Zero-sum Subsequences of Distinct Lengths

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Abstract

Let G be an additive finite abelian group, and let $\mathrm{disc}(G)$ denote the smallest positive integer t such that every sequence S over G of length $|S| \geq t$ has two nonempty zero-sum subsequences of distinct lengths. We determine $\mathrm{disc}(G)$ for some groups including the groups C_2^r, the groups of rank at most two and the groups $C_{mp^n} \oplus H$, where m, n are positive integers, p is a prime and H is a p-group with $p^n \geq D^*(H)$.

Keywords: zero-sum sequence; exponent; Davenport constant; $D^*(G)$.

1. Introduction

Throughout this paper, let G be an additive finite abelian group. Let C_n denote the cyclic group of n elements, it is well known that $|G| = 1$ or $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | n_2 \cdots | n_r$, where $r = r(G)$ is the rank of G and $n_r = \exp(G)$ is the exponent of G. Set

$$D^*(G) = 1 + \sum_{i=1}^{r} (n_i - 1).$$

Let p be a sufficiently large prime. In 1976, P. Erdős and E. Szemerédi [2] proved that if S is a sequence of length $|S| = p$ over C_p such that S takes at least three distinct values, then S has two

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nonempty zero-sum subsequences of distinct lengths, which confirmed a conjecture of Graham for sufficiently large primes. In 2010, the first author, Hamidoune and Wang [8] extended the above result to all positive integers n, and a different proof of this result was given by Grynkiewicz [13] in 2011. A natural question proposed quite recently by Girard [11] is to determine $\mathrm{disc}(G)$ which is defined as follows.

Definition 1.1 *Define $disc(G)$ to be the smallest positive integer t, such that every sequence over G of length at least t has two nonempty zero-sum subsequences of distinct lengths.*

In this paper we shall determine $\mathrm{disc}(G)$ for some groups and our main results are as follows.

Theorem 1.2 *Let p be a prime and m, n be positive integers. Then $disc(G) = D^*(G) + \exp(G)$ provided that G is one of the following groups:*

1. *$r(G) \le 2$.*
2. *$G = C_{mp^n} \oplus H$, where H is a p-group with $D^*(H) \le p^n$.*

Theorem 1.3 *Let $r \ge 1$ be an positive integer, and let $t \ge 2$ be the unique integer such that $r \in [2^t - t - 1, 2^{t+1} - t - 3]$. Then,*

1. *$disc(C_2^r) = r + t + 1$.*

2. *If S is a sequence over C_2^r of length $|S| = disc(C_2^r) - 1 = r + t$, such that all the nonempty zero-sum subsequences of S have the same length ℓ, then $\ell = 2^{t-1}$.*

2. Preliminaries

In this paper, our notations are consistent with [6, 10, 12] and we briefly present some key concepts. Let $\mathbb{Z}$ denote the set of integers, and let $\mathbb{N}$ denote the set of positive integers. For real numbers $a \le b$, we set $[a, b] = \{x \in \mathbb{Z} : a \le x \le b\}$. Let G be an additive finite abelian group. For every nonempty subset $B \subset G$, denote by $\langle B \rangle$ the subgroup of G generated by B. A sequence $S = g_1 \cdot \ldots \cdot g_l$ over G is called

- a zero-sum sequence, if $\sum_{i=1}^{l} g_i = 0 \in G$.
- zero-sum free if it has no nonempty zero-sum subsequence.
- a minimal zero-sum sequence, if it is zero-sum and has no proper zero-sum subsequence.
- a short zero-sum sequence, if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$.

Let $D(G)$ be the Davenport constant of G, which is defined as the smallest positive integer d such that every sequence over G of length at least d has a nonempty zero-sum subsequence, let $\eta(G)$ be the smallest positive integer ℓ such that every sequence over G of length at least ℓ has a short zero-sum subsequence, and let $D_2(G)$ denote the smallest positive integer t such that every sequence S over G of length $|S| \geq t$ has two disjoint nonempty zero-sum subsequences.

We have the following easy observations on $disc(G)$.

Lemma 2.1 *Let S be a zero-sum free sequence over G. If there is an element $g \in G$ that occurs* $\mathrm{ord}(g) - 1$ *times in S, then $disc(G) \geq |S| + \mathrm{ord}(g) + 1$, where $\mathrm{ord}(g)$ denotes the order of g.*

Proof. Let $T = S g^{\mathrm{ord}(g)}$. Since S is zero-sum free and $g^{\mathrm{ord}(g)-1} | S$, a nonempty zero-sum subsequence of T must have the form $g^{\mathrm{ord}(g)}$. Therefore, $\mathrm{disc}(G) \geq |T| + 1 = |S| + \mathrm{ord}(g) + 1$. □

Lemma 2.2 *$disc(G) \geq D^*(G) + \exp(G)$.*

Proof. Let $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r} = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_r \rangle$ with $1 < n_1 | n_2 \cdots | n_r$, and $ord(e_i) = n_i$ for each $i \in [1, r]$. Let

$$S = e_1^{n_1 - 1} e_2^{n_2 - 1} \cdots e_r^{n_r - 1}.$$

Clearly, S is a zero-sum free sequence of length $|S| = D^*(G) - 1$. Since e_r occurs $n_r - 1 = \mathrm{ord}(e_r) - 1$ times in S, it follows from Lemma 2.1 that $\mathrm{disc}(G) \geq |S| + n_r + 1 = D^*(G) + \exp(G)$. □

Lemma 2.3 *Let H be a proper subgroup of G. Then,*

$$disc(G) \geq disc(H) + D(G/H) - 1,$$

particularly,

$$disc(G) > disc(H).$$

Proof. Let S be a sequence over H of length $|S| = \mathrm{disc}(H) - 1$ such that every nonempty zero-sum subsequence has the same length, and let T be a sequence over $G \setminus H$ avoiding a nonempty zero-sum subsequence modulo H of length $|T| = D(G/H) - 1$. Clearly, each nonempty zero-sum subsequence of TS is in fact a subsequence of S, and therefore has the same length. Hence, $\mathrm{disc}(G) \geq 1 + |TS| = 1 + |S| + |T| = \mathrm{disc}(H) + D(G/H) - 1$. Obviously, $D(G/H) \geq 2$ since H is a proper subgroup of G. Therefore, $\mathrm{disc}(G) > \mathrm{disc}(H)$. □

3. Proof of Theorem 1.2

Lemma 3.1 *If $\eta(G) \le D(G) + \exp(G)$, then $disc(G) \le D_2(G) \le D(G) + \exp(G)$.*

Proof. Let S be an arbitrary sequence over G of length $D_2(G)$. By the definition of $D_2(G)$, S has two disjoint nonempty zero-sum subsequences T_1 and T_2. Now T_1 and T_1T_2 are two zero-sum subsequences of S of distinct lengths. Therefore, $\mathrm{disc}(G) \le D_2(G)$.

It remains to prove that $D_2(G) \le D(G) + \exp(G)$. Let S be a sequence over G of length $|S| = D(G) + \exp(G)$. We need to show that S has two disjoint nonempty zero-sum subsequences. Since $|S| = D(G) + \exp(G) \ge \eta(G)$, S has a short zero-sum subsequence T. Now $|ST^{-1}| = D(G) + \exp(G) - |T| \ge D(G)$. It follows that ST^{-1} has a nonempty zero-sum subsequence which disjoints with T. □

Lemma 3.2 *If $D(G) = D^*(G)$ and $\eta(G) \le D(G) + \exp(G)$, then $disc(G) = D_2(G) = D(G) + \exp(G)$.*

Proof. It follows from Lemma 2.2 and Lemma 3.1. □

Lemma 3.3 *[10, Corollary 5.7.5] If $G = C_n$ then $\eta(C_n) = n$.*

Lemma 3.4 *([6, Theorem 6.3],[10, Theorem 5.8.3]) , Let $G = C_m \oplus C_n$ with $1 < m|n$, then $\eta(G) = n + 2m - 2$ and $D(G) = n + m - 1$.*

Lemma 3.5 *[1] Let p be a prime and m, n be positive integers. Then $D(G) = D^*(G)$ if G is one of the following groups*

1. *G is a finite abelian p-group.*
2. *If $G = C_{mp^n} \oplus H$ with H being a finite abelian p-group and $p^n \ge D^*(H)$.*

Lemma 3.6 *([3, Lemma 2.5],[10, Proposition 5.7.11]) Let G be a finite abelian group, and let K be a subgroup of G with $\exp(G) = \exp(K)\exp(G/K)$. Then $\eta(G) \le \exp(G/K)(\eta(K) - 1) + \eta(G/K)$.*

Lemma 3.7 *[3, Lemma 2.7] Let m be a positive integer, and let H be a finite abelian group with $\exp(H)|m$ and $m \ge \mathsf{D}(H)$. Suppose that $\mathsf{D}(C_m \oplus C_m \oplus H) = 2m + \mathsf{D}(H) - 2$. Then $\eta(C_m \oplus H) \le 2m + \mathsf{D}(H) - 2$.*

Lemma 3.8 *[10, Proposition 5.1.11] Let G be a finite abelian group, and let K be a subgroup of G, then $D(G) \ge D(K) + D(G/K) - 1$.*

Proof of Theorem 1.2. *1.* If $G = C_n$ then $\eta(C_n) = D(C_n) = D^*(C_n) = n$ by Lemma 3.3. Now $\mathrm{disc}(G) = D^*(C_n) + \exp(C_n) = 2n$ follows from Lemma 3.2. If $r(G) = 2$, then $D(G) = D^*(G)$ and $\eta(G) \le D(G) + \exp(G)$ by Lemma 3.4. Therefore, again by Lemma 3.2 we arrive at $\mathrm{disc}(G) = D^*(G) + \exp(G)$.

2. Let $K = C_m$. Then $G/K = C_{p^n} \oplus H$. By Lemma 3.5 and Lemma 3.7, we have $\eta(G/K) \le 2p^n + \mathsf{D}(H) - 2$. It follows from Lemma 3.5, Lemma 3.6, Lemma 3.7 and Lemma 3.8 that $\eta(G) \le \exp(G/K)(\eta(K) - 1) + \eta(G/K) = p^n(m-1) + \eta(G/K) \le p^n(m-1) + 2p^n + D(H) - 2 = p^n m + p^n + D(H) - 2 \le \exp(G) + D^*(G) - 1$. By Lemma 3.2 and Lemma 3.5, we have $\mathrm{disc}(G) = D^*(G) + \exp(G)$. □

4. Proof of Theorem 1.3

Lemma 4.1 *Let t, r be two integers with $t \ge 2$ and $r \ge 2^t - t - 1$. Then $disc(C_2^r) \ge r + t + 1$.*

Proof. First, suppose that $r = 2^t - t - 1$. Let $e_1, \cdots, e_r$ be a basis of C_2^r, and let f be any bijection from

$$[1, r] \text{ to } \{0,1\}^t \setminus \{(0, \cdots, 0), (1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}.$$

We construct subsets $A_i \subset [1, r]$ for every $i \in [1, t]$ as follows: $j \in A_i$ if and only if the i-th coordinate of $f(j)$ equals 1. Define

$$x_i = \sum_{j \in A_i} e_j \in C_2^r.$$

Let

$$S = e_1 \cdots e_r x_1 \cdots x_t.$$

We show next that every nonempty zero-sum subsequence of S has the same length $\ell = 2^{t-1}$. Let T be any nonempty zero-sum subsequence of S. We need to show $|T| = \ell = 2^{t-1}$. By renumbering if necessary we may assume that

$$T = x_1 \cdots x_w T'$$

with $w \in [1, t]$ and $T' | e_1 \cdots e_r$.

Let $T' = \prod_{j \in A} e_j$. Then, $j \in A$ if and only if $|\{i : i \in [1, w] \text{ and } f(j)_i = 1\}|$ is odd, where $f(j)_i$ denotes the i-th coordinate of $f(j)$. Let r_k denote the number of $j \in [1, r]$ such that $f(j)_i = 1$ holds for exactly k indices $i \in [1, w]$. It follows from the definition of f that

$$|A| = \sum_{1 \le k \le w, 2 \nmid k} r_k = \Big(\sum_{1 \le k \le w, 2 \nmid k} \binom{w}{k} 2^{t-w}\Big) - w = 2^{t-1} - w.$$

Therefore, $|T| = w + |A| = 2^{t-1}$. This proves that $\mathrm{disc}(C_2^r) \ge r + t + 1$ for $r = 2^t - t - 1$. From Lemma 2.3 we infer that $\mathrm{disc}(C_2^{r+1}) \ge \mathrm{disc}(C_2^r) + 1$. Therefore, $\mathrm{disc}(C_2^{r+1}) - (r+1) \ge \mathrm{disc}(C_2^r) - r$. Hence, $\mathrm{disc}(C_2^r) \ge r + t + 1$ for all $r \ge 2^t - t - 1$. □

Lemma 4.2 *Let t and r be two positive integers with $t \ge 2$, and let $S = e_1 \cdots e_r x_1 \cdots x_t$ be a sequence of nonzero terms over C_2^r of length $r+t$, where $e_1, \cdots, e_r$ is a basis of C_2^r. For each $i \in [1,t]$, let $A_i \subset [1,r]$ be a nonempty subset such that $x_i = \sum_{j \in A_i} e_j$. If every nonempty zero-sum subsequence of S has the same length ℓ, then*

$$|\cap_{i \in I} A_i| = \frac{\ell}{2^{|I|-1}} \tag{1}$$

holds for every subset $I \subset [1,t]$ of cardinality $|I| \in [2,t]$. In particular,

$$\ell \equiv 0 \pmod{2^{t-1}}.$$

Proof. Note that $x_i \prod_{j \in A_i} e_j$ is a nonempty zero-sum subsequence of S, By the hypothesis of the lemma, we have that

$$|A_i| = \ell - 1 \text{ for every } i \in [1,t]. \tag{2}$$

Let I be any subset of $[1,t]$ with $|I| \ge 2$. Then there is a subset A_I of $[1,r]$, such that $\prod_{i \in I} x_i \prod_{j \in A_I} e_j$ is a nonempty zero-sum subsequence of S. We claim that

$$\textbf{Claim.} \quad \ell - |I| = |A_I| = \sum_{k=1}^{|I|} (-2)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le t, i_1, \cdots, i_k \in I} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

Since every nonempty zero-sum subsequence of S has the same length ℓ, we have that $\ell - |I| = |A_I|$.

Let $j \in \cup_{i \in I} A_i$ and let

$$\lambda = |\{i \in I : j \in A_i\}|.$$

Clearly, $j \in A_I$ if and only if λ is odd. Let r_j be the number of the times that j is counted in the right side of the claim above. Then,

$$r_j = \binom{\lambda}{1} - 2\binom{\lambda}{2} + 2^2\binom{\lambda}{3} - \cdots + (-2)^{\lambda-1}\binom{\lambda}{\lambda} = \frac{1-(1-2)^\lambda}{2}.$$

Therefore, $r_j = 1$ if λ is odd and $r_j = 0$ if λ is even. This proves the claim.

We prove (1) by induction on $|I|$. If $|I| = 2$, then by (2) and the claim above we obtain that $|\cap_{i \in I} A_i| = \frac{\ell}{2}$. Suppose that (1) is true for all subsets $I \subset [1,t]$ with $|I| \le m$ $(2 \le m \le t-1)$. We want to prove that (1) is true for all subsets $I \subset [1,t]$ with $|I| = m+1$. By the induction hypothesis, (2) and the claim above, we derive that

$$\begin{aligned} \ell - |I| &= -|I| + \tbinom{|I|}{1}\ell - \tbinom{|I|}{2}\ell + \tbinom{|I|}{3}\ell - \cdots + (-1)^{|I|-2}\tbinom{|I|}{|I|-1}\ell + (-2)^{|I|-1}|\cap_{i \in I} A_i| \\ &= -|I| + \tbinom{|I|}{0}\ell - (1-1)^{|I|}\ell + (-1)^{|I|}\ell + (-2)^{|I|-1}|\cap_{i \in I} A_i| \\ &= -|I| + \ell + (-1)^{|I|}\ell + (-2)^{|I|-1}|\cap_{i \in I} A_i|. \end{aligned}$$

Therefore, $|\cap_{i \in I} A_i| = \frac{\ell}{2^{|I|-1}}$ as desired. This proves (1). Now $\ell \equiv 0 \pmod{2^{t-1}}$ follows from (1) taking $I = [1,t]$. □

Proof of Theorem 1.3. 1). By Lemma 4.1 we have $\mathrm{disc}(C_2^r) \geq r+t+1$. So, it remains to prove that

$$\mathrm{disc}(C_2^r) \leq r + t + 1 \tag{3}$$

for all integers $t \geq 2$.

Let S be a sequence over C_2^r of length $|S| = r+t+1$. We need to show that S has two nonempty zero-sum subsequences of distinct lengths. Assume to the contrary that, all nonempty zero-sum subsequences of S have the same length ℓ.

If S has a zero-sum subsequence T of length $|T| \in [1,2]$. Then, $|ST^{-1}| \geq |S|-2 > r+1 = \mathsf{D}(C_2^r)$. It follows that ST^{-1} has a nonempty zero-sum subsequence W. Now W and WT are two zero-sum subsequences of S of distinct lengths, a contradiction. This proves that S has no zero-sum subsequence of length in $[1,2]$, which is equivalent to

$$(*) \quad S \text{ is a subset of } C_2^r \setminus \{0\}.$$

We proceed by induction on r. For $r = 1$, by (*) we have that $|S| \leq |C_2| - 1 = 1 < 4 = r+t+1$, a contradiction. This proves (3) for $r = 1$. Suppose that (3) is true for all smaller r and we want to prove it for r ($r \geq 2$). Suppose that $\langle S \rangle = C_2^{r_1}$. Then, $r_1 \leq r$. If $r_1 < r$ then by induction we have that S has two nonempty subsequences of distinct lengths, a contradiction. Therefore,

$$\langle S \rangle = C_2^r.$$

Let

$$S = e_1 \cdots e_r x_1 \cdots x_{t+1}$$

with $e_1, \cdots, e_r$ being a basis of C_2^r.

Suppose that

$$x_i = \sum_{j \in A_i} e_j$$

for every $i \in [1, t+1]$, where A_i is a nonempty subset of $[1,r]$ for every $i \in [1, t+1]$.
Then

$$|\cup_{i=1}^{t+1} A_i| \leq r.$$

Applying Lemma 4.2 to S we obtain that

$$\begin{aligned} r &\geq |\cup_{i=1}^{t+1} A_i| \\ &= \textstyle\sum_{1\leq i\leq t+1} |A_i| - \sum_{1\leq i<j\leq t+1} |A_i \cap A_j| + \cdots + (-1)^{t+1-1} |\cap_{i=1}^{t+1} A_i| \\ &= (t+1)(\ell-1) - \binom{t+1}{2} \cdot \frac{\ell}{2} + \cdots + (-1)^{t+1-1} \cdot \frac{\ell}{2^{t+1-1}} \\ &= \ell \cdot \frac{2^{t+1}-1}{2^{t+1-1}} - t - 1. \end{aligned}$$

By Lemma 4.2, $\ell \geq 2^t$. So $r \geq 2^{t+1} - (t+1) - 1 > r$, a contradiction.

2). Similarly to 1) we obtain that

$$S = e_1 \cdots e_r x_1 \cdots x_t$$

with $e_1, \cdots, e_r$ being a basis of C_2^r, and

$$x_i = \sum_{j \in A_i} e_j$$

for every $i \in [1, t]$, where A_i is a nonempty subset of $[1, r]$ for every $i \in [1, t]$ and

$$|\cup_{i=1}^t A_i| \le r.$$

Similarly to 1) we also get

$$r \ge |\cup_{i=1}^t A_i| = \ell \cdot \frac{2^t - 1}{2^{t-1}} - t.$$

By Lemma 4.2, $\ell = k(2^{t-1})$ for some positive integer k. Therefore, $2^{t+1} - t - 3 \ge r \ge k(2^t - 1) - t$. This forces that $k = 1$. Hence, $\ell = 2^{t-1}$.

□

5. Concluding remarks

$D_2(G)$ was first introduced by H. Halter-Koch in [14] and was studied recently by Plagne and Schmid in [15]. From Lemma 3.2 we know that the groups listed in Theorem 1.2 satisfy $\mathrm{disc}(G) = D(G) + \exp(G) = D_2(G)$. It would be quite interesting to determine all finite abelian groups G with

$$\mathrm{disc}(G) = D_2(G).$$

From Theorem 1.3 we know that $r + \log_2 r \le \mathrm{disc}(C_2^r) \le r + \log_2 r + 2$. For all sufficiently large r we have $1.261r \le D_2(C_2^r) \le 1.396r$ (For e.g. see [15, Theorem1]). So, $\mathrm{disc}(C_2^r)$ is much smaller than $D_2(C_2^r)$ for sufficiently large r.

Lemma 5.1 *[9] If G is a finite abelian group, then* $\eta(G) \le |G|$.

We can find more groups than those mentioned in Theorem 1.2 satisfying $\mathrm{disc}(G) = D_2(G)$ with the following result.

Proposition 5.2 *Let m, n be positive integers, and H be a finite abelian group with* $\exp(H) = m$. *Let $G = C_{mn} \oplus H$. If $D(G) = D^*(G)$ and $n \ge |H| - 1$, then disc$(G) = D^*(G) + \exp(G) = D_2(G)$.*

Proof. Let $K = C_n$. Then $G/K = C_m \oplus H$. Applying Lemma 3.6 and Lemma 5.1 to G and K, we obtain that $\eta(G) \le m(n-1) + \eta(G/K) \le mn - m + |G/K| = mn - m + m|H| = mn + m(|H| - 1) \le mn + mn = \exp(G) + mn \le \exp(G) + D(G)$. Now the result follows from Lemma 3.2. □

The following groups G satisfy that $D(G) = D^*(G)$:

- $G = C_2 \oplus C_{2m} \oplus C_{2n}$ with $m|n$; ([1, 7, 16])
- $G = C_3 \oplus C_{6m} \oplus C_{6n}$ with $m|n$. ([1, 7, 16])
- $G = C_{2p^a} \oplus C_{2p^b} \oplus C_{2p^c}$ with a, b, c being nonnegative integers, and p a prime; ([1])
- $G = C_2^3 \oplus C_{2n}$. ([1])

Let $m|n$. Emde Boas [1] proved that $D(G) = D^*(G)$ for $G \in \{C_2 \oplus C_{2m} \oplus C_{2n}, C_3 \oplus C_{6m} \oplus C_{6n}\}$ provided that every prime divisor p of n has the so called Property **C**, which is defined as

Definition 5.3 *We say a positive integer n has Property* **C**, *if every sequence over C_n^2 of length $3n - 3$ having no short zero-sum subsequence consists of three distinct elements each appearing $n - 1$ times.*

Property **C** has been first introduced by Emde [1] in 1969 for all primes and has been generalized to all positive integers n by the first author [4] in 2000. Emde Boas also introduced the following Property **B** for primes p and which has been generalized to all positive integers n by the first author and Geroldinger [5] in 1999.

Definition 5.4 *We say a positive integer n has Property* **B** *if every minimal zero-sum sequence over C_n^2 of length $2n - 1$ contains some element exactly $n - 1$ times.*

The first author and Geroldinger [5] also proved that Property **B** implies Property **C**. In 2010, the first author, Geroldinger and Grynkiewicz [7] proved that Property **B** is multiplicative, i.e., if both $n = k$ and $n = \ell$ have Property **B**, then $n = k\ell$ also has Property **B**. Quite recently, Reiher [16] proved that every prime p has Property **B**. So, combining the results from [1], [5], [7] and [16] we obtain that $D(G) = D^*(G)$ for $G \in \{C_2 \oplus C_{2m} \oplus C_{2n}, C_3 \oplus C_{6m} \oplus C_{6n}\}$ with $m|n$.

Now from Proposition 5.2, we obtain that $\mathrm{disc}(G) = D^*(G) + \exp(G) = D_2(G)$ for the following groups:

- $G = C_2 \oplus C_{2m} \oplus C_{2n}$ with $m|n$ and $n \geq 4m^2 - m$;
- $G = C_3 \oplus C_{6m} \oplus C_{6n}$ with $m|n$ and $n \geq 18m^2 - m$;
- $G = C_{2p^a} \oplus C_{2p^b} \oplus C_{2p^c}$ with $a \leq b \leq c$ being nonnegative integers, p a prime and $p^c \geq 4p^{a+2b} - p^b$;
- $G = C_2^3 \oplus C_{2n}$ with $n \geq 7$.

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