mathbf{y}) = 0$. Then $\mathsf{Aut}\Phi = \mathrm{P\Gamma U}(3,3)$, and Φ is a distance-transitive graph with valency 6 and diameter 3. Moreover, Φ has exactly 63 triangles. Note that the vertex set of each triangle consists of three mutually orthogonal members in U, which is called an orthogonal frame of V. Let W be the set of these orthogonal frames. Then Tutte's 12-cage can be construct on $U \cup W$ such that $u \in U$ and $w \in W$ are adjacent if and only if $u \in w$.

Let Σ be a connected graph with diameter d. For $u \in V\Sigma$ and integer i with $0 \leqslant i \leqslant d$, denote by $\Sigma_i(u)$ the set of vertices which are at distance i from u. Then the distance i graph, denoted by $\partial_i(\Sigma)$, is defined as the graph on $V\Sigma$ such that two vertices are adjacent if and only if they are at distance i from each other in Σ.

Lemma 3.4. *Let* Σ *be the graph of order* 126 *given in Example* 3.3*. Then* $\partial_3(\Sigma)$ *and* $\partial_5(\Sigma)$ *are semisymmetric graphs of valency* 12 *and* 48*, respectively.*

Proof. We continue the notation used in Example 3.3 and, without loss of generality, write $\mathsf{Aut}\Sigma = \mathrm{P\Gamma U}(3,3)$. Note that Σ has valency 3 and girth 12. It is easily shown that $|\Sigma_i(u)| = 3 \cdot 2^{i-1}$ for $u \in V\Sigma$ and $1 \leqslant i \leqslant 5$. Then the distance i graph $\partial_i(\Sigma)$ has valency $3 \cdot 2^{i-1}$, and it is connected if and only if i is odd. Clearly, $\mathsf{Aut}\Sigma \leqslant \mathsf{Aut}\partial_i(\Sigma)$.

Let $A = \mathsf{Aut}\Sigma$. We first prove that both $\partial_3(\Sigma)$ and $\partial_5(\Sigma)$ are A-edge-transitive. It suffices to show that $\Sigma_3(u)$ and $\Sigma_5(u)$ are A_u-orbits on W. Since Σ is bipartite, $\partial_2(\Sigma)$ is not connected. By the definition of Φ, we know that Φ is a connected component of $\partial_2(\Sigma)$. For $i = 2$, 4 and 6, it is easy to see that

$\Sigma_i(u) = \Phi_{\frac{i}{2}}(u)$. Recall that Φ is distance-transitive. It follow that A_u is transitive on each of $\Sigma_2(u)$, $\Sigma_4(u)$ and $\Sigma_6(u)$. Since $|\Sigma_3(u)| = 2|\Sigma_2(u)|$, we know that A_u has at most two orbits on $\Sigma_3(u)$. If A_u has two orbits on $\Sigma_3(u)$, then an A_u-orbit on $\Sigma_4(u)$ has length at most 12, a contradiction. Thus A_u is transitive on $\Sigma_3(u)$. Since $|\Sigma_5(u)| = 2|\Sigma_4(u)|$, we know that A_u has at most two orbits on $\Sigma_5(u)$. Now let B be an A_u-orbit on $\Sigma_5(u)$. Then $|B|$ is divisible by 24. Consider the subgraph Λ of Σ induced by $B \cup \Sigma_4(u)$. Then Λ is bipartite. Since B and $\Sigma_4(u)$ are A_u-orbits, Λ is semiregular. Enumerating the edges of Λ yields that $|B| = 48$, hence $\Sigma_5(u)$ is an A_u-orbit. Thus $\partial_3(\Sigma)$ and $\partial_5(\Sigma)$ are A-edge-transitive.

Let $\Gamma = \partial_3(\Sigma)$ or $\partial_5(\Sigma)$, and let X be the subgroup of $\mathsf{Aut}\Gamma$ which preserves the bipartition of Γ. Then $X \geqslant A = \mathrm{P\Gamma U}(3,3)$. Checking the subgroups of $\mathrm{P\Gamma U}(3,3)$ (refer to the Atlas [5]), we know that, for $u \in U$ and $w \in W$, the stabilizers A_u and A_w are non-conjugate maximal subgroups in A. In particular, A and hence X acts primitively on both U and W. Since Σ is not a complete bipartite graph, it is easily shown that X acts faithfully on both U and W. Note that all primitive permutation groups of degree 63 are listed in Table 1. It follows that $X = A$.

Suppose that $\mathsf{Aut}\Gamma \neq A$. Then $|\mathsf{Aut}\Gamma : A| = 2$, and so $\mathsf{Aut}\Gamma = A.\mathbb{Z}_2$. Note that $\mathrm{soc}(A) = \mathrm{PSU}(3,3)$ is a characteristic subgroup of A. It follows that $\mathrm{soc}(A)$ is normal in $\mathsf{Aut}\Gamma$. Then $\mathsf{Aut}\Gamma/\mathbf{C}_{\mathsf{Aut}\Gamma}(\mathrm{soc}(A)) = \mathbf{N}_{\mathsf{Aut}\Gamma}(\mathrm{soc}(A))/\mathbf{C}_{\mathsf{Aut}\Gamma}(\mathrm{soc}(A))$ is isomorphic to a subgroup of $\mathsf{Aut}(\mathrm{soc}(A))$. Since $\mathsf{Aut}(\mathrm{soc}(A)) \cong A$ by the Atlas [5], it follows that $\mathbf{C}_{\mathsf{Aut}\Gamma}(\mathrm{soc}(A)) \neq 1$. Since $\mathrm{soc}(A)$ is a non-abelian simple group, we know that $\mathbf{C}_{\mathsf{Aut}\Gamma}(\mathrm{soc}(A)) \cap \mathrm{soc}(A) = 1$. It implies that $\mathbf{C}_{\mathsf{Aut}\Gamma}(\mathrm{soc}(A)) \cong \mathbb{Z}_2$ and $\mathsf{Aut}\Gamma = \mathrm{soc}(A) \times \mathbf{C}_{\mathsf{Aut}\Gamma}(\mathrm{soc}(A))$. It follows that there is an involution $g \in \mathsf{Aut}\Gamma$ which centralizes A and interchanges U and W. For $u \in U$, we have that $w := u^g \in W$ and $A_w = (A_u)^g = A_u$, which is a contradiction.

Therefore, $A = \mathsf{Aut}\Gamma$, and hence Γ is semisymmetric. □

We remark that Tutte's 12-cage and its distance 3 and 5 graphs form a factorization of the complete bipartite graph $\mathsf{K}_{63,63}$.

Let Σ be a G-edge-transitive graph with bipartition subsets U and $\bar{W}$ such that $m|\bar{W}| = |U|$ for some integer $m > 1$. Define a bipartite graph $\Sigma^{1,m}$ with vertex set $U \cup (\bar{W} \times \mathbb{Z}_m)$ such that $u \in U$ and $(\bar{w}, i) \in \bar{W} \times \mathbb{Z}_m$ are adjacent if and only if $\{u, \bar{w}\} \in E\Sigma$. Each $g \in G$ can be extended to an automorphism of $\Sigma^{1,m}$, which acts on U in the same way as g on U and acts on $\bar{W} \times \mathbb{Z}_m$ as follows:

$$(\bar{w}, i)^g = (\bar{w}^g, i) \text{ for } \bar{w} \in \bar{W}, i \in \mathbb{Z}_m.$$

For each $j \in \mathbb{Z}_m$, we may define an automorphism of $\Sigma^{1,m}$, which acts on U trivially and acts on $\bar{W} \times \mathbb{Z}_m$ as follows:

$$(\bar{w}, i) \mapsto (\bar{w}, i + j);\ \bar{w} \in \bar{W}, i \in \mathbb{Z}_m.$$

Then $\mathsf{Aut}\Sigma^{1,m}$ contains an edge-transitive subgroup isomorphic to $G \times \mathbb{Z}_m$. By Lemma 2.2, it is easily shown that $\Sigma^{1,m}$ is semisymmetric while G is primitive on U and Σ is not a complete bipartite graph. In the following we list some graphs of this kind.

Example 3.5. Let U and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be as in Example 3.1. Set $U_1 = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$, $U_2 = \langle \mathbf{e}_1, \mathbf{e}_3 \rangle$ and $U_3 = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$. Let $\bar{W} = \{U_i, \pm\mathbf{e}_i + U_i \mid i = 1, 2, 3\}$.

Define a bipartite graph Σ_0 on $U \cup \bar{W}$ such that $\mathbf{x} \in U$ and $\mathbf{y} + U_i \in \bar{W}$ are adjacent if and only if $\mathbf{y} - \mathbf{x} \in U_i$. Let Σ_1 be the complement graph of Σ_0 in the complete bipartite graph with bipartition subsets U and $\bar{W}$.

By similar argument as in the proof of Lemma 3.2, we know that both Σ_0 and Σ_1 admit an edge-transitive group isomorphic to $\mathbb{Z}_3^3{:}(\mathbb{Z}_2 \times \mathrm{S}_4)$, which acts primitively on U. It follows that both $\Sigma_0^{1,3}$ and $\Sigma_1^{1,3}$ are semisymmetric. □

Example 3.6. Let $\bar{W} = \{1, 2, \cdots 19\}$, and let U be the set of 2-subsets of $\bar{W}$. Let $T = \mathrm{A}_{19}$. Then T acts primitively on both U and $\bar{W}$. For $\{i, j\} \in U$, the stabilizer $T_{\{i,j\}}$ has exactly two orbits on $\bar{W}$, which are $\{i, j\}$ and $\bar{W} \setminus \{i, j\}$.

(1) Define a bipartite graph Σ_1 on $U \cup \bar{W}$ such that $u \in U$ and $w \in \bar{W}$ are adjacent in Σ_1 if and only if $w \in u$. (Note that Σ_1 is just the vertex-edge incidence graph of the complete graph K_{19}.) Let

Σ_2 be the complement graph of Σ_1 in the complete bipartite graph $\mathsf{K}_{171,19}$. Then both Σ_1 and Σ_2 are T-edge-transitive graphs.

(2) The graphs $\Sigma_1^{1,9}$ and $\Sigma_2^{1,9}$ are semisymmetric graphs of valency 18 and 153, respectively. □

By the Atlas [5], the simple group PSL(2, 19) has two conjugacy classes of maximal subgroups isomorphic to A_5. Take $\bar{W}$ as one of these two conjugacy classes. Define a graph on $\bar{W}$ by letting $\bar{w}_1, \bar{w}_2 \in \bar{W}$ be adjacent whenever $\bar{w}_1 \cap \bar{w}_2 \cong \mathrm{D}_{10}$. Then this graph, called the Perkel graph, is a distance-transitive graph with automorphism group PSL(2, 19), order 57 and intersection array $\{6, 5, 2; 1, 1, 3\}$, refer to [3, pp.401, 13.3].

Lemma 3.7. *Assume that Σ is the Perkel graph constructed as above. Let $G = \mathsf{Aut}\Sigma$ and $\bar{w} \in V\Sigma$. Then $G_{\bar{w}}$ has exactly 6 orbits on $E\Sigma$: one has length 6, one has length 15, three have length 30 and one has length 60.*

Proof. For $1 \leqslant i \leqslant 3$, denote by $\Sigma_i(\bar{w})$ the set of vertices at distance i from $\bar{w}$. Then $|\Sigma_1(\bar{w})| = 6$, $|\Sigma_2(\bar{w})| = 30$ and $|\Sigma_3(\bar{w})| = 20$. For $1 \leqslant i \leqslant 3$ and $j = i$ or $i - 1$, denote by $\Sigma_{j,i}$ the subgraph of Σ induced by $\Sigma_j(\bar{w}) \cup \Sigma_i(\bar{w})$, where $\Sigma_0(\bar{w}) = \{\bar{w}\}$. Let $E_{j,i}$ be the edge set of $\Sigma_{j,i}$. Then $E_{0,1}$, $E_{1,2}$, $E_{2,2}$, $E_{2,3}$ and $E_{3,3}$ form a partition of $E\Sigma$. It is easily shown that $|E_{0,1}| = 6$, $|E_{1,2}| = 30$, $|E_{2,2}| = 45$, $|E_{2,3}| = 60$ and $|E_{3,3}| = 30$.

Let $H = G_{\bar{w}}$. Then $H \cong \mathrm{A}_5$. Since Σ is distance-transitive, H acts transitively on each $\Sigma_i(\bar{w})$, where $1 \leqslant i \leqslant 3$. In particular, H is transitive on $E_{0,1}$. Note that H is 2-transitive on $\Sigma_1(\bar{w})$. It follows that G acts transitively on the directed 2-paths of Σ. Then H is transitive on those 2-paths from $\bar{w}$, and hence H is transitive on $E_{1,2}$. By the construction of Σ, we know that, for an edge $\{\bar{w}_1, \bar{w}_2\}$ of Σ, the arc-stabilizer $G_{\bar{w}_1\bar{w}_2}$ is isomorphic to D_{10}. Thus, for an element $h \in H$ of order 3, if $h \in G_{\bar{w}_1}$ then h dose not fix $\bar{w}_2$. Using such an observation, it is easily shown that H is transitive on each of $E_{2,3}$ and $E_{3,3}$. Then we get 4 H-orbits on $E\Sigma$, which have length 6, 30, 60 and 30 respectively.

Consider that action of H on $E_{2,2}$. Take $\bar{v} \in \Sigma_2(\bar{w})$. Since H is transitive on $\Sigma_2(\bar{w})$, we know that $|H_{\bar{v}}| = \frac{|H|}{|\Sigma_2(\bar{w})|} = 2$. Note that $\Sigma_1(\bar{w})$ contains a unique vertex, say $\bar{u}$, adjacent to $\bar{v}$. Then $H_{\bar{v}}$ fixes $\bar{u}$, and so $H_{\bar{v}} < G_{\bar{u}\bar{v}} \cong \mathrm{D}_{10}$. Set $H_{\bar{v}} = \langle k \rangle$. Then $G_{\bar{u}\bar{v}} = \langle h, k \rangle$ for some h of order 5, and $khk = h^{-1}$. Since $G_{\bar{v}}$ is faithful on $\Sigma_1(\bar{v})$, writing h and k as permutations on $\Sigma_1(\bar{v})$, we know that h is a 5-cycle and k is a product of two disjoint transpositions. It follows that k interchanges two of the three vertices contained in $\Sigma_1(\bar{v}) \cap \Sigma_2(\bar{w})$. It implies that one of the H-orbits on $E_{2,2}$ has length at least 30. Since 45 is not a divisor of $|H|$, we know that H has at least two orbits on $E_{2,2}$. Note that a vertex-transitive graph of order 30 has at least 15 edges. It follows that H has exactly two orbits on $E_{2,2}$, which have length 30 and 15 respectively. This completes the proof. □

Example 3.8. Let Σ be the Perkel graph. Set $G = \mathsf{Aut}\Sigma$ and take $\bar{w} \in V\Sigma$. Then, for an edge $\{\bar{w}, \bar{v}\}$, the edge-stabilizer $G_{\{\bar{w},\bar{v}\}} \cong G_{\bar{w}\bar{v}}.\mathbb{Z}_2 \cong \mathrm{D}_{20}$, which is a maximal subgroup of $G = \mathrm{PSL}(2, 19)$. Thus G acts primitively one the edge set of Σ. Assume that $\Delta_i(\bar{w})$, $1 \leqslant i \leqslant 6$, are the six orbits of $G_{\bar{w}}$ acting on the edges of Σ.

(1) Let $U = E\Sigma$ and $\bar{W} = V\Sigma$. Then $\bar{W} = \{\bar{w}^g \mid g \in G\}$. For each i with $1 \leqslant i \leqslant 6$, define a bipartite graph Σ_i on $U \cup \bar{W}$ such $u \in U$ and $\bar{w}^g \in \bar{W}$ are adjacent if and only if $u^{g^{-1}} \in \Delta_i(\bar{w})$. Then each Σ_i is G-edge-transitive. (The graph of valency 6, say Σ_1, is the vertex-edge incidence graph of the Perkel graph, two of the three graphs with valency 30 are respectively the distance 3 and 7 graphs of Σ_1, and the other three graphs form a factorization of the distance 5 graph of Σ_1.)

(2) Let $\mathit{\Gamma}_i = \Sigma_i^{1,3}$, where $1 \leqslant i \leqslant 6$. Then all $\mathit{\Gamma}_i$ are semisymmetric graphs.

4 The proof of Theorem 1.1

In this section we give a proof Theorem 1.1. Our argument is based on analyzing the primitive permutation groups of degree $9p$ and $3p$.

For a positive integer $k < p$, all primitive permutation groups of degree kp are explicitly known by [18, 19]. Let X be a primitive permutation group of degree $9p$ or $3p$. Combining with [6, Appendix

Line	Degree $9p$	$T := \mathrm{soc}(X)$	Actions of T	Remark
1	45	$\mathrm{PSL}(2,9)$	cosets of D_8	$\mathrm{S}_6 \ncong X \leqslant T.\mathbb{Z}_2^2$
2	153	$\mathrm{PSL}(2,17)$	cosets of D_{16}	
3	$\frac{(c-1)c}{2}$	A_c	2-subsets	$c \in \{10,\ 18,\ 19\}$
4	27	$\mathrm{PSU}(4,2)$	isotropic lines	$T \cong \mathrm{O}^-(6,2)$
5	45	$\mathrm{PSU}(4,2)$	isotropic points	
6	63	$\mathrm{Sp}(6,2)$	points	
7	171	$\mathrm{PSL}(2,19)$	cosets of D_{20}	
8	369	$\mathrm{PSL}(2,3^4)$	cosets of $\mathrm{PGL}(2,9)$	
9	117	$\mathrm{PSL}(3,3)$	anti-flags of $\mathrm{PG}(2,3)$	$X = T.\mathbb{Z}_2$
10	657	$\mathrm{PSL}(3,8)$	flags of $\mathrm{PG}(2,8)$	$X = T.\mathbb{Z}_2, T.\mathbb{Z}_6$
11	63	$\mathrm{PSU}(3,3)$	non-isotropic points; bases	
12	117	$\mathrm{O}^+(6,3)$	one of T-orbits on non-isotropic points	$X = T, \mathrm{PGO}^+(6,3)$ $T \cong \mathrm{PSL}(4,3)$
13	$9p$	A_{9p}	natural action	2-transitive
14	18	$\mathrm{PSL}(2,17)$	points	2-transitive
15	2^e+1	$\mathrm{PSL}(2,2^e)$	points	$e = 3r$, odd prime r 2-transitive
16	63	$\mathrm{PSL}(6,2)$	points hyperplanes	2-transitive 2-transitive

Table 1 Primitive permutation groups of degree $9p$.

B], either $p = 3$ and X is of affine type, or X is one of almost simple groups listed in Tables 1 and 2.

In the following we assume that Γ is a connected G-semisymmetric graph of order $18p$, where $G \leqslant \mathsf{Aut}\Gamma$ and p is a prime. Let U and W be the orbits of G acting on $V\Gamma$. Assume that one of G^U and G^W is primitive. Without loss of generality, we assume further that G^U is primitive and that Γ is not a complete bipartite graph. By Lemma 2.2, G is faithful on W, that is, $G^W \cong G$.

The next lemma says that Theorem 1.1 holds while G^U is of affine type.

Lemma 4.1. *Assume that G^U is an affine primitive group. Then Γ is either arc-transitive or isomorphic to one of the graphs given in Examples* 3.1 *and* 3.5.

Proof. Since G^U is of affine type, $\mathrm{soc}(G^U) \cong \mathbb{Z}_3^3$. Identify U with the 3-dimensional vector space over $\mathbb{F}_3$. Write $G^U = N{:}H$, where H is an irreducible subgroup of $\mathrm{GL}(3,3)$, and $N = \mathrm{soc}(G^U)$ consists of all affine transformations of the form $\tau_{\mathbf{x}} : \mathbb{F}_3^3 \to \mathbb{F}_3^3$, $\mathbf{y} \mapsto \mathbf{y} + \mathbf{x}$. Let u be the vertex corresponding to the zero vector. Then $H = (G^U)_u$.

Let K be the kernel of G acting on U. Then K is faithful on W. Consider the quotient graph $\Sigma := \Gamma_K$ with respect to K. Identifying G^U with a subgroup of $\mathsf{Aut}\Sigma$, the graph Σ is G^U-edge-transitive. Since G is transitive on W and K is normal in G, all K-orbits on W have the same length, say m. Then either $K = 1$ or $\Gamma \cong \Sigma^{1,m}$.

Let $\bar{W}$ be the set of K-orbits on W. Since Γ is not a complete bipartite graph, G^U is faithful on $\bar{W}$ by Lemma 2.2. Suppose that N is transitive on $\bar{W}$. It is easily shown that N is regular on $\bar{W}$, and hence

Degree $3p$	X	Action or Remark
6	$\mathrm{A}_5, \mathrm{S}_5$	cosets of D_{10} in A_5
15	$\mathrm{A}_6, \mathrm{S}_6$	2-subsets
21	$\mathrm{A}_7, \mathrm{S}_7$	2-subsets
21	$\mathrm{PSL}(3,2).\mathbb{Z}_2$	point-line incedent pairs
57	$\mathrm{PSL}(2,19)$	cosets of A_5 (two actions)
15	A_7	cosets of $\mathrm{PSL}(2,7)$ (two actions)
$3p$	$\mathrm{A}_{3p}, \mathrm{S}_{3p}$	
15	$\mathrm{PSL}(4,2)$	points, hyperplanes
2^e+1	$\mathrm{PSL}(2,2^e), \mathrm{P\Gamma L}(2,2^e)$	points; odd prime e
q^2+q+1	$\mathrm{PSL}(3,q).O$	points, hyperplanes; $q \equiv 1 \,(\mathsf{mod}\ 3)$,
		$q = r^e$, prime r, $\|O\| \,\big\|\, 3e$

Table 2 Primitive permutation groups of degree $3p$ (refer to [12]).

$K=1$. By Lemma 2.1, $\mathit{\Gamma} \cong \Sigma$ is arc-transitive. Thus we assume further that N is intransitive on $\bar{W}$; in this case, $\mathit{\Gamma}$ must be semisymmetric by Lemma 2.2.

Let l be the number of N-orbits on $\bar{W}$. Then l is a proper divisor of $|\bar{W}| = \frac{27}{m}$ as N is intransitive on $\bar{W}$. Let p be an arbitrary prime divisor of H and $h \in H$ be of order p. Since G^U acts faithfully on $\bar{W}$, we know that either $\langle h \rangle$ is faithful on the set of N-orbits on $\bar{W}$, or $\langle h \rangle$ fixes every N-orbits set-wise and acts faithfully on at least one of N-orbits. It follows that p is a divisor of $l!$ or $(\frac{27}{lm})!$, and hence $p < 9$. Since $H \leqslant \mathrm{GL}(3,3) \cong \mathbb{Z}_2{\times}\mathrm{PSL}(3,3)$, checking the subgroups of $\mathrm{PSL}(3,3)$ in the Atlas [5], we know that H is isomorphic to a subgroup of $\mathbb{Z}_2{\times}\mathrm{S}_4$.

Since G^U acts transitively on $\bar{W}$, we know that H acts transitively on the l orbits of N acting on $\bar{W}$. Recall that l is a proper divisor of $\frac{27}{m}$. Then $l=3$ or 9. Since $|H|$ is not divisible by 9, we know that $l=3$. Let $\bar{W}_1$, $\bar{W}_2$ and $\bar{W}_3$ be the N-orbits on $\bar{W}$. Then $|\bar{W}_1| = |\bar{W}_2| = |\bar{W}_3| = \frac{9}{m}$, and $m=1$ or 3. For each i with $1 \leqslant i \leqslant 3$, considering the action of N on $\bar{W}_i$, there is a subspace U_i of U such that $\langle \tau_x \mid x \in U_i \rangle$ is the kernel of N acting on $\bar{W}_i$. Recall that H is transitive on $\{\bar{W}_1, \bar{W}_2, \bar{W}_3\}$. It is easily shown that $U_i \neq U_j$ for all $i \neq j$ as N is faithful on $\bar{W}$, and that H acts transitively on $\{U_1, U_2, U_3\}$. Noting that $|\bar{W}_i| = |U : U_i|$, we have $|U_i| = 3m$.

Case 1. Let $m=1$. Then $K=1$, $\mathit{\Gamma} \cong \Sigma$ and $|U_i| = 3$ for $1 \leqslant i \leqslant 3$. In particular, each U_i is a 1-dimensional subspace of U, and so we may let $U_i = \langle \mathbf{e}_i \rangle$ for a non-zero vector $\mathbf{e}_i \in U$. Recall that H is transitive on $\{\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_3 \rangle\}$. Then, since H is an irreducible subgroup of $\mathrm{GL}(3,3)$, we know that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of U. Identifying W with the set $\{\mathbf{y} + \langle \mathbf{e}_i \rangle \mid \mathbf{y} \in U, 1 \leqslant i \leqslant 3\}$ of 27 1-dimensional affine subspaces of U, the action of G on W is given by

$$(\mathbf{y} + \langle \mathbf{e}_i \rangle)^{\tau_{\mathbf{x}}} = \mathbf{y} + \mathbf{x} + \langle \mathbf{e}_i \rangle,\ (\mathbf{y} + \langle \mathbf{e}_i \rangle)^h = \mathbf{y}^h + \langle \mathbf{e}_i^h \rangle;\ \mathbf{x}, \mathbf{y} \in U, h \in H.$$

Take $h_0, h_1, h_2 \in \mathrm{GL}(3,3)$ satisfying (3.1). Then $\langle h_1, h_2, h_0 \rangle \cong \mathbb{Z}_2{\times}\mathrm{S}_4$. Without of generality, we assume that H contains h_0. Since H fixes $\{\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_3 \rangle\}$ set-wise, it is easily shown that $H \leqslant \langle h_1, h_2, h_0 \rangle$. Analyzing the irreducible subgroups of $\langle h_1, h_2, h_0 \rangle$, we conclude that H is one of the following groups:

$$\langle h_1, h_0 \rangle,\ \langle h_1 h_1^{h_0}, h_0 \rangle,\ \langle h_1, h_2, h_0 \rangle,\ \langle h_1 h_1^{h_0}, h_2, h_0 \rangle,\ \langle h_1 h_1^{h_0}, h_1 h_1^{h_0} h_1^{h_0^2} h_2, h_0 \rangle.$$

Consider the orbits of H on W. If $H = \langle h_1, h_0\rangle$ or $\langle h_1 h_1^{h_0}, h_0\rangle$ then H has 4 orbits on W, which are

$$\begin{aligned}
&\Delta_0 := \{\langle \mathbf{e}_i\rangle \mid 1 \leqslant i \leqslant 3\},\\
&\Delta_1 := \{\pm\mathbf{e}_2 + \langle \mathbf{e}_1\rangle, \pm\mathbf{e}_3 + \langle \mathbf{e}_2\rangle, \pm\mathbf{e}_1 + \langle \mathbf{e}_3\rangle\},\\
&\Delta_2 := \{\pm\mathbf{e}_3 + \langle \mathbf{e}_1\rangle, \pm\mathbf{e}_1 + \langle \mathbf{e}_2\rangle, \pm\mathbf{e}_2 + \langle \mathbf{e}_3\rangle\},\\
&\Delta_3 := \{\pm\mathbf{e}_i \pm \mathbf{e}_j + \langle \mathbf{e}_k\rangle \mid \{i,j,k\} = \{1,2,3\}\}.
\end{aligned}$$

If H is one of $\langle h_1, h_2, h_0\rangle$, $\langle h_1 h_1^{h_0}, h_2, h_0\rangle$ and $\langle h_1 h_1^{h_0}, h_1 h_1^{h_0} h_1^{h_0^2} h_2, h_0\rangle$ then H has 3 orbits on W, which are Δ_0, Δ_3 and $\Delta_4 := \Delta_1 \cup \Delta_2$. It follows that Γ is isomorphic to one of the semisymmetric graphs given in Example 3.1.

Case 2. Let $m = 3$. Then $\Gamma \cong \Sigma^{1,3}$. In this case, each U_i is a 2-dimensional subspace of U, hence for $i \neq j$ the intersection $U_i \cap U_j$ is 1-dimensional. Set $\langle \mathbf{e}_1\rangle = U_2 \cap U_3$, $\langle \mathbf{e}_2\rangle = U_1 \cap U_3$ and $\langle \mathbf{e}_3\rangle = U_1 \cap U_2$. Then $\langle \mathbf{e}_i\rangle \neq \langle \mathbf{e}_j\rangle$ for all $i \neq j$; otherwise, $\langle \mathbf{e}_1\rangle = \langle \mathbf{e}_2\rangle = \langle \mathbf{e}_3\rangle$ is H-invariant, a contradiction. Noting that H is transitive on $\{U_1, U_2, U_3\}$, it follows that H acts transitively on $\{\langle \mathbf{e}_1\rangle, \langle \mathbf{e}_2\rangle, \langle \mathbf{e}_3\rangle\}$. Thus $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of U. To determine Σ, we identify $\bar{W}$ with the set $\{\mathbf{y} + U_i \mid \mathbf{y} \in U, 1 \leqslant i \leqslant 3\}$. Then $|\bar{W}| = 9$ and the action of G^U on W is given by

$$(\mathbf{y} + U_i)^{\tau_\mathbf{x}} = \mathbf{y} + \mathbf{x} + U_i,\ (\mathbf{y} + U_i)^h = \mathbf{y}^h + U_i^h;\ \mathbf{x}, \mathbf{y} \in U, h \in H.$$

A similar argument as in Case 1 yields that H is one of

$$\langle h_1, h_0\rangle,\ \langle h_1 h_1^{h_0}, h_0\rangle,\ \langle h_1, h_2, h_0\rangle,\ \langle h_1 h_1^{h_0}, h_2, h_0\rangle,\ \langle h_1 h_1^{h_0}, h_1 h_1^{h_0} h_1^{h_0^2} h_2, h_0\rangle,$$

where $h_0, h_1, h_2 \in \mathrm{GL}(3,3)$ satisfying (3.1). It is easy to check that H has exactly 2 orbits on $\bar{W}$, say $\{U_i \mid 1 \leqslant i \leqslant 3\}$ and $\{\pm\mathbf{e}_i + U_i \mid 1 \leqslant i \leqslant 3\}$. It follows that Σ is isomorphic one of the graphs Σ_0 and Σ_1 described as in Example 3.5. Thus Γ is known. □

Next we deal with the case where G^U is almost simple, and then finish the proof of Theorem 1.1.

Lemma 4.2. *Assume that G^U is almost simple. Then Γ is either arc-transitive or isomorphic to one of Tutte's* 12*-cage, the graphs in Lemma* 3.4 *and in Examples* 3.6 (2) *and* 3.8 (2).

Proof. Recall that G is faithful on W. We shall discuss in two cases according to whether or not G acts faithfully on U.

Case 1. Assume that G is faithful on U. Then $T := \mathrm{soc}(G)$ is listed in Table 1.

Assume that G is described as in lines 13-16 of Table 1. Then G is 2-transitive on U. Moreover, G has no faithful permutation representations of degree less than $9p$ (refer to [17, pp. 175]). Thus G is also 2-transitive on W. It follows that either one of Γ and its complement in $\mathsf{K}_{9p,9p}$ is the point-hyperplane incidence graph of the projective geometry PG(5, 2), or Γ is the standard double cover of the complete graph K_{9p}. Therefore, Γ is arc-transitive.

Assume that G is described as in line 3 of Table 1. Then $T = \mathrm{soc}(G) = \mathrm{A}_c$ with $c \in \{10, 18, 19\}$. Note that G has no faithful permutation representations of degree less than c (refer to [17, pp. 175]). Suppose that G is imprimitive on W. Let B be a maximal block of G acting on W. Then $|B| = 3$ or 9, and G acts faithfully and primitively on $\Omega := \{B^g \mid g \in G\}$. Note that Table 2 gives all primitive permutation group of degree $3p$. It follows that $|\Omega| = p$, and hence $T = \mathrm{A}_{19}$ and $p = 19$. Then $T_B \cong \mathrm{A}_{18}$. It is easily shown that T is transitive on W. Then for $u \in B$ we have $|T_B : T_u| = 9$; however, A_{18} has no subgroups of index 9, a contradiction. Thus G is primitive on W. Moreover, the actions of G on U and W are equivalent, that is, G_u and G_w are conjugate in G for $u \in U$ and $w \in W$. Then $\Gamma \cong B(G, U, \Delta)$ by Lemma 2.3, where Δ is an orbital of G on U. It is easy to check that G has exactly three orbitals on U, which are self-paired. It follows Γ is arc-transitive.

Now let G be one of the groups described as in lines 1,2, 4-12 of Table 1. Checking the subgroups of G (refer to [14, Chapter II, 8.27] for $\mathrm{soc}(G) = \mathrm{PSL}(2, 3^4)$ and to [5] for others), we conclude that every subgroup of index $9p$ is maximal in G. Thus G^W is also primitive.

Suppose that the actions of G on U and W are equivalent. Then $\Gamma \cong B(G,U,\Delta)$ by Lemma 2.3, where Δ is an orbital of G on U. Checking one by one the possible participants of G, the lengths of suborbits $|\Delta(u)|$ (for a given $u \in U$) are listed in Table 4, where the non-self-paired suborbits are marked by *. (Note that, for line 1, the action of G on U is equivalent to that on the edge set of Tutte's 8-cage.)

Line	Degree	$T = \mathrm{soc}(G)$	Suborbits $\|\Delta(u)\|$	Remark	ref.
1	45	PSL(2, 9)	$4, 8, 16$ (two)		
2	153	PSL(2, 17)	4 (two), 8^* (two)		
			8 (four), 16 (six),	$G = \mathrm{PSL}(2,17)$	[27, 4.4]
			$8, 16$ (seven), 32	$G = \mathrm{PGL}(2,17)$	
4	27	PSU(4, 2)	$10, 16$		[28]
5	45	PSU(4, 2)	$12, 32$		[28]
6	63	Sp(6, 2)	$30, 32$		[28]
7	171	PSL(2, 19)	5^* (two), 10 (four),	$G = \mathrm{PSL}(2,19)$	[27, 4.4]
			10^* (four), 20 (four)		
	171		$10, 20$ (eight)	$G = \mathrm{PGL}(2,19)$	
8	369	$\mathrm{PSL}(2,3^4)$	$36, 72, 80, 90$ (two)		[27, 4.1]
9	117	PSL(3, 3)	$12, 16$ (two), $24, 48$		[18, 2.3]
10	657	PSL(3, 8)	$16, 128, 512$		[18, 2.2]
11	63	PSU(3, 3)	$6, 16$ (two), 24	bases	[28]
			$6, 24, 32$	non-isotropic points	
12	117	$\mathrm{O}^+(6,3)$	$36, 80$		[18, 2.12]

Table 3 Suborbits of some primitive groups of degree $9p$.

If Δ is self-paired, then Γ is arc-transitive. Thus we assume that $G = \mathrm{PSL}(2,p)$ with $p = 17$ or 19. It is easily shown that any two paired suborbits of G^U are merged into some self-paired suborbit of $\mathrm{PGL}(2,p)$ (acting on U), we know that $\Gamma \cong B(G,U,\Delta)$ is arc-transitive by Lemma 2.4.

Suppose that the actions of G on U and W are not equivalent. Then $T = \mathrm{soc}(G) = O^+(6,3)$ or $\mathrm{PSU}(3,3)$.

Assume that $T = \mathrm{PSU}(3,3)$. Let V be a non-degenerate 3-dimensional unitary space over $\mathbb{F}_9$. Identify U with the set of 63 non-isotropic 1-dimensional subspaces of V and W with the set of 63 orthogonal frames of V. By Example 3.3 and Lemma 3.4, Γ is isomorphic to one of Tutte's 12-cage and its distance 3 and distance 5 graphs.

Let $T = O^+(6,3)$. Then $G = T$ or $\mathrm{PGO}^+(6,3)$. Consider a non-degenerate 6-dimensional orthogonal space V over $\mathbb{F}_3$. Identify U and W respectively with two T-orbits on the 234 non-isotropic 1-dimensional subspaces of V:

$$U = \{\langle \mathbf{x} \rangle \mid \mathbf{x} \in V, Q(\mathbf{x}) = 1\},\ W = \{\langle \mathbf{x} \rangle \mid \mathbf{x} \in V, Q(\mathbf{x}) = -1\},$$

where Q is the associated quadratic form. Write

$$V = \langle \mathbf{e}_1, \mathbf{f}_1 \rangle \perp \langle \mathbf{e}_2, \mathbf{f}_2 \rangle \perp \langle \mathbf{e}_3, \mathbf{f}_3 \rangle,$$

where $(\mathbf{e}_i, \mathbf{f}_i)$ are hyperbolic pairs. Set

$$\mathbf{e} = \mathbf{e}_1 + \mathbf{f}_1,\ \mathbf{f} = \mathbf{e}_1 - \mathbf{f}_1 \text{ and } V_1 = \langle \mathbf{e}_2, \mathbf{f}_2 \rangle \perp \langle \mathbf{e}_3, \mathbf{f}_3 \rangle.$$

Then $\langle \mathbf{e} \rangle \in U$ and $\mathbf{e}^{\perp} = \langle \mathbf{f} \rangle \perp V_1$. Moreover, $G_{\langle \mathbf{e} \rangle} \cong \mathrm{O}(5,3)$ or $\mathrm{GO}(5,3)$, which has exactly two orbits on the 162 non-isotropic vectors of $\mathbf{e}^{\perp}$:

$$S_1 = \{\mathbf{x} \mid \mathbf{x} \in \mathbf{e}^{\perp}, Q(\mathbf{x}) = -1\} \text{ and } S_2 = \{\mathbf{x} \mid \mathbf{x} \in \mathbf{e}^{\perp}, Q(\mathbf{x}) = 1\}.$$

An easy calculation implies that

$$S_1 = \{\mathbf{x} \mid \mathbf{x} \in V_1, Q(\mathbf{x}) = -1\} \cup \{\pm\mathbf{f} + \mathbf{x} \mid \mathbf{x} \in V_1, Q(\mathbf{x}) = 0\} \text{ and}$$
$$S_2 = \{\mathbf{x} \mid \mathbf{x} \in V_1, Q(\mathbf{x}) = 1\} \cup \{\pm\mathbf{f} + \mathbf{x} \mid \mathbf{x} \in V_1, Q(\mathbf{x}) = -1\},$$

which have size 90 and 72, respectively. Thus $G_{\langle \mathbf{e} \rangle}$ has exactly two orbits on W:

$$\{\langle \mathbf{x} \rangle \mid \mathbf{x} \in S_1\} \text{ and } \{\langle \mathbf{e} + \mathbf{x} \rangle \mid \mathbf{x} \in S_2\}$$

with size 45 and 72, respectively. By the information about $T = O^+(6,3)$ given in the Atlas [5], we conclude that G has an automorphism σ of order 2 such that $G_{\langle \mathbf{e} \rangle}^{\sigma} = G_{\langle \mathbf{f} \rangle}$. It follows from Lemma 2.5 that Γ is arc-transitive.

Case 2. Assume that G is unfaithful on U. Then Γ is semisymmetric by Lemma 2.2 (3). Let K be the kernel of G acting on U. Set $\Sigma = \Gamma_K$. Then $\Gamma \cong \Sigma^{1,m}$, where m is the length of a K-orbit on W. Thus it suffices to determine m and Σ.

Let $\bar{W}$ be the set of K-orbits on W. Then G^U is faithful on $\bar{W}$ and, since $K \neq 1$ is faithful on W, the size of $\bar{W}$ is a proper divisor of $|W| = 9p$. This observation helps us to determine G^U as follows.

The groups in lines 13-16 of Table 1 are excluded as each of them has no faithful permutation representations of degree less than $9p$ (refer to [17, pp. 175]). If G^U is described as in line 3 of Table 1 then a similar argument as in Case 1 implies that $\mathrm{soc}(G^U) = \mathrm{A}_{19}$ and $|\bar{W}| = p = 19$. For the groups in lines 1,2 and 4-12 of Table 1, checking the subgroups of G (refer to [5] and [14, Chapter II, 8.27]), the only possible case is that $G^U = \mathrm{PSL}(2,19)$ and $G^{\bar{W}}$ is described as in Table 2.

Let $\mathrm{soc}(G^U) = \mathrm{A}_{19}$. We may identify $\bar{W}$ with the set of positive integers no more than 19 and U with the set of 2-subsets of $\bar{W}$. Then $\Sigma = \Gamma_K$ is isomorphic to one of the graphs in Example 3.6 (1), and hence Γ is known.

Let $G^U = \mathrm{PSL}(2,19)$. We may identify $\bar{W}$ and U respectively with the vertex set and edge set of the Perkel graph. Then Σ is isomorphic to one of the graphs in Example 3.8 (1), and hence Γ is known. □

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