ON 2-ARC-TRANSITIVE GRAPHS OF PRODUCT ACTION TYPE

ZAI PING LU

ABSTRACT. In this paper, we discuss the structural information about 2-arc-transitive (non-bipartite and bipartite) graphs of product action type. It is proved that a 2-arc-transitive graph of product action type requires certain restrictions on either the vertex-stabilizers or the valency. Based on the existence of some equidistant linear codes, a construction is given for 2-arc-transitive graphs of non-diagonal product action type, which produces several families of such graphs. Besides, a nontrivial construction is given for 2-arc-transitive bipartite graphs of diagonal product action type.

KEYWORDS. 2-arc-transitive graph, locally primitive graph, quasiprimitive group, product action, equidistant linear code.

1. INTRODUCTION

All graphs considered in this paper are assumed to be finite, simple and undirected.

Let $\Gamma = (V, E)$ be a connected graph with vertex set V and edge set E. An arc in Γ is an ordered pair of adjacent vertices, and a 2-arc is a triple (α, β, γ) of distinct vertices with $\{\alpha, \beta\}, \{\beta, \gamma\} \in E$. Denote by $\mathrm{Aut}(\Gamma)$ the automorphism group of Γ. For a subgroup $G \leqslant \mathrm{Aut}(\Gamma)$, the graph Γ is said to be $(G, 2)$-arc-transitive (or $(G, 2)$-arc-regular) if G acts transitively (or regularly) on the set of 2-arcs of Γ, while the group G is called a 2-arc-transitive (or 2-arc-regular) group of Γ. For a vertex $\alpha \in V$, let $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$ and $\Gamma(\alpha) = \{\beta \in V \mid \{\alpha, \beta\} \in E\}$, called the stabilizer and the neighborhood of α in G and Γ, respectively. It is elementary to show that G is 2-arc-transitive if and only if G acts transitively on V and, for $\alpha \in V$, the stabilizer G_α acts 2-transitively on $\Gamma(\alpha)$.

Assume that G is 2-arc-transitive on some connected graph $\Gamma = (V, E)$, and $\{\alpha, \beta\} \in E$. Put $G^* = \langle G_\alpha, G_\beta \rangle$, the subgroup of G generated by $G_\alpha \cup G_\beta$. Then $|G : G^*| \leqslant 2$ with the equality holds if and only if Γ is bipartite and G^* is the bipartition preserving subgroup of G, refer to [23]. Assume further that Γ is not a complete bipartite graph, and every minimal normal subgroup of G contained in G^* acts transitively on each of the G^*-orbits on V. In 1993, Praeger [19, 20] proved that, except for one case when Γ is a bipartite graph, G^* is a quasiprimitive permutation group of type HA, TW, AS or PA on each of its orbits, refer to [19, Theorem 2], [20, Theorems 2.1 and 2.3] and [21, Theorem 6.1]. (Recall that a permutation group G is quasiprimitive if every minimal normal subgroup of G is transitive.) Roughly stated, either (G, Γ) is described as in [20, Theorem 2.1 (c)], or G^* has a unique minimal normal subgroup say M, and one of the following four cases occurs for M (and G^*):

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HA (*Holomorph Affine*): M is abelian;
TW (*Twisted Wreath product*): M is nonabelian and regular on each of G^*-orbits;
AS (*Almost Simple group*): M is a nonabelian simple group;
PA (*Product Action*): $M = T_1 \times \cdots \times T_n$ for some integer $n \geqslant 2$ and isomorphic nonabelian simple groups T_i, and for $\alpha \in V$ there are isomorphic subgroups $1 \neq R_i < T_i$ such that M_α is a subdirect product of $R_1 \times \cdots \times R_n$, that is, M_α projects surjectively onto every R_i.

For convenience, we say a connected $(G, 2)$-arc-transitive graph Γ is of HA, TW, AS or PA type if the case HA, TW, AS or PA holds for M (and G^*), respectively. In addition, according to [16], the type PA is said to be diagonal if each of the projections $M_\alpha \to R_i$ is injective, and non-diagonal otherwise.

After Praeger's work, the existence of 2-arc-transitive non-bipartite graphs with HA, TW or AS type was confirmed in just a few years. For example, the classification for those graphs with HA type was given in [14], constructions and examples of graphs with TW type were given in [2] and [19, Section 6], and some classification results of graphs with AS type were given in [7, 8, 11]. The existence problem of graphs with PA type was not answered until 2006 when Li and Seress [16] constructed five families of 2-arc-transitive non-bipartite graphs, four of them consist of graphs with diagonal PA type, and the other one consists of graphs of valency 9 with non-diagonal PA type.

In this paper, we first discuss some further structural information about 2-arc-transitive (non-bipartite and bipartite) graphs with PA type. The following result is proved in Section 4, which is helpful for us to understand the behavior of M_α in the product action of a 2-arc-transitive group on some connected graph.

Theorem 1.1. *Let $\Gamma = (V, E)$ be a connected $(G, 2)$-arc-transitive graph with* PA *type, and let $M = T_1 \times \cdots \times T_n$, G^* and R_1 be defined as above. Then, for $\alpha \in V$, one of the following holds.*

(1) Γ *is of diagonal PA type.*
(2) $M_\alpha \cong (\mathbb{Z}_p^k \times \mathbb{Z}_{m_1}).\mathbb{Z}_m$, $|\Gamma(\alpha)| = p^k$, *and* $|R_1|$ *is indivisible by* p^k, *where* p *is a prime,* $m_1 \mid m$, $m \mid (p^d - 1)$ *for some divisor* d *of* k *with* $d < k$*; in addition,*
   (i) n *is divisible by some prime* r, *where either* r *is an arbitrary primitive prime divisor of* $p^k - 1$, *or* $(p, k) = (2, 6)$ *and* $r \in \{3, 7\}$*; or*
   (ii) $(p, k) = (2, 6)$, *and* M *acts regularly on either the edge set or the arc set of* Γ*; or*
   (iii) $k = 2$, *and* p *is a Mersenne prime.*

Li and Seress [16] proved that, employing an equidistant linear $[4, 2]_3$ code (see Section 5 for the definition), one can construct 2-arc-transitive graphs of valency 9 with non-diagonal PA type from connected cubic graphs which admit a simple 2-arc-regular group. This motivates us to develop a broader construction for graphs with non-diagonal PA type. In Section 5, we confirm that, for some suitable prime power q, there exist equidistant linear $[q+1, 2]_q$ codes which admit a cyclic group of order $q^2 - 1$ acting regularly on the set of nonzero codewords. This allows us to construct some qusiprimitive permutation groups of PA type with a point stabilizer isomorphic to the affine group $\mathrm{AGL}_1(q^2)$, and then give a construction for 2-arc-transitive graphs with non-diagonal PA type. Thus, in Section 6, we construct some 2-arc-transitive graphs of valency q^2 with non-diagonal PA type, which meet Theorem 1.1 (2)(i) or (iii). Then, combining [16, Lemma 5.2 and Example 5.3], we have the following result.

Theorem 1.2. *Let $q \geqslant 3$ be a prime power. Assume that $q+1$ has at most two distinct prime divisors, and either q is even or $q \equiv -1 \pmod 4$. Then there are connected 2-arc-transitive graphs of valency q^2 with non-diagonal* PA *type.*

We also construct in Section 6 some graphs of valency 2^6 and order $2^{57} \cdot 3^{42} \cdot 7^{21}$, which give examples for Theorem 1.1 (2)(ii), see Example 6.6.

For a graph $\Sigma = (V_0, E_0)$, the standard double cover $\Sigma^{(2)}$ is defined as the bipartite graph with vertex set $V_0 \times \mathbb{Z}_2$ such that $(\alpha_0, 0)$ and $(\beta_0, 1)$ are adjacent if and only if $\{\alpha_0, \beta_0\} \in E_0$. It is well-known that $\Sigma^{(2)}$ is connected if and only if Σ is connected and non-bipartite. Define a map

$$\iota : V_0 \times \mathbb{Z}_2 \to V_0 \times \mathbb{Z}_2,\ (\alpha_0, i) \mapsto (\alpha_0, i+1).$$

Then $\iota \in \mathrm{Aut}(\Sigma^{(2)})$. We view $\mathrm{Aut}(\Sigma)$ as a subgroup of $\mathrm{Aut}(\Sigma^{(2)})$ in the following way

$$(\alpha_0, i)^g = (\alpha_0^g, i),\ \alpha_0 \in V_0,\ i \in \mathbb{Z}_2,\ g \in \mathrm{Aut}(\Sigma).$$

Then $\mathrm{Aut}(\Sigma^{(2)})$ has a subgroup $\mathrm{Aut}(\Sigma) \times \langle \iota \rangle$. Thus, if Σ is $(G_0, 2)$-arc-transitive (and of some type) then $\Sigma^{(2)}$ is a $(G_0 \times \langle \iota \rangle, 2)$-arc-transitive bipartite graph (of the same type).

Employing the standard double covers of graphs, one can easily get some firsthand examples of bipartite 2-arc-transitive graphs with HA, TW, AS or PA type, which admit groups G of the form of $G^* \times \mathbb{Z}_2$, where G^* is the subgroup of G generated by the stabilizer of two adjecent vertices. In Section 7, we give a construction for 2-arc-transitive bipartite graphs of diagonal PA type, which admit certain groups G and are not standard double covers of $(G^*, 2)$-arc-transitive graphs.

2. On locally transitive graphs

In this section and the next section, we make some preparation for the proof of Theorem 1.1.

Let $\Gamma = (V, E)$ be a graph, and $G \leqslant \mathrm{Aut}(\Gamma)$. The graph Γ is said to be G-locally transitive or G-locally primitive if for every $\alpha \in V$, the stabilizer G_α acts transitively or primitively on $\Gamma(\alpha)$, respectively.

Let $\Gamma = (V, E)$ be a connected graph, $\{\alpha, \beta\} \in E$, $G \leqslant \mathrm{Aut}(\Gamma)$ and $G^* = \langle G_\alpha, G_\beta \rangle$. Assume that G_α and G_β act transitively on $\Gamma(\alpha)$ and $\Gamma(\beta)$, respectively. Then G^* acts transitively on the edge set E, and G^* acts transitively on the vertex set V if Γ is not bipartite, refer to [23, Exercise 3.8]. If Γ is not bipartite then $|G^* : G_\alpha| = |V| = |G : G_\alpha|$, yielding $G = G^*$. Suppose that Γ is bipartite with two parts, say U and W. Then G^* fixes and acts transitively on both U and W. Without loss of generality, let $\alpha \in U$ and $|U| \geqslant |W|$. We have

$$2|G^* : G_\alpha| = 2|U| \geq |V| \geqslant |G : G_\alpha|.$$

It follows that either $G = G^*$, or $|G : G^*| = 2$ and G is transitive on V. In particular, G^* is the bipartition preserving subgroup of G, and thus $G_\gamma \leqslant G^*$ for every $\gamma \in V$. Now let $\gamma \in V$. Recalling that either G^* is transitive or G^* has two orbits on V, we set $\gamma = \delta^x$ for some $x \in G^*$, where $\delta \in \{\alpha, \beta\}$. Then $\Gamma(\gamma) = \Gamma(\delta)^x$ and $G_\gamma = G_\delta^x$. This implies that G_γ acts transitively on $\Gamma(\gamma)$, and the action is primitive if and only if G_δ acts primitively on $\Gamma(\delta)$. In summary, we have the following lemma.

Lemma 2.1. *Let $\Gamma = (V, E)$ be a connected graph, $\{\alpha, \beta\} \in E$, $G \leqslant \mathrm{Aut}(\Gamma)$ and $G^* = \langle G_\alpha, G_\beta\rangle$. Assume that G_α and G_β act transitively on $\Gamma(\alpha)$ and $\Gamma(\beta)$, respectively. Then Γ is G^*-locally transitive, and Γ is G^*-locally primitive if and only if G_α and G_β act primitively on $\Gamma(\alpha)$ and $\Gamma(\beta)$, respectively. Moreover, either*

(1) *Γ is not bipartite, and $G = G^*$ is transitive on V; or*
(2) *Γ is a bipartite graph with two parts the G^*-orbits on V, and $|G : G^*| \leqslant 2$, where the equality holds if and only if G is transitive on V.*

For locally primitive graphs, the next result holds by [15, Lemmas 2.5 and 2.6], see also [10, Lemma 5.1].

Lemma 2.2. *Assume $\Gamma = (V, E)$ is a connected G-locally primitive graph, and N is a normal subgroup of G.*

(1) *If G is transitive on V and $N_\alpha \neq 1$ for some $\alpha \in V$ then Γ is N-locally transitive.*
(2) *If N is intransitive on each of the G^*-orbits on V, then either*
   (i) *N is semiregular on V, that is, $N_\alpha = 1$ for all $\alpha \in V$, and N itself is the kernel of G^* acting on the N-orbits; or*
   (ii) *G is transitive on V, N has two orbits on V, and either N is semiregular on V or Γ is N-locally transitive.*

The next lemma says that some conclusions in Lemma 2.2 are also true for a bipartite graph Γ under some weaker conditions. For $U_1, W_1 \subseteq V$, denote by $[U_1, W_1]$ the subgraph of Γ induced by $U_1 \cup W_1$.

Lemma 2.3. *Let $\Gamma = (V, E)$ be a connected G-locally transitive bipartite graph, $\{\alpha, \beta\} \in E$ and $G^* = \langle G_\alpha, G_\beta\rangle$. Assume that G_α acts primitively on $\Gamma(\alpha)$, and that G^* has a normal subgroup N which is intransitive on each of the G^*-orbits. Then N is semiregular on V, and N itself is the kernel of G^* acting on the N-orbits.*

Proof. Let U and W be the G^*-orbits containing α and β, respectively. For an arbitrary $\gamma \in U$, we have $\gamma = \alpha^x$ for some $x \in G^*$, and thus $\Gamma(\gamma) = \Gamma(\alpha)^x$ and $G_\gamma = G_\alpha^x$, it follows that G_γ acts primitively on $\Gamma(\gamma)$.

Let $\mathcal{U}$ and $\mathcal{W}$ be the sets of N-orbits on U and W, respectively. Pick $U_1 \in \mathcal{U}$ and $\gamma \in U_1$. Then $\{\Gamma(\gamma) \cap W_1 \mid W_1 \in \mathcal{W}, \Gamma(\gamma) \cap W_1 \neq \emptyset\}$ is a G_γ-invariant partition of $\Gamma(\gamma)$. Since G_γ acts primitively on $\Gamma(\gamma)$, either $\Gamma(\gamma) \subseteq W_1$ for some $W_1 \in \mathcal{W}$, or $[U_1, W_1]$ is a matching without isolated vertex for every $W_1 \in \mathcal{W}$ with $\Gamma(\gamma) \cap W_1 \neq \emptyset$.

Suppose first that $\Gamma(\gamma) \subseteq W_1$ for some $W_1 \in \mathcal{W}$. Then every vertex in U_1 has no neighbor in $W \setminus W_1$ and, since W_1 is an N-orbit, every vertex in W_1 has neighbors in U_1. Let $\delta \in W_1$, and pick two neighbors γ_1 and γ_2 of δ with $\gamma_1 \in U_1$. Let U_2 be the N-orbit containing γ_2. Then $U_1^y = U_2$, where $y \in G_\delta$ with $\gamma_1^y = \gamma_2$. Noting that

$W_1^y = W_1$, it follows that $[U_1, W_1]$ and $[U_2, W_1]$ are isomorphic. Thus every vertex in U_2 has no neighbor in $W \setminus W_1$. Let U_0 be the set of vertices which have neighbors in W_1. By the above argument, every vertex in U_0 has no neighbor in $W \setminus W_1$ and, by the choice of U_0, every vertex in W_1 has no neighbor in $U \setminus U_0$. It follows that $\Gamma = [U_0, W_1]$, and then $W_1 = W$, which contradicts that N is intransitive on W.

Now, for arbitrary $U_1 \in \mathcal{U}$ and $W_1 \in \mathcal{W}$, the subgraph $[U_1, W_1]$ is either a empty graph or a matching without isolated vertex. Let K be the kernel of G^* acting on $\mathcal{U} \cup \mathcal{W}$. We have $N \leqslant K$. In the following, we will show that $K_\gamma = 1$ for all $\gamma \in V$, and then the lemma follows.

Let γ, $\delta \in V$. Since Γ is connected, pick a path $\gamma = \alpha_0, \alpha_1, \ldots, \alpha_n = \delta$ from γ to δ. For $0 \leqslant i \leqslant n$, let V_i be the N-orbit containing α_i. Suppose that K_γ fixes α_{i-1}. Noting that K_γ fixes both V_{i-1} and V_i set-wise, since α_{i-1} has a unique neighbor in V_i, it follows that $K_\gamma \leqslant K_{\alpha_i}$. By induction, we have $K_\gamma \leqslant K_\delta$. Thus K_γ fixes V point-wise, and hence $K_\gamma = 1$. This completes the proof. □

Lemma 2.4. *Let $\Gamma = (V, E)$ be a connected G-locally transitive graph, $\{\alpha, \beta\} \in E$ and $N \trianglelefteq G$. Suppose that $(|N_\alpha|, |\Gamma(\alpha)|) = 1 = (|N_\beta|, |\Gamma(\beta)|)$. Then N is semiregular on V.*

Proof. Let γ be an arbitrary vertex of Γ. By the assumption, since G acts transitively on E, we have $(|N_\gamma|, |\Gamma(\gamma)|) = 1$. Note that $N_\gamma \trianglelefteq G_\gamma$ and G_γ acts transitively on $\Gamma(\gamma)$. Then all N_γ-orbits on $\Gamma(\gamma)$ have the same length, which is a common divisor of $|\Gamma(\gamma)|$ and $|N_\gamma|$. It follows that N_γ fixes $\Gamma(\gamma)$ point-wise. In particular, $N_\gamma \leqslant N_\delta$ for $\delta \in \Gamma(\gamma)$. Again since $(|N_\delta|, |\Gamma(\delta)|) = 1$, a similar argument implies that N_δ fixes $\Gamma(\delta)$ point-wise, and so N_γ fixes $\Gamma(\delta)$ point-wise. Thus, since Γ is connected, we conclude that N_γ fixes V point-wise, and so $N_\gamma = 1$. Then N is semiregular on V. □

3. Two elementary results on primitive affine groups

For a group X and subgroups $Y, Z \leqslant X$, we put $\mathbf{C}_Y(Z) = \{y \in Y \mid yz = zy \text{ for all } z \in Z\}$, which is called the centralizer of Z in Y.

Recall that, for positive integers p, $k > 1$, a primitive prime divisor of $p^k - 1$ is a prime which divides $p^k - 1$ but does not divide $p^i - 1$ for all $1 \leqslant i < k$. If r is a primitive prime divisor of $p^k - 1$, then k is the smallest positive integer with $p^k \equiv 1 \,(\mathsf{mod}\ r)$, and thus k is a divisor of $r - 1$; if further $r \mid (q^l - 1)$ with $l \geqslant 1$ then $k \mid l$. These facts yield a criterion for affine primitive permutation groups.

Lemma 3.1. *Let H be a permutation group on a set Ω, and $\alpha \in \Omega$. Suppose that H has a regular normal subgroup $P \cong \mathbb{Z}_p^k$, where $k \geqslant 2$ and p is a prime. Suppose that $p^k - 1$ has a primitive prime divisor r, and $|H_\alpha|$ is divisible by r. Then H is primitive on Ω.*

Proof. Let Q be a Sylow r-subgroup of H_α. Then $Q \neq 1$ as r is a divisor of $|H_\alpha|$. Set $K = PQ$. Then $K_\alpha = Q$. We next show that K is primitive on Ω, and then H is primitive. It suffices to prove that Q is a maximal subgroup of K.

By Maschke's Theorem (refer to [13, p.123, I.17.7]), since $(p, |Q|) = 1$, we have $P = P_1 \times \cdots \times P_l$, where P_i are minimal Q-invariant subgroups of P. Considering the conjugation of Q on P_i, the group Q induces a subgroup of the automorphism group $\mathrm{Aut}(P_i)$ of P_i with kernel $\mathbf{C}_Q(P_i)$. Then $\mathrm{Aut}(P_i)$ is isomorphic to the general linear group $\mathrm{GL}_{k_i}(p)$, and so

$$Q/\mathbf{C}_Q(P_i) \lesssim \mathrm{Aut}(P_i) \cong \mathrm{GL}_{k_i}(p),\ 1 \leqslant i \leqslant l.$$

Suppose that $l > 1$. Then $k_i < k$ for every i, and so $|\mathrm{GL}_{k_i}(p)|$ is indivisible by r. It follows that $Q = \mathbf{C}_Q(P_i)$ for all i, and thus Q centralizes P. Then $Q \trianglelefteq K$, which is impossible as $1 \neq Q = K_\alpha$. Therefore, $l = 1$, which yields that P is a minimal normal subgroup of K.

Let L be a maximal subgroup of K with $Q \leqslant L$. Then $K > L = PQ\cap L = (P\cap L)Q$, and so $P \cap L \neq P$. Since P is abelian and $P \trianglelefteq K$, we have $P \cap L \trianglelefteq P$ and $P \cap L \trianglelefteq L$, and thus $P \cap L \trianglelefteq \langle P, L\rangle = K$. Then $P \cap L = 1$ as P is a minimal normal subgroup of K. Thus $L = (P \cap L)Q = Q$. This says that Q is a maximal subgroup of K, and then K is primitive on Ω. Noting that $K \leqslant H$, the lemma follows. □

A transitive permutation group H on a set Ω is called a Frobenius group if $H_\alpha \neq 1$ for $\alpha \in \Omega$, and $H_{\alpha\beta} = 1$ for all $\beta \in \Omega \setminus \{\alpha\}$. The following lemma gives a characterization of imprimitive Frobenius groups with abelian socle, see [18, Lemma 2.2] for example. Recall that, for a finite group X, the socle $\mathrm{soc}(X)$ of X is generated by all minimal normal subgroups of X.

Lemma 3.2. *Let K be an imprimitive Frobenius group on Ω with $\mathrm{soc}(K) = P \cong \mathbb{Z}_p^k$, where p is a prime and $k \geqslant 2$. Then K_α is isomorphic to an irreducible subgroup of the general linear group $\mathrm{GL}_l(p)$ for some l, and $|K_\alpha|$ is a divisor of $p^d - 1$, where $2l \leqslant k$ and d is a common divisor of k and l.*

Lemma 3.3. *Let H be a 2-transitive affine group of degree 2^6 on a set Ω, and let $1 \neq K \trianglelefteq H$. Assume that $K_\alpha \neq 1$ for $\alpha \in V$, and K is imprimitive on Ω. Then*

(1) *$K_\alpha \cong \mathbb{Z}_s$ with $s \in \{3, 7\}$, and there is $x \in H_\alpha$ such that $K_\alpha\langle x\rangle \cong \mathbb{Z}_{21}$; and*
(2) *for each $x \in H_\alpha$ with $K_\alpha\langle x\rangle \cong \mathbb{Z}_{21}$, the subgroup $K\langle x\rangle$ is primitive on Ω.*

Proof. By [6, pp.215-217, Theorems 7.2C and 7.2E], K is an imprimitive Frobenius group. Applying Lemma 3.2, we get $K_\alpha \cong \mathbb{Z}_s$, where $s \in \{3, 7\}$. Calculation with GAP [9] shows that there are exactly eleven 2-transitive affine groups of degree 2^6, which contain an imprimitive normal Frobenius subgroup. Checking one by one these groups, we conclude that K_α is contained in a cyclic subgroup of order 21 in H_α. Then part (1) of this lemma follows.

Assume that $x \in H_\alpha$ with $K_\alpha\langle x\rangle \cong \mathbb{Z}_{21}$, and set $X = K\langle x\rangle$. Then $\mathrm{soc}(H) \trianglelefteq X$ and $X_\alpha \cong \mathbb{Z}_{21}$. Without loss of generality, we assume that $K_\alpha \cap \langle x\rangle = 1$, let $\langle x\rangle \cong \mathbb{Z}_r$ and write $X_\alpha = \langle y\rangle \times \langle x\rangle$ with $K_\alpha = \langle y\rangle \cong \mathbb{Z}_s$.

By Maschke's Theorem, we have $\mathbb{Z}_2^6 \cong \mathrm{soc}(H) = P_1 \times \cdots \times P_l$, where P_i are minimal X_α-invariant subgroup of $\mathrm{soc}(H)$. Since K is an imprimitive Frobenius group, y does not centralize every P_i, and s is a divisor of $|P_i|-1$, refer to [1, p.191, (35.25)]. Suppose that $l > 1$. Then either $s = 3$, $P_i \cong \mathbb{Z}_2^2$ and $l = 3$, or $s = 7$, $P_i \cong \mathbb{Z}_2^3$ and $l = 2$, where $1 \leqslant i \leqslant l$. Note that $X_\alpha/\mathbf{C}_{X_\alpha}(P_i) \lesssim \mathrm{Aut}(P_i)$. Assume first that $s = 3$. Then $r = 7$, and $\mathrm{Aut}(P_i) \cong \mathrm{GL}_2(2) \cong \mathrm{S}_3$. This implies that x centralizes every P_i. Thus $\langle x\rangle \trianglelefteq H$, which is impossible as $1 \neq \langle x \rangle \leqslant X_\alpha$. Now let $s = 7$. Then $l = 2$, and $P_1 \cong P_2 \cong \mathbb{Z}_2^3$. We have $\langle y \rangle \cong (\langle y \rangle \mathbf{C}_{X_\alpha}(P_i))/\mathbf{C}_{X_\alpha}(P_i) \leqslant X_\alpha/\mathbf{C}_{X_\alpha}(P_i) \lesssim \mathrm{Aut}(P_i) \cong \mathrm{GL}_3(2)$. By the Atlas [5], $\mathrm{GL}_3(2)$ has no element of order 21. It follows that x centralizes every P_i, which leads to a similar contradiction as above. Therefore, $l = 1$, and then $\mathrm{soc}(H)$ is a minimal normal subgroup of X. Thus X_α is a maximal subgroup of X, and part (2) of this lemma follows. □

4. The proof of Theorem 1.1

Let $\Gamma = (V, E)$ be a connected graph of valency no less than 3, and $G \leqslant \mathrm{Aut}(\Gamma)$. Let $G^* = \langle G_{\alpha_1}, G_{\alpha_2} \rangle$ for some $\{\alpha_1, \alpha_2\} \in E$, and let $M = \mathrm{soc}(G^*)$. Assume that Γ is $(G, 2)$-arc-transitive, and G^* is a quasiprimitive group of PA type on each of the G^*-orbits. Then both G^* and M have the same orbits on V. By [19, 20], we have

(I) $M = T_1 \times T_2 \times \cdots \times T_n$ is the unique minimal normal subgroup of G^*, where $n \geqslant 2$ and T_i are isomorphic nonabelian simple groups; and

(II) for $\alpha \in V$, there are subgroups $R_i < T_i$ such that $M_\alpha \leqslant R_1 \times \cdots \times R_n$ and, for every i, the projection

$$\pi_i : M_\alpha \to R_i,\ x_1x_2\cdots x_n \mapsto x_i,\ \text{ where } x_j \in R_j \text{ for all } j$$

is a surjective group homomorphism.

Note that T_1, T_2, …, T_n are all minimal normal subgroups of M, refer to [13, p.51, I.9.12]. Since M is a minimal normal subgroup of G^*, we have

(III) G_α acts transitively on $\{T_1, T_2, \ldots, T_n\}$ by conjugation.

Clearly, $M_\alpha \trianglelefteq G_\alpha$. For $1 \leqslant i, i' \leqslant n$, letting $T_i^h = T_{i'}$ for some $h \in G_\alpha$, we have

$$R_{i'} = \pi_{i'}(M_\alpha) = \pi_{i'}(M_\alpha^h) \geqslant \pi_i(M_\alpha)^h = R_i^h.$$

Similarly, since $T_{i'}^{h^{-1}} = T_i$, we have $R_i \geqslant R_{i'}^{h^{-1}}$, and so $R_i^h = R_{i'}$. It follows that

(IV) G_α acts transitively on $\{R_1, R_2, \ldots, R_n\}$ by conjugation; in particular, $R_1 \cong \cdots \cong R_n$.

For convenience, we set $N_i = \prod_{j \neq i} T_j$, where $1 \leqslant i \leqslant n$. Then

(V) $N_i \trianglelefteq M$, and the kernel $\ker(\pi_i)$ of $\pi_i : M_\alpha \to R_i$ equals to $(N_i)_\alpha$.

Note that N_1, …, N_n are all maximal normal subgroups of M, refer to [13, p.51, I.9.12]. We have

(VI) G_α acts transitively on both $\{N_1, N_2, \ldots, N_n\}$ and $\{\ker(\pi_1), \ldots, \ker(\pi_n)\}$ by conjugation; in particular, $\ker(\pi_1) \cong \cdots \cong \ker(\pi_n)$.

In addition, the following lemma holds.

Lemma 4.1. *Every N_i is intransitive on each of the M-orbits on V.*

Proof. Suppose that some N_i acts transitively on some M-orbit. Then $M = N_iM_\gamma$ for some $\gamma \in V$. Thus $T_i \cong M/N_i = N_iM_\gamma/N_i \cong M_\gamma/(N_i)_\gamma$. Then M_γ has a composition factor isomorphic to T_i, which is impossible as $M_\gamma \cong M_\alpha \leqslant R_1 \times \cdots \times R_n$. This completes the proof. □

By Lemma 2.2, Γ is M-locally transitive. If Γ is M-locally primitive, then Theorem 1.1 is true by the following simple lemma.

Lemma 4.2. *Assume Γ is M-locally primitive. Then every π_i is injective; in particular, $M_\alpha \cong R_i$ for all i.*

Proof. Suppose that some π_i is not injective. Then π_i has nontrivial kernel $\ker(\pi_i) = (N_i)_\alpha$. Then, by Lemmas 2.1 and 2.2, N_i is transitive on one of the M-orbits on V, which contradicts Lemma 4.1. This completes the proof. $\square$

We next deal with the case where Γ is not M-locally primitive. For a subgroup $X \leqslant G$, denote by $X_\alpha^{[1]}$ the kernel of X_α acting on $\Gamma(\alpha)$, and by $X_\alpha^{\Gamma(\alpha)}$ the permutation group induced by X_α on $\Gamma(\alpha)$. By [18], we have the following lemma.

Lemma 4.3. *If Γ is not M-locally primitive, then one of the following holds.*

(1) $M_\alpha \cong (\mathbb{Z}_p^k \times \mathbb{Z}_{m_1}).\mathbb{Z}_m$, $|\Gamma(\alpha)| = p^k$ *and* $M_\alpha^{[1]} \cong \mathbb{Z}_{m_1}$, *where* $m_1 \mid m$, $m \mid (p^d - 1)$ *for some divisor d of k with $d < k$;*

(2) $M_\alpha \cong (\mathbb{Z}_3^4 \times Q).\mathrm{Q}_8$, $|\Gamma(\alpha)| = 3^4$ *and* $M_\alpha^{[1]} \cong Q$, *where Q is isomorphic to a subgroup of the quaternion group* Q_8.

Together with Lemmas 4.2 and 4.3, the following lemma fulfills the proof of Theorem 1.1.

Lemma 4.4. *Assume that $|\Gamma(\alpha)| = p^k$ and M_α is described as in (1) or (2) of Lemma 4.3. Let p^l be the highest power of p that divides $|R_1|$.*

(1) *If $l = k$ then every π_i is injective.*

(2) *If $l < k$ then one of the follows holds.*

 (i) *n is divisible by some prime r, where either r is an arbitrary primitive prime divisor of $p^k - 1$, or $(p, k) = (2, 6)$ and $r \in \{3, 7\}$;*

 (ii) *$(p, k) = (2, 6)$, and M acts regularly on the edge set or arc set of Γ;*

 (iii) *$k = 2$, and p is a Mersenne prime.*

Proof. Recalling that $\pi_1 : M_\alpha \to R_1$ is a surjective homomorphism, we have $l \leqslant k$. Assume that $l = k$. Then every $\ker(\pi_i)$ has order indivisible by p. Recalling that $(N_i)_\alpha = \ker(\pi_i)$, by Lemma 2.4, $\ker(\pi_i) = 1$, and part (1) of this lemma is true.

Assume that $l < k$ from now on. If $p^k - 1$ has no primitive prime divisor then, by Zsigmondy's Theorem (see [24]), either $(p, k) = (2, 6)$, or $k = 2$ and p is a Mersenne prime. The latter case is just the case (iii) of the lemma. For $(p, k) = (2, 6)$, if $M_\alpha \cong \mathbb{Z}_2^6$ then we get the case (ii) of this lemma.

In the following, we assume further that either $(p, k) = (2, 6)$ and $M_\alpha \ncong \mathbb{Z}_2^6$, or $p^k - 1$ has a primitive prime divisor r. Noting that G_α acts 2-transitively on $\Gamma(\alpha)$, it follows that $p^k - 1$ is a divisor of $|G_{\alpha\beta}|$ for $\beta \in \Gamma(\alpha)$, and then either 21 or r is a divisor of $|G_{\alpha\beta}|$, respectively. In addition, for $(p, k) = (2, 6)$, we have $M_\alpha^{\Gamma(\alpha)} \cong \mathbb{Z}_2^6{:}\mathbb{Z}_s$ with $s \in \{3, 7\}$ by Lemma 4.3; in this case, we set $r = \frac{21}{s}$.

Claim 1. If $(p, k) = (2, 6)$ then there is an element $x \in G_{\alpha\beta}$ of order r such that $M_{\alpha\beta}\langle x\rangle = M_{\alpha\beta} \times \langle x\rangle$, where $\beta \in \Gamma(\alpha)$.

Assume that $(p, k) = (2, 6)$. By Lemma 4.3, we conclude that $M_{\alpha\beta}$ is an abelian group of order s or s^2. Then $M_{\alpha\beta} \cong \mathbb{Z}_s$, $\mathbb{Z}_s^2$ or $\mathbb{Z}_{s^2}$, and thus $\mathrm{Aut}(M_{\alpha\beta})$ has order $s - 1$, $s(s-1)(s^2-1)$ or $s(s-1)$, respectively. Since $M_{\alpha\beta} \trianglelefteq G_{\alpha\beta}$, every element in

$G_{\alpha\beta}$ induces an automorphism of $M_{\alpha\beta}$ by conjugation. If $s = 3$ then $|\mathrm{Aut}(M_{\alpha\beta})|$ is indivisible by $r = 7$, and so $M_{\alpha\beta}$ is centralized by every element of order 7 in $G_{\alpha\beta}$, our claim is true in this case.

Now let $s = 7$ and $r = 3$. Then a Sylow 3-subgroup of $\mathrm{Aut}(M_{\alpha\beta})$ is isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_3$ when $M_{\alpha\beta} \cong \mathbb{Z}_7$, $\mathbb{Z}_7^2$ or $\mathbb{Z}_{7^2}$, respectively. Noting that the 2-transitive affine group $G_\alpha^{\Gamma(\alpha)}$ has a normal subgroup isomorphic to $\mathbb{Z}_2^6{:}\mathbb{Z}_7$, calculation with GAP [9] shows that $(G_\alpha^{\Gamma(\alpha)})_\beta$ has a subgroup isomorphic to $\mathbb{Z}_9$. Pick a Sylow 3-subgroup Q of $G_{\alpha\beta}$. Then Q acts unfaithfully on $M_{\alpha\beta}$ by conjugation; otherwise, $Q \lesssim \mathbb{Z}_3^2$, which is impossible. Thus $\mathbf{C}_Q(M_{\alpha\beta}) \neq 1$, and every element of order 3 in $\mathbf{C}_Q(M_{\alpha\beta})$ is a desired x. Then Claim 1 follows.

Now fix an element $x \in G_{\alpha\beta}$ of order r, where either r is a primitive prime divisor of $p^k - 1$, or $(p, k) = (2, 6)$, $r = \frac{21}{s}$ and x is described as in Claim 1. Then $M \cap \langle x\rangle = M_\alpha \cap \langle x\rangle = 1$. Set $X = M\langle x\rangle$. Clearly, Γ is X-locally transitive, and $|X_\gamma| = r|M_\alpha|$ for all $\gamma \in V$. In addition, for $(p, k) = (2, 6)$, we have $X_{\alpha\beta} = M_{\alpha\beta} \times \langle x\rangle$.

Claim 2. Either X_α acts primitively on $\Gamma(\alpha)$, or X_β acts primitively on $\Gamma(\beta)$.

By Lemma 4.3, either $M_\alpha \cong \mathbb{Z}_2^2$ or $|\Gamma(\alpha)| \geqslant 8$. Assume first $M_\alpha \cong \mathbb{Z}_2^2$. Then $r = 3$, $X_\alpha = M_\alpha\langle x\rangle$, $X_\beta = M_\beta\langle x\rangle$ and $X_{\alpha\beta} = \langle x\rangle$. Suppose that $X_\alpha^{[1]} \neq 1 \neq X_\beta^{[1]}$. Then $X_\alpha^{[1]} = X_\beta^{[1]} = X_{\alpha\beta} = \langle x\rangle$, yielding $\langle x\rangle \trianglelefteq \langle X_\alpha, X_\beta\rangle$. Note that $\langle X_\alpha, X_\beta\rangle$ acts transitively on E, refer to [23, Exercise 3.8]. It follows that $\langle x\rangle$ fixes every edge of Γ, and thus $\langle x\rangle = 1$, a contradiction. We have $X_\alpha^{[1]} = 1$ or $X_\beta^{[1]} = 1$. Then one of $X_\alpha^{\Gamma(\alpha)}$ and $X_\beta^{\Gamma(\beta)}$ is a 2-transitive group of degree 4, and Claim 2 is true in this case.

Assume that $|\Gamma(\alpha)| \geqslant 8$. Then, by [23, Theorem 4.7], $G_\alpha^{[1]} \cap G_\beta^{[1]} = 1$, and so $X_\alpha^{[1]} \cap X_\beta^{[1]} = 1$. Considering the actions of $X_{\alpha\beta}$ on $\Gamma(\alpha)$ and $\Gamma(\beta)$, we have

$$X_{\alpha\beta}^{\Gamma(\alpha)} \cong X_{\alpha\beta}/X_\alpha^{[1]},\ X_{\alpha\beta}^{\Gamma(\beta)} \cong X_{\alpha\beta}/X_\beta^{[1]}.$$

If neither $|X_{\alpha\beta}^{\Gamma(\alpha)}|$ nor $|X_{\alpha\beta}^{\Gamma(\beta)}|$ is divisible by r, then all Sylow r-subgroups are contained in both $X_\alpha^{[1]}$ and $X_\beta^{[1]}$, which contradicts that $X_\alpha^{[1]} \cap X_\beta^{[1]} = 1$. Without loss of generality, we assume that $|X_{\alpha\beta}^{\Gamma(\alpha)}|$ is divisible by r. If r is a primitive prime divisor of $p^k - 1$, then $X_\alpha^{\Gamma(\alpha)}$ is primitive by Lemma 3.1. Now let $(p, k) = (2, 6)$. Noting that $\mathbb{Z}_s \cong M_{\alpha\beta}^{\Gamma(\alpha)} \trianglelefteq X_{\alpha\beta}^{\Gamma(\alpha)}$ and $X_{\alpha\beta}\langle x\rangle = M_{\alpha\beta} \times \langle x\rangle$, we have $X_{\alpha\beta}^{\Gamma(\alpha)} \cong \mathbb{Z}_{21}$. Then $X_\alpha^{\Gamma(\alpha)}$ is primitive by Lemma 3.3. Thus Claim 2 follows.

Finally, consider the action of $\langle x\rangle$ on $\{T_1, \ldots, T_n\}$ by conjugation. Suppose that some T_i, say T_1 without loss of generality, is normalized by x. Then $N_1 = \prod_{j\neq 1} T_j$ is also normalized by x, and thus $N_1 \trianglelefteq X$. Note that N_1 is intransitive on each of the M-orbits, see Lemma 4.1. Assume that Γ is not bipartite. Then, by Claim 2 and Lemma 2.2, N_1 is semiregular on V, and so $\ker(\pi_1) = (N_1)_\alpha = 1$, yielding $M_\alpha \cong R_1$. Thus $k = l$, which is not the case. If Γ is bipartite then, by Claim 2 and Lemma 2.3, N_1 is semiregular on V, we have a similar contradiction as above. Therefore, $\langle x\rangle$ acts faithfully and semiregularly on $\{T_1, \ldots, T_n\}$. Then r is a divisor of n, and case (i) of this lemma follows. This completes the proof. $\square$

5. A construction of equidistant linear codes

Let $q = p^f$ for some prime p and integer $f \geqslant 1$. Denote by $\mathbb{F}_q$ the field of order q, and $\mathbb{F}_q^n$ the n-dimensional row vector space over $\mathbb{F}_q$, where $n \geqslant 1$. For a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{F}_q^n$, letting $\mathsf{supp}(\mathbf{v}) = \{i \mid v_i \neq 0,\, 1 \leqslant i \leqslant n\}$, the weight $\mathsf{wt}(\mathbf{v})$ is defined as $|\mathsf{supp}(\mathbf{v})|$, i.e., the number of nonzero coordinates of $\mathbf{v}$.

Let k be an integer with $1 \leqslant k \leqslant n$. Every k-dimensional subspace $\mathcal{C}$ of $\mathbb{F}_q^n$ is called a linear $[n,k]_q$ code, where n is called the length of $\mathcal{C}$, and the vectors in $\mathcal{C}$ are called codewords. A linear $[n,k]_q$ code $\mathcal{C}$ is said to be equidistant if all nonzero codewords have the same weight say ω, while ω is called the weight of $\mathcal{C}$ and write $\mathsf{wt}(\mathcal{C}) = \omega$.

Let $\mathcal{C}$ be an equidistant linear $[n,2]_q$ code with $\mathsf{wt}(\mathcal{C}) = \omega$. For $\mathbf{0} \neq \mathbf{w} \in \mathcal{C}$, define

$$\mathcal{C}_{\mathbf{w}} = \{\mathbf{u} \in \mathcal{C} \mid \mathsf{supp}(\mathbf{u}) = \mathsf{supp}(\mathbf{w}) \text{ or } \emptyset\}.$$

Then it is easily shown that $\mathcal{C}_{\mathbf{w}}$ is a 1-dimensional subspace of $\mathcal{C}$, and every 1-dimensional subspace of $\mathcal{C}$ is obtained in the form of $\mathcal{C}_{\mathbf{w}}$. Let $1 \leqslant \ell \leqslant q+1$, and choose $\mathbf{w}_\ell \in \mathcal{C}$, with $\mathcal{C} = \cup_{\ell=1}^{q+1} \mathcal{C}_{\mathbf{w}_\ell}$. Set $\Delta = \cup_{\ell=1}^{q+1} \mathsf{supp}(\mathbf{w}_\ell)$, and view $\mathcal{C}$ as an $[m,2]_q$ code, where $m = |\Delta|$. Then $\omega \leqslant m-1$ by the Singleton bound, refer to [12, p.73, Corollary 2.50]. Consider the linear maps $\pi_i : \mathcal{C} \to \mathbb{F}_q$ given by $(v_1, \ldots, v_n) \mapsto v_i$, where $i \in \Delta$. Clearly, every π_i is surjective, and $\ker(\pi_i)$ is 1-dimensional. Then, for each $i \in \Delta$, there is some $\mathbf{w}_\ell$ with $i \notin \mathsf{supp}(\mathbf{w}_\ell)$ and $\ker(\pi_i) = \mathcal{C}_{\mathbf{w}_\ell}$. It follows that the subsets $\Delta \setminus \mathsf{supp}(\mathbf{w}_\ell)$ of Δ are pairwise disjoint. Noting that $|\Delta \setminus \mathsf{supp}(\mathbf{w}_\ell)| = m - \omega$, we have

$$q + 1 \leqslant \frac{m}{m-\omega} \leqslant \frac{n}{n-\omega}.$$

Then we get the following fact.

Lemma 5.1. *Let $\mathcal{C}$ be an equidistant linear $[n,2]_q$ code with $\mathsf{wt}(\mathcal{C}) = \omega$. If $n = q+1$ then $\omega = n-1$, and $\ker(\pi_i)$, $1 \leqslant i \leqslant n$, are distinct 1-dimensional subspaces of $\mathcal{C}$.*

Let $n = q+1$ from now on. Denote by $\mathbb{F}_q^*$ the multiplicative group of $\mathbb{F}_q$, and write $\mathbb{F}_q^* = \langle \eta, \lambda \rangle$, where λ has odd order, and η has order a power of 2. Clearly, $\mathbb{F}_q^* = \langle \eta\lambda \rangle = \langle \eta\lambda^2 \rangle$. Note that $\eta = 1$ if q is even, and $(\eta\lambda)^{\frac{q-1}{2}} = -1 = (\eta\lambda^2)^{\frac{q-1}{2}}$ if q is odd. Pick two invertible $n \times n$ matrices over $\mathbb{F}_q$:

$$\mathbf{D} = \begin{pmatrix} \eta\lambda & 0 & \mathbf{0} \\ 0 & \lambda & \mathbf{0} \\ 0 & 0 & \eta\lambda \mathbf{I}_{n-2} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{n-1} \\ 1 & \mathbf{0} \end{pmatrix},$$

where $\mathbf{I}_m$ denotes the identity matrix of order m. Let $\mathbf{A} = \mathbf{DP}$. Then

$$\mathbf{A}^n = \eta^{n-1}\lambda^n \mathbf{I}_n = \eta\lambda^2 \mathbf{I}_n.$$

In particular, $\mathbf{A}$ has order $n(q-1)$ as an element of the general linear group $\mathrm{GL}_n(q)$.

View $\mathbf{A}$ as the linear transformation of $\mathbb{F}_q^n$ given by right multiplication on the row vectors

$$(u_1, u_2, \ldots, u_n)\mathbf{A} = (\eta\lambda u_n, \eta\lambda u_1, \lambda u_2, \eta\lambda u_3, \eta\lambda u_4, \ldots, \eta\lambda u_{n-1}).$$

Then we have an action of the cyclic group $\langle \mathbf{A} \rangle$ on $\mathbb{F}_q^n$. A linear $[n,k]_q$ code $\mathcal{C}$ is said to be $\langle \mathbf{A} \rangle$-invariant if $\mathbf{uA} \in \mathcal{C}$ for all $\mathbf{u} \in \mathcal{C}$, and $\langle \mathbf{A} \rangle$-irreducible if further $\mathcal{C}$

does not contain $\langle \mathbf{A}\rangle$-invariant linear $[n,k']_q$ codes for some $1 \leqslant k' < k$. An $\langle \mathbf{A}\rangle$-invariant linear $[n,k]_q$ code $\mathcal{C}$ is said to be faithful if $\langle \mathbf{A}\rangle$ acts faithfully on $\mathcal{C}$, that is, no nonidentity matrix in $\langle \mathbf{A}\rangle$ fixes $\mathcal{C}$ point-wise.

Lemma 5.2. *Let $\mathcal{C}$ be an $\langle \mathbf{A}\rangle$-irreducible linear $[n,k]_q$ code, where $n = q+1$. Then either $k = 2$, or $k = 1$, q is even and $\mathcal{C}$ is spanned by the vector $(1,1,\ldots,1)$. If further $\mathcal{C}$ is faithful, then $\langle \mathbf{A}\rangle$ is regular on the nonzero codewords; in particular, $\mathcal{C}$ is an equidistant $[n,2]_q$ code of weight q.*

Proof. Assume that $\mathbf{A}$ induces an invertible linear transformation of order m on $\mathcal{C}$. Then m is a divisor of the order q^2-1 of $\mathbf{A}$ in $\mathrm{GL}_n(q)$. Now k is the smallest positive integer such that $q^k - 1 \equiv 0 \,(\mathsf{mod}\ m)$, refer to [13, p.165, II.3.10]. Thus $k \leqslant 2$.

Suppose that $k = 1$. Then m is a divisor of $q-1$, and the kernel of $\langle \mathbf{A}\rangle$ acting on $\mathcal{C}$ contains the unique subgroup $\langle \mathbf{A}^{q-1}\rangle$ of order $q+1$. Thus $\mathbf{u}\mathbf{A}^{q-1} = \mathbf{u}$ for all $\mathbf{u} \in \mathcal{C}$. If q is odd, then $(\mathbf{A}^{q-1})^{\frac{q+1}{2}} = (\mathbf{A}^n)^{\frac{q-1}{2}} = (\eta\lambda^2)^{\frac{q-1}{2}}\mathbf{I}_n = -\mathbf{I}_n$, yielding $\mathbf{u} = \mathbf{u}(\mathbf{A}^{q-1})^{\frac{q+1}{2}} = -\mathbf{u}$, which is impossible. Therefore, q is a even. For an arbitrary codeword $\mathbf{u} = (u_1,u_2,\ldots,u_n) \in \mathcal{C}$, calculation shows that

$$(u_1,u_2,\ldots,u_n)\mathbf{A}^{q-1} = (u_3,u_4,u_5,\ldots,u_n,u_1,u_2).$$

Since $\mathbf{u}\mathbf{A}^{q-1} = \mathbf{u}$, we have $u_1 = u_2 = \cdots = u_n$. Then $\mathcal{C}$ is spanned by the vector $(1,1,\ldots,1)$, and the first part of this lemma follows.

Now assume that $\mathcal{C}$ is faithful. Then $k = 2$. Noticing the Singleton bound, we may choose a nonzero codeword $\mathbf{w}_1$ with $\mathsf{wt}(\mathbf{w}_1) \leq n-1$. Let $\mathbf{w}_2 = \mathbf{w}_1\mathbf{A}$. Recalling that $\mathcal{C}_{\mathbf{w}_1}$ is 1-dimensional, it is not $\langle \mathbf{A}\rangle$-invariant, and thus $\mathcal{C}_{\mathbf{w}_1} \neq \mathcal{C}_{\mathbf{w}_2}$. In particular, we have $\mathcal{C} = \mathcal{C}_{\mathbf{w}_1} \oplus \mathcal{C}_{\mathbf{w}_2}$. Assume that $\mathbf{A}^i$ fixes $\mathbf{w}_1$ for some i. Then $\mathbf{w}_2\mathbf{A}^i = \mathbf{w}_1\mathbf{A}^{i+1} = \mathbf{w}_1\mathbf{A} = \mathbf{w}_2$. It follows that $\mathbf{A}^i$ fixes $\mathcal{C}$ point-wise. This implies that $\mathbf{A}^i = \mathbf{I}_n$. Then $\langle \mathbf{A}\rangle$ is regular on $\mathcal{C} \setminus \{\mathbf{0}\}$, and the lemma follows from Lemma 5.1 □

Theorem 5.3. *Assume that $n = q+1 = 2^s r^t$ for some odd prime r and integers $s,t \geqslant 0$. Then there exists a faithful $\langle \mathbf{A}\rangle$-irreducible linear $[n,2]_q$ code. If q is a Mersenne prime then $\mathbb{F}_q^n$ is a direct sum of faithful $\langle \mathbf{A}\rangle$-irreducible linear $[n,2]_q$ codes.*

Proof. Appealing to Maschke's Theorem, refer to [13, p.123, I.17.7], we write

$$\mathbb{F}_q^n = \oplus_{i=1}^m \mathcal{C}_i,$$

where $\mathcal{C}_i$ are $\langle \mathbf{A}\rangle$-irreducible $[n,k_i]_q$ codes. By Lemma 5.2, we assume that $k_1 = \cdots = k_{m-1} = 2$, and either $k_m = 2$ or q is even and $\mathcal{C}_m$ is spanned by $(1,1,\ldots,1)$.

Let K_i be the kernel of $\langle \mathbf{A}\rangle$ acting on $\mathcal{C}_i$, where $1 \leqslant i \leqslant m$. Recalling that $\mathbf{A}^n = \eta\lambda^2\mathbf{I}_n$, we know that $\langle \mathbf{A}^n\rangle$ is semiregular on the set of nonzero codewords of every $\mathcal{C}_i$, and thus $K_i \cap \langle \mathbf{A}^n\rangle = 1$. Then $|K_i|$ is a divisor of $q+1$. Now it suffices to show that $|K_i| = 1$ for some i, and if q is a Mersenne prime then $|K_i| = 1$ for all i.

Assume first q is even. Then $n = r^t$, and $\langle \mathbf{A}\rangle$ contains a unique subgroup of order r. It follows that either $|K_i| = 1$ for some i, or all K_i contains a common subgroup of order r. The latter case implies that $\langle \mathbf{A}\rangle$ is unfaithful on $\mathbb{F}_q^n$, which is impossible.

Now let q be odd. Then $\langle \mathbf{A}^n\rangle$ has even order $q-1$. Recalling that $K_i \cap \langle \mathbf{A}^n\rangle = 1$ for all i, since $\langle \mathbf{A}\rangle$ has a unique involution, it follows that every $|K_i|$ is an odd divisor of $q+1$. Thus, since $\langle \mathbf{A} \rangle$ is faithful on $\mathbb{F}_q^n$, we have $|K_i| = 1$ for some i. If further q is a Merdenne prime, then $|K_i| = 1$ for all i. This completes the proof. □

Remark 5.4. *By the definitions, it is easy to see that each faithful $\langle \mathbf{A} \rangle$-irreducible linear $[n, 2]_q$ code gives a minimal normal subgroup of the semidirect product $\mathbb{F}_q^n{:}\langle \mathbf{A} \rangle$ isomorphic to $\mathbb{Z}_p^{2f}$.*

6. A construction of graphs with non-diagonal PA type

For a finite group G and $H \leqslant G$, denote by $[G : H]$ the set of right cosets of H in G. Assume that H is core-free in G, that is, $\cap_{g \in G} H^g = 1$. Then we have a faithful and transitive action of G on $[G : H]$ by right multiplication, and thus we identify G with a transitive permutation group on $[G : H]$. For an element $g \in G \setminus H$ with $g^2 \in H$, the coset graph $\mathrm{Cos}(G, H, g)$ is defined as the graph with vertex set $[G : H]$ such that Hx and Hy are adjacent if and only if $yx^{-1} \in HgH$. It is well-known that $\mathrm{Cos}(G, H, g)$ is G-arc-transitive and of valency $|H : (H \cap H^g)|$, and that up to isomorphism every arc-transitive graph is constructed in this way. As a graph automorphism, the element g maps the vertex H to one of its neighbors, it follows that $\mathrm{Cos}(G, H, g)$ is connected if and only if $G = \langle H, g \rangle$, refer to [3, p.118, 17B].

In the following, for some prime power q, we will construct a quasiprimitive group G of (non-diagonal) PA type with a point stabilizer H isomorphic to the affine group $\mathrm{AGL}_1(q^2)$, and then produce a connected coset graph $\mathrm{Cos}(G, H, g)$ of valency q^2. If this is so then, noting that H acts 2-transitively on $[H : (H \cap H^g)]$ by right multiplication, $\mathrm{Cos}(G, H, g)$ is $(G, 2)$-arc-transitive by [7, Theorem 2.1]; of course, such a graph satisfies Theorem 1.1 (2).

For the rest of this section, we always assume that

(C1) $q = p^f$ for some prime p and integer $f \geqslant 1$, and $n := q + 1 = 2^s r^t > 3$, where $t \geqslant 0$, r is an odd prime, and either $s \geqslant 2$ or q is even;

(C2) X is an almost simple group with socle T, $|X : T| \leqslant 2$, X has a subgroup $R = F{:}(\langle b \rangle \times \langle c \rangle)$ isomorphic to $\mathrm{AGL}_1(q)$, where $F \cong \mathbb{Z}_p^f$, b has order $\frac{q-1}{(2,q-1)}$, c has order $(2, q-1)$, and if $|X : T| = 2$ then q is odd and $c \notin T$;

(C3) $\tau = (1, 2, \ldots, n) \in \mathrm{S}_n$, and $W = X \wr \langle \tau \rangle$, the wreath product of X by $\langle \tau \rangle$, where
$$(x_1, x_2, \ldots, x_n)^\tau = (x_n, x_1, x_2, \ldots, x_{n-1}) \text{ for } x_i \in X,\ 1 \leqslant i \leqslant n;$$

(C4) $\pi_i : (x_1, x_2, \ldots, x_n) \mapsto x_i$, $1 \leqslant i \leqslant n$, are the projections of X^n onto X.

The next lemma follows easily from (C1) and (C2).

Lemma 6.1. $R \cap T = F{:}\langle b, c^{|X:T|} \rangle$.

Proof. Note that F is the unique minimal normal subgroup of R. Since $T \trianglelefteq X$, we have $F \cap T \trianglelefteq R$, yielding $F \cap T = 1$ or $F \leqslant T$. If $F \cap T = 1$ then $|X|$ is divisible by $|F||T| = q|T|$, yielding $|X : T| \geqslant q \geqslant 3$, a contradiction. Thus $F \leqslant T$. Since $|X : T| \leqslant 2$, we have $b \in T$. Then
$$R \cap T = F{:}(\langle b, c \rangle \cap T) = F\langle b \rangle(\langle c \rangle \cap T) = F{:}\langle b, c^{|X:T|} \rangle,$$

as desired. This completes the proof. $\square$

For $Y \leqslant X$, we always deal with the direct product Y^n of n copies Y as a subgroup of W. Also, $\langle\tau\rangle$ is viewed as a subgroup of W, so that $W = X^n{:}\langle\tau\rangle$. Sometimes, we use boldface type for the elements in X^n. Pick three elements in R^n as follows:

$$\mathbf{b} = (b, b, \dots, b),\ \mathbf{c} = (c, c, \dots, c),\ \mathbf{d}_0 = (bc, b, bc, bc, \dots, bc).$$

Then $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}_0$ have order $\frac{q-1}{(2,q-1)}$, $(2, q-1)$ and $q-1$, respectively. Let

$$\theta = \mathbf{d}_0\tau.$$

Calculation shows that

$$\theta^n = (b^2c, b^2c, b^2c, \dots, b^2c) = \mathbf{b}^2\mathbf{c}.$$

It follows that θ has order $n(q-1) = q^2-1$, and $\langle\theta^n\rangle = \langle\mathbf{b}^2\rangle \times \langle\mathbf{c}\rangle$.

It is easy to check that $\mathbf{C}_{\langle\theta\rangle}(F^n) = 1$, $F^n \cap \langle\theta\rangle = 1$ and F^n is normalized by θ. In addition, for $(x_1, x_2, \dots, x_n) \in F^n$, we have

$$(x_1, x_2, \dots, x_n)^\theta = (x_n^{bc}, x_1^{bc}, x_2^{b}, x_3^{bc}, x_4^{bc}, \dots, x_{n-1}^{bc}).$$

Recall that $R = F{:}(\langle b\rangle \times \langle c\rangle) \cong \mathrm{AGL}_1(q)$. If we deal with F as the field $\mathbb{F}_q$, then b and c may be chosen such that $x^b = \lambda x$ and $x^c = \eta x$ for $x \in F$, where $\lambda, \eta \in \mathbb{F}_q^*$ have order $\frac{q-1}{(2,q-1)}$ and $(2, q-1)$ respectively. Thus, viewing F^n as the n-dimensional vector space $\mathbb{F}_q^n$, the next lemma follows directly from Lemma 5.1, Theorem 5.3 and Remark 5.4.

Lemma 6.2. *$F^n{:}\langle\theta\rangle$ has a minimal normal subgroup E such that*

(1) $E \cong \mathbb{Z}_p^{2f}$;
(2) *$\langle\theta\rangle$ acts transitively on $E\backslash\{1\}$ by conjugation, in particular, $E{:}\langle\theta\rangle \cong \mathrm{AGL}_1(q^2)$;*
(3) *$\pi_i(E) = F$ and $\ker(\pi_i) \cap E \neq \ker(\pi_j) \cap E$, where $1 \leqslant i < j \leqslant n$.*

Using Lemma 6.2, we can easily construct a quasiprimitive permutation group of PA type, which is described as in the following result.

Theorem 6.3. *Let $G = T^n\langle\theta\rangle$, and let E be a minimal normal subgroup of $F^n{:}\langle\theta\rangle$ satisfying (1)-(3) of Lemma 6.2. Let $H = E{:}\langle\theta\rangle$. Then G is a quasiprimitive group on $[G : H]$ of (non-diagonal) PA type, where $T^n \cap H$ is a subdirect product of $(R \cap T)^n$.*

Proof. First, it is easily shown that $\mathbf{C}_{\langle\theta\rangle}(T^n) = 1$, and $\langle\theta\rangle$ normalizes T^n and acts transitively by conjugation on the set of simple direct factors of T^n. This implies that G is a group and has a unique minimal normal subgroup T^n, and hence H is core-free in G. Thus it suffices to show that $\pi_i(T^n \cap H) = R \cap T$ for $1 \leqslant i \leqslant n$.

Calculation shows that $\theta^m \in T^n$ if and only if m is divisible by $n|X : T|$. It follows that $T^n \cap \langle\theta\rangle = \langle\theta^{n|X:T|}\rangle = \langle\mathbf{b}^{2|X:T|}, \mathbf{c}^{|X:T|}\rangle$. Since either $q \equiv -1 \pmod 4$ or q is even, $\mathbf{b}$ has odd order $\frac{q-1}{(2,q-1)}$. Noting that $2|X : T|$ is a divisor of 4, we have $\langle\mathbf{b}^{2|X:T|}\rangle = \langle\mathbf{b}\rangle$. Then $T^n \cap \langle\theta\rangle = \langle\mathbf{b}, \mathbf{c}^{|X:T|}\rangle$. Now

$$T^n \cap H = T^n \cap (E{:}\langle\theta\rangle) = E{:}(T^n \cap \langle\theta\rangle) = E{:}(\langle\mathbf{b}, \mathbf{c}^{|X:T|}\rangle).$$

By Lemmas 6.1 and 6.2, we have

$$R \cap T = F(\langle b, c^{|X:T|}\rangle) = \pi_i(E)\pi_i(\langle\mathbf{b}, \mathbf{c}^{|X:T|}\rangle) = \pi_i(T^n \cap H),$$

as desired. This completes the proof. □

Now we are ready to give a construction for graphs of non-diagonal PA type.

Theorem 6.4. *Let G and H be as in Theorem 6.3. Suppose that $\mathbf{N}_X(\langle b,c\rangle)$ contains an involution o of T such that $X=\langle F,b,c,o\rangle$. Let $\mathbf{o}=(o,o,\ldots,o)$ and $\Gamma(X)=\mathrm{Cos}(G,H,\mathbf{o})$. Then $\Gamma(X)$ is connected, $(G,2)$-arc-transitive and of valency q^2.*

Proof. Noting that $H\cong \mathrm{AGL}_1(q^2)$, if $H\cap H^{\mathbf{o}}$ has order q^2-1 then $\Gamma(X)$ have valency q^2, which then yields the 2-arc-transitivity of G on the graph $\Gamma(X)$. Thus it suffices to confirm that $|H\cap H^{\mathbf{o}}|=q^2-1$ and $G=\langle H,\mathbf{o}\rangle$.

By the choice of o, we know that o centralizes c and normalizes $\langle b\rangle$. Let $\mathbf{c}_0=(c,1,c,c,\ldots,c)$. Then $\mathbf{o}$ centralizes $\mathbf{c}_0$ and normalizes $\langle\mathbf{b}\rangle$. Clearly, $\mathbf{o}$ centralizes τ. Then $\mathbf{o}$ centralizes $\mathbf{c}_0\tau$. Noting that $\langle\theta\rangle=\langle\mathbf{d}_0\tau\rangle=\langle\mathbf{b}\rangle\times\langle\mathbf{c}_0\tau\rangle$, it follows that $\langle\theta\rangle^{\mathbf{o}}=\langle\theta\rangle$, and so $\langle\theta\rangle\leqslant H\cap H^{\mathbf{o}}$. Suppose that $|H\cap H^{\mathbf{o}}|>q^2-1$. Since $\langle\theta\rangle$ is maximal in H, we have $H\cap H^{\mathbf{o}}=H$, which yields that E is normalized by $\mathbf{o}$. Then $\pi_1(E)=F$ is normalized by o. Since $\langle F,b,c,o\rangle=X$, we have $F\trianglelefteq X$, which is impossible. Thus $|H\cap H^{\mathbf{o}}|=q^2-1$, as desired.

By the choice of (X,T,o), we have $T=\langle F,b,c^{|X:T|},o\rangle$. Recalling that $\theta^n=\mathbf{b}^2\mathbf{c}$, since $\mathbf{b}$ has odd order, we have $\langle\theta^n\rangle=\langle\mathbf{b}\rangle\times\langle\mathbf{c}\rangle$. By Lemma 6.2, $\pi_i(E)=F$ for all i. We have $\pi_i(T^n\cap\langle H,\mathbf{o}\rangle)\geqslant\langle F,b,c^{|X:T|},o\rangle=T$, yielding $\pi_i(T^n\cap\langle H,\mathbf{o}\rangle)=T$, where $1\leqslant i\leqslant n$. Let $K_i=\ker(\pi_i)\cap T^n$. Then

$$(T^n\cap\langle H,\mathbf{o}\rangle)/K_i\cong T,\ 1\leqslant i\leqslant n.$$

Again by Lemma 6.2, $\ker(\pi_1)\cap E$, $\ker(\pi_2)\cap E,\ldots,\ker(\pi_n)\cap E$ are distinct. Then $K_1,\ldots,K_n$ are distinct normal subgroups of $T^n\cap\langle H,\mathbf{o}\rangle$. It follows that $T^n\cap\langle H,\mathbf{o}\rangle\cong T^n$, refer to [6, p.113, Lemma 4.3A]. Then $T^n\cap\langle H,\mathbf{o}\rangle=T^n$, and so $\langle H,\mathbf{o}\rangle\geqslant\langle T^n,\theta\rangle=G$. Thus $G=\langle H,\mathbf{o}\rangle$, as desired. This completes the proof. □

The following example collects some almost simple groups, which support Theorem 6.4. Thus there do exist 2-arc-transitive graphs which satisfy (2) of Theorem 1.1 (i) or (iii).

Example 6.5. (1) Let $X=\mathrm{S}_p$ and $T=\mathrm{A}_p$, where $7\leqslant p\equiv -1\,(\mathsf{mod}\ 4)$, and $p+1$ has at most two distinct prime divisors. Note that S_p has a maximal subgroup $F{:}\langle a\rangle$ isomorphic to $\mathrm{AGL}_1(p)$, refer to [17], where $F\cong\mathbb{Z}_p$, and a is a $(p-1)$-cycle. Let $b=a^2$ and $c=a^{\frac{p-1}{2}}$. Then $F\langle b\rangle$ is a maximal subgroup of A_p except for $p\in\{7,11,23\}$, and c is a product of $\frac{p-1}{2}$ disjoint transpositions. It is easy to see that S_p contains an element d, which is a product of $\frac{p-1}{2}$ disjoint transpositions and inverts a by conjugation. Clearly, $cd=dc$. Let $o=cd$. We have $oc=co$, $o\in\mathrm{A}_p$ and $\langle F,b,c,o\rangle=\mathrm{S}_p$. Thus, by Theorem 6.4, we get a connected 2-arc-transitive graph $\Gamma(X)$ of valency p^2.

(2) Let $X=\mathrm{PGL}_2(q)$ and $T=\mathrm{PSL}_2(q)$, where either $q\geqslant 4$ is even or $7\leqslant q\equiv -1\,(\mathsf{mod}\ 4)$, and $q+1$ has at most two distinct prime divisors. Note that all subgroups of X and T are explicitly known, refer to [4] and [13, p.213, II.8.27], respectively. In particular, X has a maximal subgroup $F{:}\langle a\rangle$ isomorphic to $\mathrm{AGL}_1(q)$, where $|F|=q$, and a has order $q-1$. Let $b=a^{(2,q-1)}$ and $c=a^{\frac{q-1}{(2,q-1)}}$. Then $F\langle b\rangle$ is a maximal

subgroup of T. Let $N = \mathbf{N}_X(\langle a\rangle)$. Then N is a dihedral group of order $2(q-1)$, and $N \cap T$ is a dihedral group of order $\frac{2(q-1)}{(2,q-1)}$. Pick an involution o in $N \cap T$. Then $oc = co$ and $X = \langle F, b, c, o\rangle$. By Theorem 6.4, we get a connected 2-arc-transitive graph $\Gamma(X)$ of valency q^2. □

We end this section by an example, which gives some graphs satisfying Theorem 1.1 (2) (ii).

Example 6.6. Let $\mathrm{PSL}_2(8) = T < X = T.3 \cong \mathrm{Ree}(3)$, and let F be a Sylow 2-subgroup of T. By the Atlas [5], we have $\mathbf{N}_T(F) \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7$ and $\mathbf{N}_X(F) \cong \mathbb{Z}_2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$. Pick an element b of order 3 in $\mathbf{N}_X(F)$. Let τ be the 21-cycle $(1, 2, \ldots, 21)$ in S_{21}.

It is easily shown that the wreath product $X \wr \langle\tau\rangle$ has a normal subgroup $G = T^{21}{:}\langle\theta\rangle$, where $\theta = (b, 1, b, \ldots, b)\tau$ has order 63. Let $M = T^{21}$. Then M is the unique minimal normal subgroup of G. Note that F^{21} is a $\langle\theta\rangle$-invariant subgroup of M. Considering the conjugation of $\langle\theta\rangle$ on F^{21}, calculation with GAP [9] shows that

(1) F^{21} has exactly 13 minimal $\langle\theta\rangle$-invariant subgroups: one of them has order 2, one of them has order 2^2, two of them have order 2^3, and the other ones have order 2^6; in fact, F^{21} is the direct product of these 13 subgroups;
(2) among those 9 subgroups of order 2^6 in (1), there are exactly 6 subgroups such that $\langle\theta\rangle$ acts regularly on the nonidentity elements, that is, each of these 6 subgroups together with θ generates a group isomorphic to $\mathrm{AGL}_1(2^6)$.

We fix a minimal $\langle\theta\rangle$-invariant subgroup E of F^{21} with $E\langle\theta\rangle \cong \mathrm{AGL}_1(2^6)$, and let $H = E\langle\theta\rangle$. Then $M \cap H = E \cong \mathbb{Z}_2^6$, and G is a quasiprimitive group of (non-diagonal) PA type on $[G : H]$. Consider the normalizer of $\langle\theta\rangle$ in G. We have $\mathbf{N}_G(\langle\theta\rangle) = \mathbf{N}_M(\langle\theta\rangle)\langle\theta\rangle$. Again confirmed by GAP [9], we conclude that $\mathbf{N}_M(\langle\theta\rangle) \cong \mathrm{S}_3$, $\mathbf{N}_G(\langle\theta\rangle) = \mathbf{N}_M(\langle\theta\rangle) \times \langle\theta\rangle$, and there is a unique 2-element $g \in \mathbf{N}_M(\langle\theta\rangle)$ (up to the double coset HgH) such that $G = \langle H, g\rangle$ and $g^2 \in H$. Thus we have a connected $(G, 2)$-arc-transitive graph $\mathrm{Cos}(G, H, g)$ of valency 2^6 and order $2^{57} \cdot 3^{42} \cdot 7^{21}$, where M acts regularly on the arc set of this graph.

Note, there are 6 choices for the group E, and so we may obtain 6 graphs. However, we do not know whether there are isomorphic ones among these graphs. □

7. A construction of bipartite graphs with diagonal PA type

This section aims to construct some 2-arc-transitive bipartite graphs with diagonal PA type, which admit certain groups G and are not standard double covers of $(G^*, 2)$-arc-transitive graphs.

Lemma 7.1. *Let $\Gamma = (V, E)$ is a connected bipartite graph, $G \leqslant \mathrm{Aut}(\Gamma)$. Let G^* be the bipartition preserving subgroup of G. Assume that G is transitive on V. If Γ is the standard double cover of some graph admitting G^*, then $\{G_\alpha \mid \alpha \in V\}$ is a conjugacy class of subgroups in G^*.*

Proof. Clearly, $G_\alpha \leqslant G^*$ for all $\alpha \in V$. Let U and W be the G^*-orbits on V. Then $\{G_\alpha \mid \alpha \in U\}$ and $\{G_\beta \mid \beta \in W\}$ are conjugacy classes of subgroups in G^*. Assume that Γ is the standard double cover of some graph admitting G^*. Then $\mathrm{Aut}(\Gamma)$ has an involution ι which centralizes G^* and interchanges U and W. Let $\alpha \in U$ and $\beta = \alpha^\iota$. We have $\beta \in W$. Replacing G by $G^* \times \langle \iota \rangle$ if necessary, we have $G_\beta = G_{\alpha^\iota} = G_\alpha^\iota = G_\alpha$. It follows that $\{G_\alpha \mid \alpha \in U\} = \{G_\beta \mid \beta \in W\}$, and the lemma follows. $\square$

From now on, let $p \geqslant 5$ be a prime, and let $\tau = (1, 2, \ldots, p-1) \in \mathrm{S}_{p-1}$. Let $X = \mathrm{PGL}(2, p)$ or S_p with socle T. We will define a subgroup G of the wreath product $W = X \wr \langle \tau \rangle$, and construct connected $(G, 2)$-arc-transitive bipartite graphs.

Note that X has a subgroup R isomorphic to $\mathrm{AGL}_1(p)$, and $T \cap R \cong \mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$. Choose $a, b \in R$ with order p and $p-1$, respectively. Then $R = \langle a \rangle{:}\langle b \rangle$. It is easily shown that b is contained in a dihedral subgroup D of X with order $2(p-1)$, which has the center $\langle b^{\frac{p-1}{2}} \rangle$ and intersects with T at a dihedral group of order $p-1$. Thus both T and $X \setminus T$ contain involutions which invert b and centralize $b^{\frac{p-1}{2}}$. Choose an involution $c \in X$ with $b^c = b^{-1}$ and $cb^{\frac{p-1}{2}} \notin T$. We have $D = \langle b, c \rangle$, and $X = \langle a, b, c \rangle$.

Pick three elements in W as follows:

$$\mathbf{a} = (a, a, \ldots, a),\ \mathbf{b} = (b, b, \ldots, b),\ \mathbf{o} = (c, bc, b^2c, \ldots, b^{p-2}c).$$

Clearly, τ centralizes both $\mathbf{a}$ and $\mathbf{b}$, and all coordinates of $\mathbf{o}$ are distinct. Since b has order $p-1$, we have $b^{p-2} = b^{-1}$. Calculation shows that

$$\mathbf{o}^\tau = \mathbf{b}^{-1}\mathbf{o},\ \mathbf{b}^{\mathbf{o}} = \mathbf{b}^{-1},\ \tau^{\mathbf{o}} = \mathbf{b}^{-1}\tau,\ \langle \mathbf{a}, \mathbf{b}, \tau \rangle = \langle \mathbf{a} \rangle{:}\langle \mathbf{b} \rangle \times \langle \tau \rangle,\ \langle \mathbf{a}, \mathbf{b} \rangle \cap T^{p-1} = \langle \mathbf{a}, \mathbf{b}^2 \rangle.$$

Let

$$G^* = T^{p-1}\langle \mathbf{b}, \tau \rangle.$$

Suppose that $\mathbf{o} \in G^*$. We have $\mathbf{o} = (t_1, t_2, \ldots, t_{p-1})\mathbf{b}^i$ for some i and $t_1, t_2, \ldots, t_{p-1} \in T$. Then $(t_1, t_2, \ldots, t_{p-1}) = \mathbf{o}\mathbf{b}^{-i} = (b^{-i}c, b^{1-i}c, \ldots, b^{p-2-i}c)$. It follows that $b = b^{1-i}cb^{-i}c = t_2t_1 \in T$, a contradiction. Therefore, $\mathbf{o} \notin G^*$.

Let

$$G = G^*{:}\langle \mathbf{o} \rangle,\ H = \langle \mathbf{a}, \mathbf{b}, \tau \rangle.$$

Then $\mathrm{AGL}_1(p) \times \mathbb{Z}_{p-1} \cong H < G^*$, and it is easily shown that T^{p-1} is the unique minimal normal subgroup of G^* and G. Thus we have the following lemma.

Lemma 7.2. *The group G acts faithfully on $[G:H]$ by right multiplication, G^* has two orbits on $[G:H]$, G^* is a quasiprimitive group with diagonal PA type on each of its orbits, and $T^{p-1} \cap H = \langle \mathbf{a}, \mathbf{b}^2 \rangle$ is a diagonal subgroup of $(T \cap R)^{p-1}$.*

Theorem 7.3. *Let G, H and $\mathbf{o}$ be as above, and let $\Gamma = \mathrm{Cos}(G, H, \mathbf{o})$. Then Γ is a connected $(G, 2)$-arc-transitive bipartite graph of valency p, and Γ is not the standard double cover of some graph whose automorphism group contains G^*.*

Proof. Let $K = \langle \mathbf{b}, \tau \rangle$. Then $|H : K| = p$, and $\mathbf{o}$ normalizes K. Thus $H \cap H^{\mathbf{o}} \geqslant K$. Suppose that $H \cap H^{\mathbf{o}} > K$. Then $H = H^{\mathbf{o}}$. Noting that $\langle \mathbf{a} \rangle$ is characteristic in H, it follows that $\mathbf{o}$ normalizes $\langle \mathbf{a} \rangle$, and so c normalizes $\langle a \rangle$. Then $\langle a \rangle \trianglelefteq \langle a, b, c \rangle = X$, a contradiction. Thus $H \cap H^{\mathbf{o}} = K$. It is easily shown that H acts 2-transitively on $[H : K]$ by right multiplication. Then Γ is $(G, 2)$-arc-transitive and of valency p.

We next show that Γ is connected, that is, $G = \langle H, \mathbf{o} \rangle$. Let $G_0 = \langle \mathbf{a}, \mathbf{b}, \mathbf{o} \rangle$. Clearly, G_0 is a subgroup of X^{p-1} and normalized by τ. We have $G_0{:}\langle \tau \rangle = \langle \mathbf{a}, \mathbf{b}, \tau, \mathbf{o} \rangle = \langle H, \mathbf{o} \rangle$. Then it suffices to show $T^{p-1} \leqslant G_0$.

For $x \in X$, denote by $\mathbf{e}_{i,x}$ the element of X^{p-1} with the ith coordinate x and all other coordinates 1. Write $X^{p-1} = X_1 \times X_2 \times \cdots \times X_{p-1}$ and $T^{p-1} = T_1 \times T_2 \times \cdots \times T_{p-1}$, where

$$X_i = \{\mathbf{e}_{i,x} \mid x \in X\},\ T_i = \{\mathbf{e}_{i,t} \mid t \in T\},\ 1 \leqslant i \leqslant p-1.$$

For $1 \leqslant i < j \leqslant p-1$, let π_i be the projection of G_0 to X_i, and define a group homomorphism:

$$\pi_{ij} : G_0 \to X_i \times X_j,\ \mathbf{e}_{1,x_1}\mathbf{e}_{2,x_2} \cdots \mathbf{e}_{p-1,x_{p-1}} \mapsto \mathbf{e}_{i,x_i}\mathbf{e}_{j,x_j}.$$

It is easy to see that

$$\pi_i(\ker(\pi_j)) \times \pi_j(\ker(\pi_i)) \leqslant \pi_{ij}(G_0).$$

In addition,

$$\pi_i(G_0) = \langle \mathbf{e}_{i,a}, \mathbf{e}_{i,b}, \mathbf{e}_{i,b^{i-1}c} \rangle = X_i \cong X,\ 1 \leqslant i \leqslant p-1.$$

Suppose that $\ker(\pi_i) = \ker(\pi_j)$ for some $1 \leqslant i \leqslant j \leqslant p-1$. Define $\theta : X_i \to X_j$, $\pi_i(\mathbf{x}) \mapsto \pi_j(\mathbf{x})$, where $\mathbf{x}$ runs over the elements of G_0. It is easily shown that θ is a bijection and preserves the operations of groups. Then θ is an isomorphism, and

$$\theta : \mathbf{e}_{i,a} \mapsto \mathbf{e}_{j,a},\ \mathbf{e}_{i,b} \mapsto \mathbf{e}_{j,b},\ \mathbf{e}_{i,b^{i-1}c} \mapsto \mathbf{e}_{j,b^{j-1}c}.$$

It follows that X has an automorphism σ with

$$\sigma : a \mapsto a,\ b \mapsto b,\ b^{i-1}c \mapsto b^{j-1}c.$$

Note that every automorphism of X is induced by the conjugation of some element in X. Then there is $x \in X$ such that

$$a^x = a,\ b^x = b,\ (b^{i-1}c)^x = b^{j-1}c.$$

The only possibility is that $x = 1$. Then $b^{i-1}c = b^{j-1}c$, yielding $i = j$. Therefore, $\ker(\pi_i) \neq \ker(\pi_j)$ for $1 \leqslant i < j \leqslant p-1$.

Recalling that G_0 is normalized by τ, it is easily shown that

$$(\ker(\pi_i))^\tau = \ker(\pi_{i^\tau}),\ 1 \leqslant i \leqslant p-1.$$

In particular, we have $\ker(\pi_i) \neq 1$ for all i. Let $1 \leqslant i < j \leqslant p-1$. Since $\ker(\pi_i) \trianglelefteq G_0$, we have $\pi_j(\ker(\pi_i)) \trianglelefteq \pi_j(G_0) = X_j$. Noting that $\mathrm{soc}(X_j) = T_j \cong T$, either $T_j \leq \pi_j(\ker(\pi_i))$, or $\pi_j(\ker(\pi_i)) = 1$. The latter case implies that $\ker(\pi_i) = \ker(\pi_j)$, a contradiction. Thus $T_j \leq \pi_j(\ker(\pi_i))$. Similarly, we have $T_i \leq \pi_i(\ker(\pi_j))$. Then

$$T_i \times T_j \leqslant \pi_i(\ker(\pi_j)) \times \pi_j(\ker(\pi_i)) \leqslant \pi_{ij}(G_0).$$

By [22, p.79, Lemma 4.10], we have $T^{p-1} = T_1 \times T_2 \times \cdots \times T_{p-1} \leqslant G_0$, as desired.

Now Γ is a connected $(G,2)$-arc-transitive graph of valency p. Note that $H \leqslant G^*$, and G^* has two orbits on $[G:H]$, see Lemma 7.2. Then Γ is bipartite. Suppose that H and $H^{\mathbf{o}}$ are conjugate in G^*. Since $G^* = T^{p-1}H$, there is some $\mathbf{t} \in T^{p-1}$ such that $H^{\mathbf{t}} = H^{\mathbf{o}}$. Note that H has center $\langle \tau \rangle$, and $H^{\mathbf{o}}$ has center $\langle \tau^{\mathbf{o}} \rangle$. Recalling that $\tau^{\mathbf{o}} = \mathbf{b}^{-1}\tau$, we have $(\tau^i)^{\mathbf{t}} = \mathbf{b}^{-1}\tau$ for some integer $i > 0$. Noting that τ^{-1} normalizes T^{p-1}, there exists $\mathbf{t}' \in T^{p-1}$ such that $\mathbf{t}^{\tau^{-1}} = \mathbf{t}'$, and so $\tau\mathbf{t} = \mathbf{t}'\tau$. Thus, $(\tau^i)^{\mathbf{t}} = \mathbf{t}^{-1}\tau^i\mathbf{t} = \mathbf{t}^{-1}\tau^{i-1}\mathbf{t}'\tau$. By induction on i, we have $(\tau^i)^{\mathbf{t}} = \mathbf{t}''\tau^i$ for some $\mathbf{t}'' \in T^{p-1}$. Then $\mathbf{b}^{-1}\tau = \mathbf{t}''\tau^i$. It follows that $i \equiv 1 \pmod{p-1}$, and $\mathbf{b}^{-1} = \mathbf{t}'' \in T^{p-1}$, yielding $\mathbf{b} \in T^{p-1}$. Then $\langle \mathbf{a}, \mathbf{b}^2 \rangle = T^{p-1} \cap H \geqslant \langle \mathbf{a}, \mathbf{b} \rangle$, which is impossible as $\mathbf{b}$ has even order $p-1$. Therefore, H and $H^{\mathbf{o}}$ are not conjugate in G^*. By Lemma 7.1, Γ is not a standard double cover. This complete the proof. □

Remark 7.4. Let G, H, $\mathbf{o}$ and Γ be as in Theorem 7.3. Although Γ is not the standard double cover of any graph admitting G^*, we do not know if there exists a graph Σ such that $\Gamma \cong \Sigma^{(2)}$. This brings us to an interesting problem as follows.

Problem 7.5. Constructing or characterizing 2-arc-transitive bipartite graphs of (diagonal) PA type, which is not the standard double cover of any graph.

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Z.P. Lu, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China
Email address: lu@nankai.edu.cn