CONGRUENCES FOR BROKEN 3-DIAMOND AND 7 DOTS BRACELET PARTITIONS

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Abstract. Andrews and Paule introduced broken k-diamond partitions by using MacMahon's partition analysis. Later, Fu found a generalization which he called k dots bracelet partitions. In this paper, with the aid of Farkas and Kra's partition theorem and a p-dissection identity of $f(-q)$, we derive many congruences for broken 3-diamond partitions and 7 dots bracelet partitions.

1. Introduction

Andrews and Paule [1] studied broken k-diamond partitions by using MacMahon's partition analysis, and gave the generating function for $\Delta_k(n)$ which denotes the number of broken k-diamond partitions of n:

$$\sum_{n=0}^{\infty} \Delta_k(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty^2(-q^{2k+1};q^{2k+1})_\infty}.$$

They proved that for $n \geq 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod 3. \tag{1.1}$$

Later, Hirschhorn and Sellers [15] discussed congruences for $\Delta_1(n)$ and $\Delta_2(n)$. Mortenson [17] and Fu [11] supplied combinatorial proofs of (1.1). For more congruences for $\Delta_2(n)$, the reader may wish to see [5, 6, 18, 21]. Recently, Xiong [24] and Jameson [16] proved four conjectures for $\Delta_3(n)$ and $\Delta_5(n)$ posed by Paule and Radu [18]. Radu and Sellers [19, 22] obtained some parity results for $\Delta_k(n)$.

In the process of studying broken 1-diamond partitions, Fu [11] introduced a generalization which he called k dots bracelet partitions and found some general congruences. The number of this kind of partitions of n is denoted by $\mathfrak{B}_k(n)$, and the generating function is given by

$$\sum_{n=0}^{\infty} \mathfrak{B}_k(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty^{k-1}(-q^k;q^k)_\infty},\ k \geq 3.$$

Radu and Sellers [20, Theorem 1.4] found that for $n \geq 0$,

$$\mathfrak{B}_5(10n+7) \equiv 0 \pmod{5^2},$$

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$$\mathfrak{B}_7(14n+11) \equiv 0 \pmod{7^2}, \tag{1.2}$$
$$\mathfrak{B}_{11}(22n+21) \equiv 0 \pmod{11^2}.$$

Using the method in [7], the authors [8] deduced some congruences modulo 2 for $\mathfrak{B}_5(n)$ and modulo p for $\mathfrak{B}_k(n)$ with a given prime p.

The work in this paper is inspired by the following Farkas and Kra partition theorem:

$$\mathbf{O}\left((-q;q^2)_\infty(-q^7;q^{14})_\infty\right) = q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty, \tag{1.3}$$

where $\mathbf{O}$ denotes the odd part of the series. Warnaar [23] deduced the above result by finding a generalized theorem. Later, Hirschhorn [13,14] proved (1.3) by using Jacobi's triple product identity [3, Theorem 1.3.3].

In view of an identity [2, Equation (36.8), p.69]

$$\begin{aligned}\psi\left(q^{\mu+\nu}\right)\psi\left(q^{\mu-\nu}\right) &= \varphi\left(q^{\mu(\mu^2-\nu^2)}\right)\psi\left(q^{2\mu}\right)\\ &+\sum_{m=1}^{\mu/2-1} q^{\mu m^2-\nu m} f\left(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)}\right) f\left(q^{2\nu m}, q^{2\mu-2\nu m}\right)\\ &+q^{\mu^3/4-\mu\nu/2}\psi\left(q^{2\mu(\mu^2-\nu^2)}\right) f\left(q^{\mu\nu}, q^{2\mu-\mu\nu}\right),\end{aligned} \tag{1.4}$$

where μ is even, we deduce that (1.3) holds. Meanwhile, we derive that

$$\mathbf{E}\left((-q;q^2)_\infty(-q^7;q^{14})_\infty\right) = \frac{\varphi(q^{28})\psi(q^8)+q^6\varphi(q^4)\psi(q^{56})}{f(-q^4)f(-q^{28})}, \tag{1.5}$$

where $\mathbf{E}$ denotes the even part of the series. An equivalent form of (1.5) is

$$(-q;q^2)_\infty(-q^7;q^{14})_\infty + (q;q^2)_\infty(q^7;q^{14})_\infty = 2\frac{\varphi(q^{28})\psi(q^8)+q^6\varphi(q^4)\psi(q^{56})}{f(-q^4)f(-q^{28})}.$$

The authors [7, Theorem 2.2] studied that for any prime $p \geq 5$,

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}). \tag{1.6}$$

Further, for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p-1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p},$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & p\equiv 1 \pmod{6},\\ \frac{-p-1}{6}, & p\equiv -1 \pmod{6}.\end{cases}$$

Setting $p=7$ in (1.6), we get the 7-dissection identity of $f(-q)$ which was stated by Cao [4, Eq. (2.21)]:

$$f(-q) = \frac{f(-q^{14},-q^{35})f(-q^{49})}{f(-q^7,-q^{42})} - q\frac{f(-q^{21},-q^{28})f(-q^{49})}{f(-q^{14},-q^{35})}$$

$$-q^2 f(-q^{49}) + q^5 \frac{f(-q^7,-q^{42})f(-q^{49})}{f(-q^{21},-q^{28})}. \tag{1.7}$$

The objective of this paper is to find applications of (1.3), (1.5), and (1.6) to congruences for partition functions. In section 2, we prove two congruence relations related to (1.3) and (1.5). In section 3, we derive many new congruences for $\Delta_3(n)$ and $\mathfrak{B}_7(n)$.

Here and in what follows, we have made use of the standard q-series notation [12]

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a_1,a_2,\ldots,a_m;q)_\infty = \prod_{j=1}^{m}(a_j;q)_\infty, \quad |q|<1.$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows:

$$f(a,b) = (-a,-b,ab;ab)_\infty.$$

Thus,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = (-q;q^2)_\infty (q^4;q^4)_\infty,$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

The Legendre symbol is defined as follows:

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

2. Preliminaries

In order to obtain the main results, we first prove the following lemma.

Lemma 2.1. *We have*

$$\mathbf{O}((q;q^2)_\infty (q^7;q^{14})_\infty) \equiv q f(-q^2) f(-q^{14}) \pmod{2}, \tag{2.1}$$

$$\mathbf{E}((q;q^2)_\infty (q^7;q^{14})_\infty) \equiv \frac{f(-q^4)^5}{f(-q^{28})} + q^6 \frac{f(-q^{28})^5}{f(-q^4)} \pmod{2}. \tag{2.2}$$

Proof. Setting $\mu = 4$ and $\nu = 3$ in (1.4), we have

$$\psi(q)\psi(q^7) = \varphi(q^{28})\psi(q^8) + q^{10}\psi(q^{56}) f(q^{-4},q^{12}) + q\psi(q^2)\psi(q^{14}). \tag{2.3}$$

With the aid of

$$f(a,b)=a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n,b(ab)^{-n}),$$

it follows that

$$q^4f(q^{-4},q^{12})=f(q^4,q^4)=\varphi(q^4).$$

So we rewrite (2.3) as

$$\psi(q)\psi(q^7)=q\psi(q^2)\psi(q^{14})+\left(\varphi(q^{28})\psi(q^8)+q^6\varphi(q^4)\psi(q^{56})\right).$$

Since

$$\psi(q)\psi(q^7)=(-q;q^2)_\infty(q^4;q^4)_\infty(-q^7;q^{14})_\infty(q^{28};q^{28})_\infty,$$

we have

$$(-q;q^2)_\infty(-q^7;q^{14})_\infty=q\frac{\psi(q^2)\psi(q^{14})}{f(-q^4)f(-q^{28})}+\frac{\varphi(q^{28})\psi(q^8)+q^6\varphi(q^4)\psi(q^{56})}{f(-q^4)f(-q^{28})}.$$

Therefore, we prove (1.3) and (1.5). It is obvious to get (2.1) from (1.3). Applying the following facts

$$\varphi(q)\equiv 1 \pmod 2 \quad \text{and} \quad \psi(q)\equiv f(-q)^3 \pmod 2$$

to (1.5), we deduce (2.2). □

3. Main results

3.1. Congruences for broken 3-diamond partitions.

Lemma 3.1. *For $\alpha\geq 0$ and $n\geq 0$, we have*

$$\sum_{n=0}^{\infty}\Delta_3\left(2\cdot 7^\alpha n+\frac{2\cdot 7^\alpha+1}{3}\right)q^n\equiv f(-q)f(-q^7) \pmod 2. \tag{3.1}$$

Proof. We prove the lemma by induction on α. First,

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_3(n)q^n&=\frac{(q^2;q^2)_\infty}{(-q^7;q^7)_\infty(q;q)_\infty^3}\\&\equiv\frac{1}{(q;q)_\infty(q^7;q^7)_\infty} \pmod 2\\&\equiv(q;q^2)_\infty(q^7;q^{14})_\infty \pmod 2.\end{aligned} \tag{3.2}$$

In light of (2.1), it can be seen that

$$\sum_{n=0}^{\infty}\Delta_3(2n+1)q^n\equiv f(-q)f(-q^7) \pmod 2. \tag{3.3}$$

This is the case when $\alpha=0$. Suppose that the lemma holds for α. Then applying (1.7) yields

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_3\left(2\cdot 7^\alpha(7n+2)+\frac{2\cdot 7^\alpha+1}{3}\right)q^n&=\sum_{n=0}^{\infty}\Delta_3\left(2\cdot 7^{\alpha+1}+\frac{2\cdot 7^{\alpha+1}+1}{3}\right)q^n\\&\equiv f(-q)f(-q^7) \pmod 2.\end{aligned}$$

Therefore, the congruence holds for $\alpha+1$. we complete the proof. □

Theorem 3.2. *For $\alpha \geq 0$ and $n \geq 0$, we have*

$$\Delta_3\left(2\cdot 7^{\alpha+1}n+\frac{20\cdot 7^{\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$
$$\Delta_3\left(2\cdot 7^{\alpha+1}n+\frac{26\cdot 7^{\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$
$$\Delta_3\left(2\cdot 7^{\alpha+1}n+\frac{38\cdot 7^{\alpha}+1}{3}\right)\equiv 0 \pmod 2.$$

Proof. With the aid of (1.7), we find that there are no terms with powers of q congruent to $3, 4, 6$ modulo 7 on the right-hand side of (3.1). This completes the proof. □

Note that setting $\alpha = 0$ in Theorem 3.2 yields $\Delta_3(14n+7, 9, 13) \equiv 0 \pmod 2$ which were proved by Radu and Sellers [19].

Lemma 3.3. *For $\alpha \geq 0$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty}\Delta_3\left(2\cdot 4^{\alpha}n+\frac{2\cdot 4^{\alpha}+1}{3}\right)q^n\equiv f(-q)f(-q^7) \pmod 2.$$

Proof. We rewrite (3.3) as

$$\sum_{n=0}^{\infty}\Delta_3(2n+1)q^n\equiv (q^2;q^2)_\infty(q;q^2)_\infty(q^7;q^{14})_\infty(q^{14};q^{14})_\infty \pmod 2.$$

Using (2.1) again, we show that

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_3(2(2n+1)+1)q^n &= \sum_{n=0}^{\infty}\Delta_3(4n+3)q^n\\ &\equiv f(-q)^2f(-q^7)^2 \pmod 2\\ &\equiv f(-q^2)f(-q^{14}) \pmod 2. \end{aligned}\tag{3.4}$$

Then

$$\sum_{n=0}^{\infty}\Delta_3(4(2n)+3)q^n=\sum_{n=0}^{\infty}\Delta_3(8n+3)q^n\equiv f(-q)f(-q^7) \pmod 2. \tag{3.5}$$

So we deduce that

$$\Delta_3(2n+1)\equiv\Delta_3(8n+3) \pmod 2. \tag{3.6}$$

By induction on α, we deduce the lemma based on the above relation. □

Theorem 3.4. *For $\alpha \geq 0$ and $n \geq 0$, we have*

$$\Delta_3\left(14\cdot 4^{\alpha}n+\frac{5\cdot 4^{\alpha+1}+1}{3}\right)\equiv 0 \pmod 2,$$
$$\Delta_3\left(14\cdot 4^{\alpha}n+\frac{26\cdot 4^{\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$
$$\Delta_3\left(14\cdot 4^{\alpha}n+\frac{38\cdot 4^{\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$

$$\Delta_3\left(2\cdot 4^{\alpha+1}n+\frac{5\cdot 4^{\alpha+1}+1}{3}\right)\equiv 0 \pmod 2.$$

Proof. Based on Lemma 3.3, the proof of the first three congruences is similar to that of Theorem 3.2. From the proof of Lemma 3.3, we find that

$$\sum_{n=0}^{\infty}\Delta_3\left(2\cdot 4^{\alpha}(2n+1)+\frac{2\cdot 4^{\alpha}+1}{3}\right)q^n\equiv f(-q^2)f(-q^{14}) \pmod 2.$$

Then

$$\Delta_3\left(4^{\alpha+1}(2n+1)+\frac{2\cdot 4^{\alpha+1}+1}{3}\right)\equiv 0 \pmod 2.$$

□

Invoking (1.6), we get more congruences for $\Delta_3(n)$.

Lemma 3.5. *For a given prime $p\geq 5$, $\left(\frac{-7}{p}\right)=-1$, $\alpha\geq 0$, and $n\geq 0$, we have*

$$\sum_{n=0}^{\infty}\Delta_3\left(2\cdot p^{2\alpha}n+\frac{2\cdot p^{2\alpha}+1}{3}\right)q^n\equiv f(-q)f(-q^7) \pmod 2.$$

Proof. We prove the lemma by induction on α. It is obvious that (3.3) is the case when $\alpha=0$. Suppose that the lemma holds for α. Then for a prime $p\geq 5$ and $-(p-1)/2\leq k,m\leq (p-1)/2$, we solve

$$\frac{3k^2+k}{2}+7\cdot\frac{3m^2+m}{2}\equiv\frac{p^2-1}{3} \pmod p,$$

namely,

$$(6k+1)^2+7(6m+1)^2\equiv 0 \pmod p.$$

Since $\left(\frac{-7}{p}\right)=-1$, we get $k=m=\frac{\pm p-1}{6}$. Therefore,

$$\sum_{n=0}^{\infty}\Delta_3\left(2\cdot p^{2\alpha}\left(pn+\frac{p^2-1}{3}\right)+\frac{2\cdot p^{2\alpha}+1}{3}\right)q^n\equiv f(-q^p)f(-q^{7p}) \pmod 2, \tag{3.7}$$

$$\sum_{n=0}^{\infty}\Delta_3\left(2\cdot p^{2\alpha}\left(p(pn)+\frac{p^2-1}{3}\right)+\frac{2\cdot p^{2\alpha}+1}{3}\right)q^n\equiv f(-q)f(-q^7) \pmod 2.$$

Therefore, the lemma holds for $\alpha+1$. □

Employing Lemma 3.5, we have the following theorem.

Theorem 3.6. *For a given prime $p\geq 5$, $\left(\frac{-7}{p}\right)=-1$, $\alpha\geq 1$, and $n\geq 0$, we have*

$$\Delta_3\left(2\cdot p^{2\alpha}n+\frac{(6i+2p)p^{2\alpha-1}+1}{3}\right)\equiv 0 \pmod 2,$$

where $i=1,2,\ldots,p-1$.

Proof. Applying (3.7) yields

$$\Delta_3\left(2\cdot p^{2\alpha}\left(p(pn+i)+\frac{p^2-1}{3}\right)+\frac{2\cdot p^{2\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$

where $i=1,2,\ldots,p-1$. □

Lemma 3.7. *For $\alpha\geq 0$ and $n\geq 0$, we have*

$$\sum_{n=0}^{\infty}\Delta_3\left(8\cdot 7^{2\alpha}n+\frac{5\cdot 7^{2\alpha}+1}{3}\right)q^n\equiv qf(-q)f(-q^{28}) \pmod 2. \tag{3.8}$$

Proof. In view of (2.2) and (3.2), we get that

$$\sum_{n=0}^{\infty}\Delta_3(2n)q^n\equiv\frac{f(-q^2)^5}{f(-q^{14})}+q^3\frac{f(-q^{14})^5}{f(-q^2)} \pmod 2. \tag{3.9}$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_3\left(2(2n+1)\right)q^n&=\sum_{n=0}^{\infty}\Delta_3(4n+2)q^n\\&\equiv q(q;q^2)_\infty(q^7;q^7)_\infty(q^7;q^7)_\infty^4 \pmod 2\\&\equiv q(q;q^2)_\infty(q^7;q^{14})_\infty(q^{14};q^{14})_\infty(q^{14};q^{14})_\infty^2 \pmod 2\\&=q(q;q^2)_\infty(q^7;q^{14})_\infty(q^{14};q^{14})_\infty^3.\end{aligned} \tag{3.10}$$

In light of (2.1), we show that

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_3(4(2n)+2)q^n&=\sum_{n=0}^{\infty}\Delta_3(8n+2)q^n\\&\equiv qf(-q)f(-q^7)^4 \pmod 2\\&\equiv qf(-q)f(-q^{28}) \pmod 2.\end{aligned} \tag{3.11}$$

Substituting (1.7) into (3.11) yields

$$\sum_{n=0}^{\infty}\Delta_3\left(8(7n+3)+2\right)q^n=\sum_{n=0}^{\infty}\Delta_3(56n+26)q^n\equiv f(-q^4)f(-q^7) \pmod 2.$$

Using (1.7) again, we have

$$\sum_{n=0}^{\infty}\Delta_3\left(56(7n+1)+26\right)q^n=\sum_{n=0}^{\infty}\Delta_3(392n+82)q^n\equiv qf(-q)f(-q^{28}) \pmod 2.$$

So we derive that

$$\Delta_3(8n+2)\equiv\Delta_3(392n+82) \pmod 2.$$

By induction on α, we conclude the lemma based on the above relation. □

Theorem 3.8. *For $\alpha\geq 0$ and $n\geq 0$, we have*

$$\Delta_3\left(8\cdot 7^{2\alpha+1}n+\frac{5\cdot 7^{2\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$

$$\Delta_3\left(8\cdot 7^{2\alpha+1}n+\frac{101\cdot 7^{2\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$
$$\Delta_3\left(8\cdot 7^{2\alpha+1}n+\frac{125\cdot 7^{2\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$
$$\Delta_3\left(8\cdot 7^{2\alpha+2}n+\frac{59\cdot 7^{2\alpha+1}+1}{3}\right)\equiv 0 \pmod 2,$$
$$\Delta_3\left(8\cdot 7^{2\alpha+2}n+\frac{83\cdot 7^{2\alpha+1}+1}{3}\right)\equiv 0 \pmod 2,$$
$$\Delta_3\left(8\cdot 7^{2\alpha+2}n+\frac{131\cdot 7^{2\alpha+1}+1}{3}\right)\equiv 0 \pmod 2.$$

Proof. Since there are no terms with powers of q congruent to $0,4,5$ modulo 7 on the right-hand side of (3.8), we obtain the first three congruences. Combing (1.7) and (3.8), we show that

$$\sum_{n=0}^{\infty}\Delta_3\left(8\cdot 7^{2\alpha}(7n+3)+\frac{5\cdot 7^{2\alpha}+1}{3}\right)q^n\equiv f(-q^4)f(-q^7) \pmod 2.$$

Using (1.7) again, we derive the other three congruences. □

We point out that Radu and Sellers [22] deduced (3.3)-(3.6), (3.10), and (3.11) by using modular forms.

3.2. **Congruences for 7 dots bracelet partitions.**

Lemma 3.9. *For $\alpha\geq 0$ and $n\geq 0$, we have*

$$\sum_{n=0}^{\infty}\mathfrak{B}_7\left(4\cdot 7^{2\alpha}n+\frac{5\cdot 7^{2\alpha}+1}{2}\right)q^n\equiv f(-q)f(-q^{14}) \pmod 2. \tag{3.12}$$

Proof. First, we show that

$$\sum_{n=0}^{\infty}\mathfrak{B}_7(n)q^n=\frac{\sum_{n=0}^{\infty}\Delta_3(n)q^n}{f(-q)_\infty^4}\equiv\frac{\sum_{n=0}^{\infty}\Delta_3(n)q^n}{f(-q^4)} \pmod 2. \tag{3.13}$$

Applying (2.1) and (3.3), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_7(2n+1)q^n&\equiv\frac{\sum_{n=0}^{\infty}\Delta_3(2n+1)q^n}{f(-q^2)} \pmod 2\\ &\equiv\frac{f(-q)f(-q^7)}{f(-q^2)} \pmod 2\\ &=(q;q^2)_\infty(q^7;q^{14})_\infty(q^{14};q^{14})_\infty.\end{aligned}$$

Utilizing (2.1) again, we obtain that

$$\sum_{n=0}^{\infty}\mathfrak{B}_7\left(2(2n+1)+1\right)q^n=\sum_{n=0}^{\infty}\mathfrak{B}_7(4n+3)q^n\equiv f(-q)f(-q^{14}) \pmod 2. \tag{3.14}$$

With the aid of (1.7), it follows that

$$\sum_{n=0}^{\infty} \mathfrak{B}_7\left(4(7n+2)+3\right)q^n = \sum_{n=0}^{\infty} \mathfrak{B}_7(28n+11)q^n \equiv f(-q^2)f(-q^7) \pmod{2}. \quad (3.15)$$

Using (1.7) again, we get that

$$\sum_{n=0}^{\infty} \mathfrak{B}_7\left(28(7n+4)+11\right)q^n = \sum_{n=0}^{\infty} \mathfrak{B}_7(196n+123)q^n \equiv f(-q)f(-q^{14}) \pmod{2}.$$

Then

$$\mathfrak{B}_7(4n+3) \equiv \mathfrak{B}_7(196n+123) \pmod{2}.$$

By induction on α, we prove the lemma based on the above relation. □

Theorem 3.10. *For $\alpha \geq 0$ and $n \geq 0$, we have*

$$\mathfrak{B}_7\left(4\cdot 7^{2\alpha+1}n + \frac{29\cdot 7^{2\alpha}+1}{2}\right) \equiv 0 \pmod{2},$$
$$\mathfrak{B}_7\left(4\cdot 7^{2\alpha+1}n + \frac{37\cdot 7^{2\alpha}+1}{2}\right) \equiv 0 \pmod{2},$$
$$\mathfrak{B}_7\left(4\cdot 7^{2\alpha+1}n + \frac{53\cdot 7^{2\alpha}+1}{2}\right) \equiv 0 \pmod{2},$$
$$\mathfrak{B}_7\left(4\cdot 7^{2\alpha+2}n + \frac{11\cdot 7^{2\alpha+1}+1}{2}\right) \equiv 0 \pmod{2},$$
$$\mathfrak{B}_7\left(4\cdot 7^{2\alpha+2}n + \frac{43\cdot 7^{2\alpha+1}+1}{2}\right) \equiv 0 \pmod{2},$$
$$\mathfrak{B}_7\left(4\cdot 7^{2\alpha+2}n + \frac{51\cdot 7^{2\alpha+1}+1}{2}\right) \equiv 0 \pmod{2}.$$

Proof. According to (1.7), we show that there are no terms with powers of q congruent to $3, 4, 6$ modulo 7 on the right-hand side of (3.12). Therefore, we obtain the first three congruences. From (1.7) and Lemma 3.9, it follows that

$$\sum_{n=0}^{\infty} \mathfrak{B}_7\left(4\cdot 7^{2\alpha}(7n+2) + \frac{5\cdot 7^{2\alpha}+1}{2}\right)q^n \equiv f(-q^2)f(-q^7) \pmod{2}.$$

In view of (1.7), we obtain the left congruences. □

Notice that

$$\mathfrak{B}_7\left(14\left(2\cdot 7^{2\alpha}n + \frac{29\cdot 7^{2\alpha-1}-3}{4}\right)+11\right) = \mathfrak{B}_7\left(4\cdot 7^{2\alpha+1}n + \frac{29\cdot 7^{2\alpha}+1}{2}\right).$$

Therefore, combining (1.2) and Theorem 3.10, we can obtain many infinite families of congruences modulo 98 for $\mathfrak{B}_7(n)$.

Lemma 3.11. *For any prime $p \geq 5$, $\left(\frac{-14}{p}\right) = -1$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \mathfrak{B}_7\left(4p^{2\alpha}n + \frac{5p^{2\alpha}+1}{2}\right)q^n \equiv f(-q)f(-q^{14}) \pmod{2}.$$

Proof. We prove the lemma by induction on α. Notice that (3.14) is the case when $\alpha = 0$. Suppose that the lemma holds for α. Then, for any prime $p \geq 5$ and $-(p-1)/2 \leq k,\ m \leq (p-1)/2$, we solve

$$\frac{3k^2+k}{2} + 14 \cdot \frac{3m^2+m}{2} \equiv \frac{5p^2-5}{8} \pmod{p},$$

namely,

$$(6k+1)^2 + 14(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-14}{p}\right) = -1$, we have $k = m = \frac{\pm p - 1}{6}$. Therefore, using (1.6), we deduce that

$$\sum_{n=0}^{\infty} \mathfrak{B}_7\left(4p^{2\alpha}\left(pn + \frac{5p^2-5}{8}\right) + \frac{5p^{2\alpha}+1}{2}\right) q^n \equiv f(-q^p)f(-q^{14p}) \pmod{2}, \tag{3.16}$$

$$\sum_{n=0}^{\infty} \mathfrak{B}_7\left(4p^{2\alpha}\left(p(pn) + \frac{5p^2-5}{8}\right) + \frac{5p^{2\alpha}+1}{2}\right) q^n \equiv f(-q)f(-q^{14}) \pmod{2}.$$

So the lemma holds for $\alpha + 1$. □

With the aid of Lemma 3.11, we get the following congruences.

Theorem 3.12. *For any prime $p \geq 5$, $\left(\frac{-14}{p}\right) = -1$, $\alpha \geq 1$, and $n \geq 0$, we have*

$$\mathfrak{B}_7\left(4p^{2\alpha}n + \frac{(8i+5p)p^{2\alpha-1}+1}{2}\right) \equiv 0 \pmod{2},$$

$$\mathfrak{B}_7\left(28p^{2\alpha}n + \frac{(56i+21p)p^{2\alpha-1}+1}{2}\right) \equiv 0 \pmod{2},$$

where $i = 1, 2, \ldots, p-1$.

Proof. Based on (3.16), it follows that for $i = 1, 2, \ldots, p-1$,

$$\mathfrak{B}_7\left(4p^{2\alpha+1}(pn+i) + \frac{5p^{2\alpha+2}+1}{2}\right) \equiv 0 \pmod{2}.$$

In addition, combining (1.7) and Lemma 3.11 yields

$$\sum_{n=0}^{\infty} \mathfrak{B}_7\left(4p^{2\alpha}(7n+2) + \frac{5p^{2\alpha}+1}{2}\right) q^n \equiv f(-q^2)f(-q^7) \pmod{2}.$$

Then

$$\sum_{n=0}^{\infty} \mathfrak{B}_7\left(28p^{2\alpha}\left(pn + \frac{3p^2-3}{8}\right) + \frac{21p^{2\alpha}+1}{2}\right) q^n \equiv f(-q^{2p})f(-q^{7p}) \pmod{2}.$$

Therefore, for $i = 1, 2, \ldots, p-1$,

$$\mathfrak{B}_7\left(28p^{2\alpha+1}(pn+i) + \frac{21p^{2\alpha+2}+1}{2}\right) \equiv 0 \pmod{2}.$$

□

Theorem 3.13. *For $n \geq 0$, we have*

$$\mathfrak{B}_7(28n+8) \equiv \mathfrak{B}_7(28n+12) \equiv \mathfrak{B}_7(28n+20) \equiv 0 \pmod{2}.$$

Proof. From (3.9) and (3.13), it follows that

$$\sum_{n=0}^{\infty} \mathfrak{B}_7(4n)q^n \equiv \frac{\sum_{n=0}^{\infty} \Delta_3(4n)q^n}{f(-q)} \equiv \frac{f(-q)^5}{f(-q^7)f(-q)} \equiv \frac{f(-q^4)}{f(-q^7)} \pmod{2}.$$

In view of (1.7), we show that

$$\mathfrak{B}_7\left(4(7n+i)\right) \equiv 0 \pmod{2},$$

where $i = 2, 3, 5$. □

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