On the relationship between $(A^f)_\alpha$-spectral radii of graphs with starlike branch tree or bouquet branch graph and its linearly order

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Abstract

For a graph G and a vertex v of G, let $G_v(n_1, n_2, \ldots, n_d)$ be the graph obtained from G by linking the paths on $n_1, n_2, \ldots, n_d$ vertices to the vertex v of G, respectively. We denote by $d_G(v_i)$ (or d_i for short) the degree of the vertex v_i in G. Let $f(x, y) > 0$ be a real symmetric function in x and y. The function-weighted adjacency matrix $A^f(G)$ of a graph G is a square matrix, where the (i, j)-entry is equal to $f(d_i, d_j)$ if the vertices v_i and v_j are adjacent and 0 otherwise, in which d_i is the degree of the vertex v_i. In [Discrete Math. 347 (2024) 113772.], Shan and Liu showed that the A_α-spectral radius of $G_v(n_1, n_2, \ldots, n_d)$ will increase according to the shortlex ordering of $(n_1, n_2, \ldots, n_d)$. However, we find some mistakes in their proof. In this paper, we will correct their proof, and moreover, extend their results from the A_α-spectral radius to the $(A^f)_\alpha$-spectral radius. In addition, let $G^c_v(n_1, n_2, \ldots, n_d)$ be the graph obtained from G by identifying a vertex from each of the cycles on $n_1, n_2, \ldots, n_d$ vertices and the vertex v of G, respectively. We will show that the $(A^f)_\alpha$-spectral radius of $G^c_v(n_1, n_2, \ldots, n_d)$ will decrease according to the majorization ordering of $(n_1, n_2, \ldots, n_d)$.

Keywords: A_α-spectral radius; function-weighted adjacency matrix; shortlex ordering; majorization ordering

1. Introduction

Let $G = (V(G), E(G))$ be a finite, undirected, and simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. We denote by $|G| = |V(G)|$ the order of G. An edge $e \in E(G)$ with end vertices v_i and v_j is usually denoted by v_iv_j. For $i = 1, 2, \ldots, n$, we denote by $d_G(v_i)$ (or d_i for short) the degree of the vertex v_i in G. A vertex of degree 1 is called a *pendent vertex*.

For a weighted graph $G = (V, E, \omega)$ of order n, where $\omega : V \times V \to \mathbb{R}_{\geq 0}$ is the edge weight function such that $\omega(v_i, v_j) = \omega(v_j, v_i) > 0$ if and only if $v_iv_j \in E(G)$, the matrix $A^w(G) = (a_{ij})_{n\times n}$ with $a_{ij} = w(v_i, v_j)$ if $v_iv_j \in E(G)$ and 0 otherwise is called the weighted adjacency matrix of G. Since G is a connected graph and $\omega(v_i, v_j) > 0$, then $A^w(G)$ is an $n \times n$ nonnegative and irreducible matrix. By Perron-Frobenius theorem, its spectral radius is the largest eigenvalue of $A^w(G)$, denote by $\rho_w(G)$. As a special case, $A^w(G)$ is equal to the adjacency matrix $A(G)$ when $w(v_i, v_j) = 1$ for each edge $v_iv_j \in E(G)$.

In molecular graph theory, the topological indices of molecular graphs are used to reflect chemical properties of chemical molecules. There are many topological indices and among them there is a family of degree-based indices. The degree-based index $TI_f(G)$ of G with positive symmetric function $f(x, y)$ is defined as

$$TI_f(G) = \sum_{v_iv_j \in E(G)} f(d_i, d_j).$$

Gutman [7] collected many important and well-studied chemical or topological indices; see them in Table 1. In order to study the discrimination property, Rada [20] introduced the exponentials of the best known degree-based chemical or topological indices; see them in Table 2.

Each index maps a molecular graph into a single number. One of the authors [14] proposed that if we use a matrix to represent the structure of a molecular graph with weights separately on its pairs of adjacent vertices, it will keep more structural information of the graph. For example, the Randić matrix [21, 22], the Atom-Bond-Connectivity matrix [6], the Arithmetic-Geometric matrix [25] and the Sombor matrix [12] were considered separately. Many different function-weighted adjacency matrixes which correspond to the indexes have been studied one by one, but without using unified approaches. In this paper, our main purpose is to use

Function f(x,y)	The corresponding index
$x+y$	first Zagreb index
xy	second Zagreb index
$(x+y)^2$	first hyper-Zagreb index
$(xy)^2$	second hyper-Zagreb index
$x^{-3}+y^{-3}$	modified first Zagreb index
$\|x-y\|$	Albertson index
$(x/y+y/x)/2$	extended index
$(x-y)^2$	sigma index
$1/\sqrt{xy}$	Randić index
$\sqrt{xy}$	reciprocal Randić index
$1/\sqrt{x+y}$	sum-connectivity index
$\sqrt{x+y}$	reciprocal sum-connectivity index
$2/(x+y)$	harmonic index
$\sqrt{(x+y-2)/(xy)}$	atom-bond-connectivity (ABC) index
$(xy/(x+y-2))^3$	augmented Zagreb index
x^2+y^2	forgotten index
$x^{-2}+y^{-2}$	inverse degree
$2\sqrt{xy}/(x+y)$	geometric-arithmetic (GA) index
$(x+y)/(2\sqrt{xy})$	arithmetic-geometric (AG) index
$xy/(x+y)$	inverse sum index
$x+y+xy$	first Gourava index
$(x+y)xy$	second Gourava index
$(x+y+xy)^2$	first hyper-Gourava index
$((x+y)xy)^2$	second hyper-Gourava index
$1/\sqrt{x+y+xy}$	sum-connectivity Gourava index
$\sqrt{(x+y)xy}$	product-connectivity Gourava index
$\sqrt{x^2+y^2}$	Sombor index

Table 1: Some well-studied chemical or topological indices

Function f(x,y)	The corresponding index
e^{x+y}	exponential first Zagreb index
e^{xy}	exponential second Zagreb index
$e^{1/\sqrt{xy}}$	exponential Randić index
$e^{\sqrt{(x+y-2)/(xy)}}$	exponential ABC index
$e^{2\sqrt{xy}/(x+y)}$	exponential GA index
$e^{2/(x+y)}$	exponential harmonic index
$e^{1/\sqrt{x+y}}$	exponential sum-connectivity index
$e^{(xy/(x+y-2))^3}$	exponential augmented Zagreb index

Table 2: Some well-known exponential chemical or topological indices

unified approaches to consider the spectral properties of these matrices. Based on these examples, the function-weighted adjacency matrix which is denoted by $A^f(G)$ in this paper first appeared in Das et al. [5], and it is defined as

$$A^f(G)(i,j) = \begin{cases} f(d_i, d_j), & v_i v_j \in E(G); \\ 0, & otherwise. \end{cases}$$

For $i = 1, 2, \ldots, n$, we denote by $d_G^f(v_i)$ the sum of the weight of all the edges incident with the vertex v_i in G. Let $D^f(G) = (d_{ij})_{n\times n}$ be the function-weighted diagonal matrix with $d_{ii} = d_G^f(v_i)$. If $f(x,y) \equiv 1$, then $D^f(G)$ is exactly the degree diagonal matrix of G, which is denoted by $D(G)$. $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G. As usual, denote by $\rho(G)$ and $\mu(G)$ the spectral radius, signless Laplacian spectral radius of G, respectively. To unify the study of $A(G)$ and $Q(G)$, Nikiforov [16] put forward the concept of the A_α-matrix of a graph G, denote by $A_\alpha(G)$, where

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

and $0 \le \alpha < 1$. The spectral radius of $A_\alpha(G)$ is denoted by $\rho_\alpha(G)$ and called A_α-spectral radius of G. From the definition, we have $\rho(G) =$

$\rho_0(G)$ and $2\rho_{0.5}(G) = \mu(G)$.

Here we should point out that in literature A_f is usually used for the function-weighted adjacency matrix A^f. However, we also have the matrix A_α. In order to avoid confusion, we adopt A^f for A_f.

In this paper, we consider the A_α-spectral radius of the function-weighted adjacency matrix $A^f(G)$. We define it as

$$(A^f)_\alpha(G) = \alpha D^f(G) + (1-\alpha)A^f(G)$$

where $0 \leq \alpha < 1$. Throughout this paper, the spectral radius of $(A^f)_\alpha(G)$ is denoted by $\rho^f_\alpha(G)$ and called $(A^f)_\alpha$-spectral radius of the graph G. For $\alpha = 0$, $(A^f)_\alpha(G)$ is the function-weighted adjacency matrix $A^f(G)$, the spectral radius of $A^f(G)$ is denoted by $\rho^f(G)$ and called the function-weighted adjacency spectral radius of graph G. By Perron-Frobenius theorem, there exist positive real vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\intercal$ such that $A^f_\alpha(G)\mathbf{x} = \rho^f_\alpha(G)\mathbf{x}$. Throughout this paper, we choose $\mathbf{x}$ such that $\|\mathbf{x}\|_2 = 1$ and x_i corresponds to vertex v_i, and we call the unique unit positive vector $\mathbf{x}$ the principal eigenvector of G.

Let $\mathcal{P}(k)$ denote the set of all vectors whose elements are nondecreasing sequences of positive integers with sums all equal to k. Let $\mathcal{P}(k,p)$ be the subset of $\mathcal{P}(k)$ such that each vector of which contains exactly p elements, where $p \leq k$. Throughout this paper, we will use $\mathbf{a} = [a_1, a_2, \ldots, a_p]$ to indicate a vector belonging to $\mathcal{P}(k,p)$. The shortlex ordering of vectors in $\mathcal{P}(k)$ is defined as follows:

Definition 1.1. *For two vectors $\mathbf{a} = [a_1, a_2, \ldots, a_p]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_q]$ in $\mathcal{P}(k)$, we write $\mathbf{a} \prec_{lex} \mathbf{b}$ if either $p < q$ or, when $p = q, a_i < b_i$ holds for the smallest index i at which the two vectors differ, and we say that $\mathbf{a} \prec_{lex} \mathbf{b}$ satisfies the shortlex ordering.*

Let $X = (x_1, \ldots, x_k)$ and $Y = (y_1, \ldots, y_k)$, where $x_1 \geq \cdots \geq x_k$ and $y_1 \geq \cdots \geq y_k$ are real. We say X majorizes Y and let $X \succeq_M Y$, if for every $j, 1 \leq j \leq k, \sum_{i=1}^{j} x_i \geq \sum_{i=1}^{j} y_i$, with equality if $j = k$. The name majorization appeared first in 1959 by Hardy, Littlewood and Polya. The theory of majorization is very useful in so many diverse fields. For more details on this concept we refer to Arnold [1]. In this paper, we consider the majorization ordering of vectors in $\mathcal{P}(k)$, in which the elements are with nondecreasing sequences, then we have the following definition:

Definition 1.2. *For two vectors* $\mathbf{a} = [a_1, a_2, \ldots, a_p]$ *and* $\mathbf{b} = [b_1, b_2, \ldots, b_p]$ *in* $\mathcal{P}(k, p)$*, we write* $\mathbf{a} \succeq_M \mathbf{b}$ *if for every* $j, 1 \le j \le p, \sum_{i=1}^{j} a_i \le \sum_{i=1}^{j} b_i$*, withe equality if* $j = p$*, and we say that* $\mathbf{a}$ *majorizes* $\mathbf{b}$*.*

Clearly, for any two vectors $\mathbf{a} \neq \mathbf{b}$ in $\mathcal{P}(k, p)$, if $\mathbf{a} \succeq_M \mathbf{b}$, then for every $j, 1 \le j \le p, \sum_{i=1}^{j} a_i \le \sum_{i=1}^{j} b_i$. Thus $a_i < b_i$ holds for the smallest index i at which the two vectors differ. We have $\mathbf{a} \prec_{\text{lex}} \mathbf{b}$. These two orderings are contrary relationship.

Let $\mathbf{a}+\mathbf{b}$ denote the concatenation of two vectors $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{a}*s$ denote the concatenation of s copies of vector $\mathbf{a}$. For example, if $\mathbf{a} = [1, 2, 4]$ and $\mathbf{b} = [3, 2]$, then $\mathbf{a} + \mathbf{b} = [1, 2, 4, 3, 2]$ and $\mathbf{a}*3 = [1, 2, 4, 1, 2, 4, 1, 2, 4]$. Here, we agree that $\mathbf{a} = [a_1, a_2, \ldots, a_p] = [a_1, a_2, \ldots, a_p, 0, \ldots, 0]$, and $\mathbf{a} + [0] = \mathbf{a}$.

Let G be a connected graph and v be a vertex of G ($V(G) = \{v\}$ is allowed). Let $G_v(\mathbf{a})$ with $\mathbf{a} = [n_1, n_2, \ldots, n_d]$ be the graph obtained from G by linking the paths on $n_1, n_2, \ldots, n_d$ vertices to the vertex v of G, respectively. Then $|G_v(\mathbf{a})| = |G| + \sum_{i=1}^{d} n_i$ and $G_v(\mathbf{a})$ is called the graph with starlike branch tree. In what follows, if $G_v(\mathbf{a})$ is a starlike tree, then we always simplify rewrite $G_v(\mathbf{a})$ as $S_v(\mathbf{a})$.

The graph operation that decreases or increases its spectral radius plays an important role in the study of spectral graph theory. For $1 \le a \le b$, let $\mathbf{a} = [a, b], \mathbf{b} = [a - 1, b + 1]$. If $a \ge 2$, then $\mathbf{a}$ and $\mathbf{b}$ belong to $\mathcal{P}(k, 2)$ and $\mathbf{b} \prec_{\text{lex}} \mathbf{a}$, Li and Feng firstly showed that:

Theorem 1.3. *[13] Let* G *be a connected graph with* $v \in V(G)$*.Then*

$$\rho\left(G_v(\mathbf{b})\right) < \rho\left(G_v(\mathbf{a})\right).$$

The theorem is also well-known as the Li-Feng Grafting Theorem. Later, Cvetković and Simić [4] showed that the Li-Feng Grafting Theorem also holds for the signless Laplacian spectral radius, that is,

Theorem 1.4. *[4] Let* G *be a connected graph with* $v \in V(G)$*. Then*

$$\mu\left(G_v(\mathbf{b})\right) < \mu\left(G_v(\mathbf{a})\right).$$

By employing the A_α-spectral radius, Nikiforov and Rojo [17] considered the unified result of Theorems 1.3 and 1.4. They conjectured that

Conjecture 1.5. *[17] Let G be a connected graph with $v \in V(G)$ and $0 \le \alpha < 1$. Then*

$$\rho_\alpha\left(G_v(\mathbf{b})\right) < \rho_\alpha\left(G_v(\mathbf{a})\right).$$

In the same paper [17], Nikiforov and Rojo mentioned that they can show Conjecture 1.5 for $\rho_\alpha\left(G_v(\mathbf{a})\right) \ge \frac{9}{4}$. In the sequel, Lin, Huang and Xue [11] completely proved Conjecture 1.5 (independently, Guo and Zhou in [8]). Recently, in [9], we considered the property of the function-weighted adjacency spectral radius and obtained

Theorem 1.6. *[9] Assume that $f(x, y) > 0$ is increasing in variable x. If $5 \le a + 2 \le b$, then*

$$\rho^f(G_v(\mathbf{b})) < \rho^f(G_v(\mathbf{a})).$$

Oboudi [19] considered the problem for the adjacency spectral radius of starlike trees, which are obtained by appending at least three paths or varying number of paths and the order of this paths satisfies the majorization ordering. They obtained the following theorem with a contrary relationship.

Theorem 1.7. *[19] Suppose that $\mathbf{a}, \mathbf{b} \in \mathcal{P}(k, p)$ and $S_v(\mathbf{a}) \ncong S_v(\mathbf{b})$. Assume that $\mathbf{a} \preceq_M \mathbf{b}$. Then $\rho(S_v(\mathbf{a})) \ge \rho(S_v(\mathbf{b}))$.*

In [18], Oliveira, Stevanović and Trevisan considered the problem for the starlike trees such that the order of the paths satisfies the shortlex ordering and they obtained the following theorem.

Theorem 1.8. *[18] Suppose that $\mathbf{a}, \mathbf{b} \in \mathcal{P}(k)$ and $S_v(\mathbf{a}) \ncong S_v(\mathbf{b})$. Then, $\rho(S_v(\mathbf{a})) < \rho(S_v(\mathbf{b}))$ if and only if $\mathbf{a} \prec_{lex} \mathbf{b}$.*

Recently, Li and Guo [15] considered the problem for the adjacency spectral radius of the general graph $G_v(\mathbf{a})$ and got

Theorem 1.9. *[15] Let G be a connected graph with vertex v. Then*

$$\rho\left(G_v(\mathbf{a})\right) < \rho\left(G_v(\mathbf{b})\right)$$

if and only if $\mathbf{a} \prec_{lex} \mathbf{b}$.

In [23], Shan and Liu extended Theorem 1.9 from the spectral radius to A_α-spectral radius for $\alpha \in [0, 1)$ and proved

Theorem 1.10. *[23] Let G be a connected graph with vertex v. If $\alpha \in [0,1)$ and $\mathbf{a} \prec_{lex} \mathbf{b}$, then*

$$\rho_\alpha\left(G_v(\mathbf{a})\right) \leq \rho_\alpha\left(G_v(\mathbf{b})\right)$$

where the equality holds if and only if $G_v(\mathbf{a}) \cong G_v(\mathbf{b})$ is a path graph.

The proof of Theorem 1.10 depends on the following theorem.

Theorem 1.11. *[23] Let $\mathbf{e} = [a] * s + [b]$ and $\mathbf{f} = [c] * s + [d]$, where a, b, c, d and s are five integers such that $a + b = c + d, a > \max\{c, d\} \geq \min\{c, d\} > b \geq 0$ and $s > 0$. If v is a vertex of the connected graph G and $\alpha \in [0, 1)$, then*

$$\rho_\alpha\left(G_v(\mathbf{e})\right) \leq \rho_\alpha\left(G_v(\mathbf{f})\right)$$

where the equality holds if and only if $\alpha = 0$ and G is a trivial graph.

From this result, we obtain that if $\alpha = 0$ and G is a trivial graph, then $\rho_\alpha\left(G_v(\mathbf{e})\right) = \rho_\alpha\left(G_v(\mathbf{f})\right)$. However, if we take

$$\mathbf{e} = [5] * 2 + [2] \text{ and } \mathbf{f} = [4] * 2 + [3],$$

by calculation, we get

$$\rho_0\left(S_v(\mathbf{e})\right) \approx 2.084 < \rho_0\left(S_v(\mathbf{f})\right) \approx 2.0928,$$

that is to say the result dose not hold. In Section 3, we will correct it and get the following theorem.

Theorem 1.12. *Let $\mathbf{e} = [a]*s+[b]$ and $\mathbf{f} = [c]*s+[d]$, where a, b, c, d and s are five integers such that $a + b = c + d, a > \max\{c, d\} \geq \min\{c, d\} > b \geq 0$ and $s > 0$. If v is a vertex of the connected graph G and $\alpha \in [0, 1)$, then*

$$\rho_\alpha\left(G_v(\mathbf{e})\right) \leq \rho_\alpha\left(G_v(\mathbf{f})\right)$$

where the equality holds if and only if G is a trivial graph, $\alpha = 0$ and $s = 1$ or $s > 1$ with $c = b + 1$.

In addition, we will extend Theorem 1.10 for the $(A^f)_\alpha$-spectral radius and obtain the following result.

Theorem 1.13. *Let G be a connected graph with vertex v and $\alpha \in [0, 1)$. Assume that the weight function $f(x, y) > 0$ satisfies that $f(1, x) \leq$*

$f(1,2) = f(2,x)$ for any x. For two vectors $\mathbf{a} = [a_1, a_2, \ldots, a_p]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_p] \in \mathcal{P}(k,p)$ with $\mathbf{a} \neq \mathbf{b}$, if $\mathbf{a} \prec_{lex} \mathbf{b}$, then

$$\rho_\alpha^f\left(G_v(\mathbf{a})\right) \leq \rho_\alpha^f\left(G_v(\mathbf{b})\right).$$

where the equality holds if and only if $G_v(\mathbf{a}) \cong G_v(\mathbf{b})$ is a path graph.

Let $G_v^c(\mathbf{a})$ with $\mathbf{a} = [n_1, n_2, \ldots, n_d]$, $n_i \geq 3$ be the graph obtained from G by identifying the vertex v of G and a vertex from each of the cycles on $n_1, n_2, \ldots, n_d$ vertices, respectively. From the definition, if $V(G) = \{v\}$ (that is, G is a trivial graph with vertex v), then $G_v^c(\mathbf{a})$ is isomorphic to a bouquet graph. Xue et al. [24] considered the cycle version of Li-Feng transformation of the spectral radius and they obtained the following result.

Theorem 1.14. *[24] Let $\mathbf{a} = [a, b]$ and $\mathbf{b} = [a-1, b+1]$. If $4 \leq a \leq b$, then $\rho(G_v^c(\mathbf{a})) < \rho(G_v^c(\mathbf{b}))$.*

In [9], the authors considered the property of the function-weighted adjacency spectral radius and obtained

Theorem 1.15. *[9] Let $\mathbf{a} = [a, b]$ and $\mathbf{b} = [a-1, b+1]$. If $4 \leq a \leq b$ and $2f(2,2) < \rho^f(G_v^c(\mathbf{a}))$, then $\rho^f(G_v^c(\mathbf{a})) < \rho^f(G_v^c(\mathbf{b}))$.*

For the vectors belonging to $\mathcal{P}(k,p)$, in section 4 we will consider the relationship between the majorization ordering of vectors and the $(A^f)_\alpha$-spectral radii of graphs $G_v^c(\mathbf{a})$, and we obtain the following result.

Theorem 1.16. *Let G be a connected graph and $v \in V(G)$. Assume that the weight function $f(x,y) > 0$ satisfies that $f(2,2) \leq f(2,x)$ for any $x \geq 2$. For two vectors $\mathbf{a} = [a_1, a_2, \ldots, a_p]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_p]$ in $\mathcal{P}(k,p)$ with $3 \leq a_1$ and $3 \leq b_1$, if $\alpha \in [0,1)$ and $\mathbf{a} \preceq_M \mathbf{b}$, then*

$$\rho_\alpha^f\left(G_v^c(\mathbf{a})\right) \leq \rho_\alpha^f\left(G_v^c(\mathbf{b})\right)$$

where the equality holds if and only if $\mathbf{a} = \mathbf{b}$.

The rest of this paper is organized as follows. We will introduce some preliminary results in Section 2. Based on this, in Section 3 we will correct Theorem 1.11 and give it a more detailed proof, and then extend the result from the A_α-spectral radius to $(A^f)_\alpha$-spectral radius, as well as give the proof of Theorem 1.13. In section 4, we firstly consider the cycle version of the Li-Feng transformation of $(A^f)_\alpha$-spectral radius and prove Theorem 1.16.

2. Some preliminary results

In this section, we provide some knowledge on matrix theory for nonnegative matrices, the characteristic polynomial of weighted graphs and the majorization ordering of two vectors, which will be used in the sequel.

Theorem 2.1. *[10] Let A and B be $n \times n$ nonnegative symmetric matrices. Then $\rho(A+B) \geq \rho(A)$. Furthermore, if A is irreducible and B is not null, then $\rho(A+B) > \rho(A)$.*

Theorem 2.2. *[2] Let A be an $n \times n$ real symmetric matrix and B be a principal submatrix of A. Then $\rho(A) \geq \rho(B)$.*

Let G_1 and G_2 be two disjoint weighted graphs with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. The coalescence of G_1 and G_2, denoted by $G_1(v_1) \cdot G_2(v_2)$ (or $G_1 \cdot G_2$ for short), is obtained from G_1 and G_2 by identifying v_1 and v_2 to form a new vertex u and the edge weight function ω of $G_1(v_1) \cdot G_2(v_2)$ will be defined as

$$\omega(e) = \begin{cases} \omega_{G_i}(e), & \text{if } e \in E(G_i) \setminus \{(v_i, v_i)\} \text{ for } i = 1, 2 \\ \omega_{G_1}(v_1, v_1) + \omega_{G_2}(v_2, v_2) & \text{if } e = (u, u) \\ 0, & \text{otherwise} \end{cases}$$

Assume that the weighted characteristic polynomial of a weighted graph G is defined by $\phi(G) = \det(xI - A^w(G))$. Recently, Shan et al. [23] proved

Theorem 2.3. *Let G, H be two nontrivial connected weighted graphs with $u \in V(G)$ and $v_1, v_2 \in V(H)$ Take $G_i = G(u) \cdot H(v_i)$ for $i = 1,$ 2. If $\phi(H - v_2) > \phi(H - v_1)$ for $x \geq \rho_w(H - v_1)$, then $\rho_w(G_2) > \rho_w(G_1)$.*

For $j, k \in \{1, \ldots, n\}$ with $j \geq k$, let $e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{0, \ldots, 0}_{n-j})$ and

$$e_{j,k} = e_j - e_k = (\underbrace{0, \ldots, 0}_{k-1}, -1, \underbrace{0, \ldots, 0}_{j-1-k}, 1, \underbrace{0, \ldots, 0}_{n-j}).$$

For any

$$\mathbf{a} = [a_1, a_2, \ldots, a_k, \ldots, a_j, \ldots, a_q] \in \mathcal{P}(k, q) \text{ with } q \geq j \geq k$$

denote

$$\mathbf{a} + e_{j,k} = [a_1, a_2, \ldots, a_k - 1, \ldots, a_j + 1, \ldots, a_q] \in \mathcal{P}(k, q).$$

In [19], Oliveira considered the majorization ordering of two vectors and obtained the following theorem.

Theorem 2.4. *Let* $\mathbf{a}, \mathbf{b} \in \mathcal{P}(k, q)$ *and* $\mathbf{a} \neq \mathbf{b}$. *Assume that* $\mathbf{a} \preceq_M \mathbf{b}$. *Then there exists a sequence* $\mathbf{a} = \mathbf{a}_0 \preceq_M \mathbf{a}_1 \preceq_M \cdots \preceq_M \mathbf{a}_t = \mathbf{b}$ *such that for every* $i \in \{1, \ldots, t\}, \mathbf{a}_i = \mathbf{a}_{i-1} + e_{j_i,k_i}$ *for some* $j_i > k_i$.

3. Proof of Theorem 1.13

Firstly, we will give some lemmas which can be proven by using the same ideas as Proposition 10, Lemma 11 and the analysis in Section 3 in [23] and so we omit their proofs here.

Suppose that $n \geq 2$ and u is an end vertex of the path P_{n+1} on $n+1$ vertices. Let B_n be the principal submatrix of $A_\alpha^f(P_{n+1})$ obtained by deleting the row and column corresponding to the vertex u. Let $h_n(x)$ (for short h_n) be the characteristic polynomial of B_n and θ_n be the maximum eigenvalue of B_n. We have the following result.

Lemma 3.1. *Let* p, q, l *be three positive integers with* $q = p + l$ *and the weight function* $f(x, y)$ *satisfy* $f(1, 2) = f(2, 2)$. *Assume that* $h_0 = 1, h_1 = x - f(2, 2)\alpha$. *Then for* $\alpha \in [0, 1)$, *when* $x \geq \theta_l$ *it holds that*

$$h_{q-1}h_p > h_q h_{p-1}.$$

Let $\mathbf{a} = [a] * s + [b]$. For the graph $G_v(\mathbf{a})$, suppose that $D_1^f(G)$ is the diagonal matrix obtained from $D^f(G)$ by replacing the diagonal entry corresponding to vertex v with $d_G^f(v) + (s-1)f(2, 2)$. Take $Q'(G) = \alpha D_1^f(G) + (1-\alpha)A^f(G)$ and denote the weighted graph associated with $Q'(G)$ by $\widetilde{G}$. Notice that $\widetilde{G}$ is trivial if and only if $\alpha = 0$ and G is trivial.

Let $P_s(a, b)$ be the weighted path on the vertex set $\{u_0, u_1, \ldots, u_{a+b}\}$ whose edge weights are $f(2, 2)$ except for $\omega(u_{a-1}, u_a) = \sqrt{s}f(2, 2)$. Take $Q'(P_s(a, b)) = \alpha D^f(P_s(a, b)) + (1-\alpha)A^f(P_s(a, b))$ and denote the weighted graph associated with $Q'(P_s(a,b))$ by $\widetilde{P}_s(a,b)$. Let $G_s(v;a,b)$ be the weighted graph obtained from $\widetilde{G}$ and $\widetilde{P}_s(a,b)$ by coalescing vertices v and u_a.

If the weight function $f(x,y) > 0$ satisfies that $f(1,x) \le f(1,2) = f(2,x)$ for any x, then we get

Lemma 3.2. *For the graph $G_v(\mathbf{a})$ with $\mathbf{a} = [a] * s + [b]$, we have*

$$\rho_\alpha^f\left(G_v(\mathbf{a})\right) \le \rho_\alpha^f\left(G_s(v;a,b)\right)$$

where the equality holds if and only if a and b are greater than or equal to 2, or $f(1,x) = f(1,2)$ for any x, or $G_v(\mathbf{a})$ is a path graph.

In the proof of Theorem 1.11 in [23], if $\alpha = 0$ and G is a trivial graph but $s \ne 1$, the weighted graphs $G_s(v;a,b)$ and $G_s(v;c,d)$ are not both isomorphic to $P_s(a,b)$. Moreover, by the proof we can only get

$$\rho\left(G_s(v;a,b)\right) < \rho\left(G_s(v;b+1,a-1)\right)$$

rather than

$$\rho\left(G_s(v;a,b)\right) < \rho\left(G_s(v;a-1,b+1)\right).$$

In the following, we will give a detailed proof of Theorem 1.12 and extend the result from the A_α-spectral radius to the $(A^f)_\alpha$-spectral radius. As well, we will extend Theorem 1.10 from the A_α-spectral radius to the $(A^f)_\alpha$-spectral radius for the vectors belonging to $\mathcal{P}(k,p)$.

3.1. The proof of Theorem 1.12

From Lemma 3.2, we have

$$\rho_\alpha\left(G_v(\mathbf{e})\right) = \rho\left(G_s(v;a,b)\right) \quad \text{and} \quad \rho_\alpha\left(G_v(\mathbf{f})\right) = \rho\left(G_s(v;c,d)\right).$$

If G is a trivial graph, $\alpha = 0$ and $s = 1$. The weighted graphs $G_s(v;a,b)$ and $G_s(v;c,d)$ are both isomorphic to $P_s(a,b)$. It follows that $\rho_\alpha\left(G_v(\mathbf{e})\right) = \rho_\alpha\left(G_v(\mathbf{f})\right)$.

Next, we will distinguish the cases that G is a trivial graph, $\alpha = 0$ with $s > 1$, or G is a nontrivial graph, or $\alpha > 0$.

Case 1. G is a trivial graph, $\alpha = 0$ with $s > 1$.

Subcase 1.1 $a = b + 2$.

Since $a = b + 2$, we have $c = d = a - 1 = b + 1$. By

$$G_s(v; a, b) \cong G_s(v; b + 1, a - 1) \cong G_s(v; c, d),$$

we have

$$\rho_\alpha\left(G_s(v; a, b)\right) = \rho_\alpha\left(G_s(v; c, d)\right)$$

i.e., $\rho_\alpha\left(G_v(\mathbf{e})\right) = \rho_\alpha\left(G_v(\mathbf{f})\right)$.

Subcase 1.2 $a > b + 2$ and $\max\{c, d\} = a - 1$.

If $c = b + 1 < a - 1 = d$, then

$$G_s(v; a, b) \cong G_s(v; b + 1, a - 1) \cong G_s(v; c, d).$$

We have $\rho_\alpha\left(G_v(\mathbf{e})\right) = \rho_\alpha\left(G_v(\mathbf{f})\right)$.

If $c = a - 1 > b + 1 = d$, then

$$\begin{aligned}\rho_\alpha\left(G_s(v; a, b)\right) &= \rho_\alpha\left(G_s(v; b + 1, a - 1)\right)\\ &< \rho_\alpha\left(G_s(v; a - 1, b + 1)\right)\\ &= \rho_\alpha\left(G_s(v; c, d)\right).\end{aligned}$$

Thus we obtain $\rho_\alpha\left(G_v(\mathbf{e})\right) < \rho_\alpha\left(G_v(\mathbf{f})\right)$.

Subcase 1.3 $a > b + 2$ and $\max\{c, d\} < a - 1$.

Similarly, if $a - 1 > c \geq d > b + 1$, then, by

$$\begin{aligned}\rho_\alpha\left(G_s(v; a, b)\right) &= \rho_\alpha\left(G_s(v; b + 1, a - 1)\right)\\ &< \rho_\alpha\left(G_s(v; a - 1, b + 1)\right),\end{aligned}$$

we have

$$\rho_\alpha\left(G_s(v; a - 1, b + 1)\right) \leq \rho_\alpha\left(G_s(v; c, d)\right).$$

Thus, $\rho_\alpha\left(G_v(\mathbf{e})\right) < \rho_\alpha\left(G_v(\mathbf{f})\right)$.

Analogously, if $a - 1 > d > c > b + 1$, we have

$$\begin{aligned}\rho_\alpha\left(G_s(v; c, d)\right) &> \rho_\alpha\left(G_s(v; d + 1, c - 1)\right)\\ &\geq \rho_\alpha\left(G_s(v; a - 1, b + 1)\right)\\ &> \rho_\alpha\left(G_s(v; a, b)\right).\end{aligned}$$

Hence, $\rho_\alpha\left(G_v(\mathbf{e})\right) < \rho_\alpha\left(G_v(\mathbf{f})\right)$.

Case 2. G is a nontrivial graph or $\alpha > 0$.

Let $H = \widetilde{P}_s(a, b)$ and rewrite vertices u_a, u_{a-1} as v_1, v_2, respectively. Then

$$G_s(v; a, b) = \widetilde{G}(v) \cdot H(v_1) \text{ and } G_s(v; b+1, a-1) = \tilde{G}(v) \cdot H(v_2).$$

Since

$$\phi(H - v_1) = h_a h_b \text{ and } \phi(H - v_2) = h_{a-1} h_{b+1},$$

when $x \geq \rho_\alpha(G_s(v; a, b)) > \theta_a$, by Lemma 3.1 we have

$$h_a h_b < h_{a-1} h_{b+1}.$$

Hence,

$$\phi(H - v_1) < \phi(H - v_2).$$

According to Theorem 2.3, it follows that

$$\rho_\alpha(G_s(v; a, b)) < \rho_\alpha(G_s(v; b+1, a-1)). \tag{1}$$

by equation Because $a > b$, by theorems 2.1 and 2.2, we get

$$\rho_\alpha(G_s(v; a, b)) < \rho_\alpha(G_s(v; a-1, b+1)).$$

If $c \geq d$, it is easy to see that

$$\rho_\alpha(G_v(\mathbf{e})) < \rho_\alpha(G_v(\mathbf{f})).$$

Otherwise, $c < d$. Then $c - 1 < d + 1$. By equation (1), we have

$$\rho_\alpha(G_s(v; d+1, c-1)) < \rho_\alpha(G_s(v; c, d)).$$

Because $a \geq d + 1 \geq c - 1 \geq b$, we obtain

$$\rho_\alpha(G_s(v; a, b)) \leq \rho_\alpha(G_s(v; d+1, c-1)).$$

Thus, we have $\rho_\alpha(G_v(\mathbf{e})) < \rho_\alpha(G_v(\mathbf{f}))$.

Next we extend the result from the A_α-spectral radius to the $(A^f)_\alpha$-spectral radius with $b \geq 1$.

Lemma 3.3. *Let* $\mathbf{e} = [a] * s + [b]$ *and* $\mathbf{f} = [c] * s + [d]$*, where* a, b, c, d *and* s *are five integers such that* $a + b = c + d, a > \max\{c, d\} \geq \min\{c, d\} > b \geq$

1 and $s > 0$. Let the weight function $f(x, y) > 0$ satisfy that $f(1, x) \leq f(1, 2) = f(2, x)$ for any x. If v is a vertex of the connected graph G and $\alpha \in [0, 1)$, then

$$\rho_\alpha^f (G_v(\mathbf{e})) \leq \rho_\alpha^f (G_v(\mathbf{f}))$$

where the equality holds if and only if one of the following conditions hold:

(i) G is a trivial graph, $\alpha = 0$ and $s = 1$;

(ii) G is a trivial graph, $\alpha = 0$, $s > 1$, $c = b + 1$ and $b \geq 2$ or $f(1, d_G(v) + s + 1) = f(2, d_G(v) + s + 1)$.

Proof. If G is a trivial graph, $\alpha = 0$ and $s = 1$, then $G_v(\mathbf{e}) \cong G_v(\mathbf{f})$ is a path graph. We have $\rho_\alpha^f (G_v(\mathbf{e})) = \rho_\alpha^f (G_v(\mathbf{f}))$.

If $b \geq 2$ or $f(1, d_G(v)+s+1) = f(2, d_G(v)+s+1)$, the proof is similar to that of Theorem 1.12.

Next, we consider the case that $b = 1$ and $f(1, d_G(v) + s + 1) < f(2, d_G(v) + s + 1) = f(1, 2)$. By Theorems 2.1 and 2.2, we have

$$\rho_\alpha^f (G_v(\mathbf{e})) < \rho_\alpha^f (G_s(v; a, b)).$$

From

$$\rho_\alpha^f (G_s(v; a, b)) \leq \rho_\alpha^f (G_v(\mathbf{f})),$$

the proof follows immediately. □

3.2. The proof of Theorem 1.13

For two vectors $\mathbf{a} = [a_1, a_2, \ldots, a_p]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_p] \in \mathcal{P}(k, p)$ with $\mathbf{a} \neq \mathbf{b}$, since $(\mathcal{P}(k), \prec_{\text{lex}})$ is linearly ordered set, by the property of shortlex ordering, there exists some $1 \leq i \leq p-1$ such that $a_j = b_j$ for any $j < i$, $b_i = b_{i+1} = \cdots = b_{p-1} = a_i + 1$. And, $b_p = a_p + \sum_{j=i}^{p-1} (a_j - a_i - 1)$ such that $b_p + 1 \geq a_p$.

Take $\mathbf{c} = [a_1, a_2, \ldots, a_{i-1}, a_i] + [b_p + 1] * (p - i)$.

If $a_{i+1} = a_{i+2} = \cdots = a_p = b_p + 1$, then

$$\rho_\alpha^f (G_v(\mathbf{a})) = \rho_\alpha^f (G_v(\mathbf{c})).$$

Assume that $p - i \geq 2$. Then $b_p = a_i$, which contradicts the fact that the elements of $\mathbf{b}$ are nondecreasing. Thus $p - i + 1 = 2$. We have that $\mathbf{a} = [a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}]$ and $\mathbf{b} = [a_1, a_2, \ldots, a_{i-1}, a_i + 1, a_{i+1} - 1]$.

By Lemma 3.3, if $G_v(\mathbf{a}) \cong G_v(\mathbf{b})$ is a path graph, then we have

$$\rho_\alpha^f \left(G_v(\mathbf{a})\right) = \rho_\alpha^f \left(G_v(\mathbf{b})\right)$$

Otherwise, we obtain

$$\rho_\alpha^f \left(G_v(\mathbf{a})\right) < \rho_\alpha^f \left(G_v(\mathbf{b})\right).$$

If $a_{i+1} = a_{i+2} = \cdots = a_p = b_p + 1$ dose not hold, then since the weight function $f(x, y) > 0$ satisfies that $f(1, x) \leq f(1, 2) = f(2, x)$ for any x, by Theorems 2.1 and 2.2 we get that

$$\rho_\alpha^f \left(G_v(\mathbf{a})\right) < \rho_\alpha^f \left(G_v(\mathbf{c})\right).$$

Similarly, by Lemma 3.3 it is easy to see that

$$\rho_\alpha^f \left(G_v(\mathbf{c})\right) \leq \rho_\alpha^f \left(G_v(\mathbf{b})\right).$$

Thus we obtain

$$\rho_\alpha^f \left(G_v(\mathbf{a})\right) < \rho_\alpha^f \left(G_v(\mathbf{b})\right).$$

Remark 3.4. *This result works for the weighted adjacency matrices defined by the augmented Zagreb index and exponential augmented Zagreb index listed in Tables 1 and 2. It is interesting to consider the weighted adjacency matrices correspond to other indices.*

Remark 3.5. *We not only use unified approaches to consider the spectral properties of these matrices, but also obtain Theorem 1.9 by taking $\alpha = 0$, $f(x, y) \equiv 1$ and Theorem 1.10 by taking $f(x, y) \equiv 1$ directly.*

4. Proof of Theorem 1.16

Recall that a path $P = v_0 v_1 \dots v_k$ in G is an internal path if $d_G(v_i) = 2$ for every $1 \leq i \leq k - 1$. Suppose that the spectral radius of G is $\rho_\alpha^f(G) > 2f(2, 2)$. Let $\mathbf{x}$ be a principal eigenvector of G, and x_{v_i} be the entry of $\mathbf{x}$ corresponding to v_i. Note that x_{v_i} satisfies that

$$f(2,2)(1-\alpha)x_{v_{i-1}} - [\rho_\alpha^f(G) - 2f(2,2)\alpha]x_{v_i} + f(2,2)(1-\alpha)x_{v_{i+1}} = 0$$

for $2 \le i \le k-2$, that is

$$x_{v_{i-1}} - \frac{[\rho_\alpha^f(G) - 2f(2,2)\alpha]}{f(2,2)(1-\alpha)} x_{v_i} + x_{v_{i+1}} = 0.$$

Clearly, this is a recurrence relation, and the characteristic equation is

$$t^2 - \frac{[\rho_\alpha^f(G) - 2f(2,2)\alpha]}{f(2,2)(1-\alpha)} t + 1 = 0.$$

Suppose that $t_2 \ge t_1$ are roots of the above equation. Since $\rho_\alpha^f(G) > 2f(2,2)$, we have $t_2 > 1 > t_1 > 0$ and $t_1 t_2 = 1$. Let x_{v_1} and $x_{v_{k-1}}$ be the initial conditions. By solving this recurrence relation, we have

$$x_{v_i} = \frac{1}{t_2^{k-1} - t_1^{k-3}} \left[\left(x_{v_1} t_2^k - x_{v_{k-1}} t_2^2 \right) t_1^i + \left(x_{v_{k-1}} - x_{v_1} t_1^{k-2} \right) t_2^i \right]$$

for $1 \le i \le k-1$. If $x_{v_1} = x_{v_{k-1}}$, then

$$\begin{aligned} x_{v_i} &= \frac{x_{v_1}}{t_2^{k-1} - t_1^{k-3}} \left(t_2^{k-i} - t_1^{i-2} + t_2^i - t_1^{k-i-2} \right) \\ &= \frac{x_{v_1}}{t_2^{k-2} - t_1^{k-2}} \left(t_2^{k-i-1} - t_1^{k-i-1} + t_2^{i-1} - t_1^{i-1} \right). \end{aligned}$$

Firstly, we prove the following proposition.

Proposition 4.1. *For $t_2 > 1 > t_1 > 0$, $t_1 t_2 = 1$ and $2 \le i \le k-2$,*

$$x_{v_i} = \frac{x_{v_1}}{t_2^{k-2} - t_1^{k-2}} \left(t_2^{k-i-1} - t_1^{k-i-1} + t_2^{i-1} - t_1^{i-1} \right)$$

is strictly decreasing in k for fixed $x_{v_1} > 0$.

Proof. Let

$$g(k) = \frac{t_2^{k-i-1} - t_1^{k-i-1} + t_2^{i-1} - t_1^{i-1}}{t_2^{k-2} - t_1^{k-2}}.$$

Due to $t_1 t_2 = 1$, we have

$$g(k) = \frac{t_2^{2k-i-3} + t_2^{k+i-3} - t_2^{k-i-1} - t_2^{i-1}}{t_2^{2k-4} - 1}.$$

We obtain

$$\begin{aligned} g'(k) &= \frac{\left[\left(2t_2^{2k-i-3}+t_2^{k+i-3}-t_2^{k-i-1}\right)\left(t_2^{2k-4}-1\right)\right]\ln t_2}{(t_2^{2k-4}-1)^2} \\ &\quad - \frac{\left[(t_2^{2k-i-3}+t_2^{k+i-3}-t_2^{k-i-1}-t_2^{i-1})2t_2^{2k-4}\right]\ln t_2}{(t_2^{2k-4}-1)^2} \\ &= \frac{\left[\left(t_2^{2i-2}-1\right)\left(2t_2^{2k-i-3}-t_2^{3k-i-5}-t_2^{k-i-1}\right)\right]\ln t_2}{(t_2^{2k-4}-1)^2} < 0. \end{aligned}$$

Thus we get

$$x_{v_i} = \frac{x_{v_1}}{t_2^{k-2}-t_1^{k-2}}\left(t_2^{k-i-1}-t_1^{k-i-1}+t_2^{i-1}-t_1^{i-1}\right)$$

is strictly decreasing in k for fixed $x_{v_1} > 0$. □

Next, we will use Proposition 4.1 to prove the cycle version of Li-Feng Grafting Theorem for $(A^f)_\alpha$-spectral radius. The proofs of Theorem 1.14 and Theorem 1.15 are divided into four cases. Here, we firstly prove the property of principal eigenvector. Then we only need to consider two cases.

Lemma 4.2. *Let* $\mathbf{a} = [a, b]$ *and* $\mathbf{b} = [a-1, b+1]$*. If* $\alpha \in [0, 1)$*,* $4 \le a \le b$ *and the weight function* $f(x, y) > 0$ *satisfies that* $f(2, 2) \le f(2, x)$ *for any* $x \ge 2$*, then* $\rho_\alpha^f(G_v^c(\mathbf{a})) < \rho_\alpha^f(G_v^c(\mathbf{b}))$*.*

Proof. Let $H = G_v^c(\mathbf{a})$. Since the weight function $f(x, y) > 0$ satisfies that $f(2, 2) \le f(2, x)$ for any $x \ge 2$, we have $\rho_\alpha^f(H) > 2f(2, 2)$. Then by taking equation

$$x_{v_i} = \frac{x_{v_1}}{t_2^{k-2}-t_1^{k-2}}\left(t_2^{k-i-1}-t_1^{k-i-1}+t_2^{i-1}-t_1^{i-1}\right)$$

in mind, it is easy to see that x_{v_i} is monotonically decreasing for $1 \le i \le l = \lfloor \frac{a}{2} \rfloor$. That is

$$x_{u_1} > x_{u_2} > \cdots > x_{u_{l-1}} > x_{u_l}.$$

Similarly, we have

$$x_{v_1} > x_{v_2} > \cdots > x_{v_{l-1}} > x_{v_l}.$$

Suppose that the pendant cycle of length a is $C = vu_1u_2 \dots u_{a-1}v$ and the pendant cycle of length b is $C' = vv_1v_2 \dots v_{b-1}v$.

Firstly, we show $x_v \geq x_{u_1}$ by negation. If $x_v < x_{u_1}$, then

$$\begin{aligned} & (1-\alpha)\left[2f(2,d_H(v))x_{u_1} + 2f(2,d_H(v))x_{v_1}\right] \\ \leq & \left[\rho_\alpha^f(H) - 2f(2,d_H(v))\alpha\right] x_v \\ < & \left\{\rho_\alpha^f(H) - \alpha\left[f(2,d_H(v)) + f(2,2)\right]\right\} x_v \\ < & \left\{\rho_\alpha^f(H) - \alpha\left[f(2,d_H(v)) + f(2,2)\right]\right\} x_{u_1} \\ = & (1-\alpha)\left[f(2,d_H(v))x_v + f(2,2)x_{u_2}\right]. \end{aligned}$$

By $x_{u_1} > x_{u_2}$, we have $x_v > x_{u_1}$, which contradicts $x_v < x_{u_1}$. Hence $x_v \geq x_{u_1}$.

In addition, we prove $x_{u_1} \geq x_{v_1}$. Since for the pendant cycle C, we have

$$\begin{aligned} & \left\{\rho_\alpha^f(H) - \alpha\left[f(2,d_H(v)) + f(2,2)\right]\right\} x_{u_1} \\ = & f(2,d_H(v))(1-\alpha)x_v + f(2,2)(1-\alpha)x_{u_2} \\ = & f(2,d_H(v))(1-\alpha)x_v + \frac{f(2,2)(1-\alpha)\left(t_2^{a-3} - t_1^{a-3} + t_2 - t_1\right)x_{u_1}}{t_2^{a-2} - t_1^{a-2}}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \left\{\rho_\alpha^f(H) - \alpha\left[f(2,d_H(v)) + f(2,2)\right] - \frac{f(2,2)(1-\alpha)\left(t_2^{a-3} - t_1^{a-3} + t_2 - t_1\right)}{t_2^{a-2} - t_1^{a-2}}\right\} x_{u_1} \\ = & f(2,d_H(v))(1-\alpha)x_v. \end{aligned}$$

By an argument similar to the proof in Proposition 4.1, it is easy to see that $x_{u_1} \geq x_{v_1}$ for $a \leq b$.

Then we consider the following two cases.

Case 1. a is even.

Then $a = 2l$. By Proposition 4.1 and $x_{u_1} \geq x_{v_1}$, we have $x_{u_l} \geq x_{v_l}$. If $x_{v_1} \geq x_{u_l}$, then there exists an integer $1 \leq i \leq l-1$ such that $x_{v_i} \geq x_{u_l} \geq x_{v_{i+1}}$. Let H' be a graph obtained from H by deleting edges $u_lu_{l-1}, u_lu_{l+1}, v_iv_{i+1}$ and adding edges $u_{l-1}u_{l+1}, u_lv_i, u_lv_{i+1}$. Then

$H' \cong G_v^c(\mathbf{b})$. We have

$$\begin{aligned}
&\frac{\left\langle A_\alpha^f\left(H'\right)\mathbf{x},\mathbf{x}\right\rangle - \langle A_\alpha^f(H)\mathbf{x},\mathbf{x}\rangle}{2f(2,2)(1-\alpha)} \\
&= x_{u_{l-1}}x_{u_{l+1}} + x_{u_l}x_{v_i} + x_{u_l}x_{v_{i+1}} - x_{u_l}x_{u_{l-1}} - x_{u_l}x_{u_{l+1}} - x_{v_i}x_{v_{i+1}} \\
&= x_{u_{l-1}}^2 + x_{u_l}x_{v_i} + x_{u_l}x_{v_{i+1}} - 2x_{u_l}x_{u_{l-1}} - x_{v_i}x_{v_{i+1}} \\
&= \left(x_{u_{l-1}} - x_{u_l}\right)^2 + \left(x_{v_i} - x_{u_l}\right)\left(x_{u_l} - x_{v_{i+1}}\right) \\
&> 0
\end{aligned}$$

and so,

$$\rho_\alpha^f\left(H'\right) \geq \left\langle A_\alpha^f\left(H'\right)\mathbf{x},\mathbf{x}\right\rangle > \langle A_\alpha^f(H)\mathbf{x},\mathbf{x}\rangle = \rho_\alpha^f(H).$$

That is,

$$\rho_\alpha^f(G_v^c(\mathbf{a})) < \rho_\alpha^f(G_v^c(\mathbf{b})).$$

Otherwise, $x_v \geq x_{u_l} > x_{v_1}$. Let H' be a graph obtained from H by deleting edges $u_lu_{l-1}, u_lu_{l+1}, vv_1$ and adding edges $u_{l-1}u_{l+1}, u_lv, u_lv_1$. Then $H' \cong G_v^c(\mathbf{b})$. We have

$$\begin{aligned}
&\left\langle A_\alpha^f\left(H'\right)\mathbf{x},\mathbf{x}\right\rangle - \langle A_\alpha^f(H)\mathbf{x},\mathbf{x}\rangle \\
&= [2f(2,2)(1-\alpha)]\left[x_{u_{l-1}}x_{u_{l+1}} + x_{u_l}x_{v_1} - x_{u_l}x_{u_{l-1}} - x_{u_l}x_{u_{l+1}}\right] \\
&\quad + 2f(2,d_H(v))(1-\alpha)\left(x_{u_l}x_v - x_vx_{v_1}\right) \\
&\quad + \alpha\left[f(2,d_H(v)) - f(2,2)\right]\left[x_{u_l}^2 - x_{v_1}^2\right] \\
&\geq 2f(2,2)(1-\alpha)\left[\left(x_{u_l} - x_{u_{l-1}}\right)^2 + \left(x_v - x_{u_l}\right)\left(x_{u_l} - x_{v_1}\right)\right] \\
&\quad + \alpha\left[f(2,d_H(v)) - f(2,2)\right]\left[x_{u_l}^2 - x_{v_1}^2\right] \\
&> 0.
\end{aligned}$$

We obtain

$$\rho_\alpha^f(G_v^c(\mathbf{a})) < \rho_\alpha^f(G_v^c(\mathbf{b})).$$

Case 2. a is odd.

Then $a = 2l+1$. Similarly, if $x_{v_1} \geq x_{u_l}$, then there exists an integer $1 \leq i \leq l-1$ such that $x_{v_i} \geq x_{u_l} \geq x_{v_{i+1}}$. Let H' be a graph obtained from H by deleting edges $u_lu_{l-1}, u_lu_{l+1}, v_iv_{i+1}$ and adding edges

$u_{l-1}u_{l+1}, u_lv_i, u_lv_{i+1}$. Then $H' \cong G_v^c(\mathbf{b})$. And we obtain

$$\begin{aligned}
&\frac{\left\langle A_\alpha^f\left(H'\right)\mathbf{x},\mathbf{x}\right\rangle-\langle A_\alpha^f(H)\mathbf{x},\mathbf{x}\rangle}{2f(2,2)(1-\alpha)}\\
&=x_{u_{l-1}}x_{u_{l+1}}+x_{u_l}x_{v_i}+x_{u_l}x_{v_{i+1}}-x_{u_l}x_{u_{l-1}}-x_{u_l}x_{u_{l+1}}-x_{v_i}x_{v_{i+1}}\\
&=x_{u_l}x_{v_i}+x_{u_l}x_{v_{i+1}}-x_{u_l}^2-x_{v_i}x_{v_{i+1}}\\
&=\left(x_{v_i}-x_{u_l}\right)\left(x_{u_l}-x_{v_{i+1}}\right)\\
&\geq 0.
\end{aligned}$$

If $\left\langle A_\alpha^f\left(H'\right)\mathbf{x},\mathbf{x}\right\rangle-\langle A_\alpha^f(H)\mathbf{x},\mathbf{x}\rangle = 0$, then either $x_{u_l}=x_{v_i}$ or $x_{u_l}=x_{v_{i+1}}$ (suppose that $x_{v_0}=x_v$), which implies that $\mathbf{x}$ is not a principal eigenvector of H'. Thus

$$\rho_\alpha^f\left(H'\right)>\left\langle A_\alpha^f\left(H'\right)\mathbf{x},\mathbf{x}\right\rangle\geq\langle A_\alpha^f(H)\mathbf{x},\mathbf{x}\rangle=\rho_\alpha^f(H).$$

Otherwise, $x_v \geq x_{u_l} > x_{v_1}$. Let H' be a graph obtained from H by deleting edges $u_lu_{l-1}, u_lu_{l+1}, vv_1$ and adding edges $u_{l-1}u_{l+1}, u_lv, u_lv_1$. Then $H' \cong G_v^c(\mathbf{b})$. Similarly, we have

$$\left\langle A_\alpha^f\left(H'\right)\mathbf{x},\mathbf{x}\right\rangle-\langle A_\alpha^f(H)\mathbf{x},\mathbf{x}\rangle>0.$$

Consequently, we get $\rho_\alpha^f(G_v^c(\mathbf{a})) < \rho_\alpha^f(G_v^c(\mathbf{b}))$. □

According to Theorem 2.4 and Lemma 4.2, it is easy to obtain Theorem 1.16.

Remark 4.3. *This result works for the weighted adjacency matrices defined by almost half of the indices listed in Tables 1 and 2. Such as modified first Zagreb index, extended index, Randić index, sum-connectivity index, harmonic index, atom-bond-connectivity (ABC) index, augmented Zagreb index, arithmetic-geometric (AG) index, inverse sum index, sum-connectivity Gourava index, exponential Randić index, exponential ABC index, exponential harmonic index, exponential sum-connectivity index and exponential augmented Zagreb index. It is interesting to consider the weighted adjacency matrices correspond to other indices.*

5. Concluding remarks

In [23], Shan and Liu showed that the A_α-spectral radius of $G_v(n_1, n_2, \ldots, n_d)$ will increase according to the shortlex ordering of $(n_1, n_2, \ldots, n_d)$. In this paper, we improve their result and extend it from A_α-spectral radius to the $(A^f)_\alpha$-spectral radius. In addition, we consider the cycle version of Li-Feng Grafting Theorem of $(A^f)_\alpha$-spectral radius and obtain the relationship between majorization ordering of $(n_1, n_2, \ldots, n_d)$ and the $(A^f)_\alpha$-spectral radii of $G_v^c(n_1, n_2, \ldots, n_d)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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