The Li-Feng transformation of weighted adjacency matrices for graphs with degree-based edge-weights*

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Abstract

For a graph G, let d_v be the degree of a vertex v. Given a symmetric real function $f(x, y)$, the weight of edge uv in graph G is equal to the value $f(d_u, d_v)$. The degree-based weighted adjacency matrix is defined as $A_f(G)$, in which the (u, v)-entry is equal to $f(d_u, d_v)$ if uv is an edge of G and 0 otherwise. In this paper, we consider the Li-Feng transformation and show that if a graph G contains two pendant paths on a common vertex, the uniform distribution of pendant paths increases the largest eigenvalue of $A_f(G)$, when $f(x, y)$ is increasing in x and the length of two pendant paths should be at least 2. We also consider the cycle version of Li-Feng transformation and show that if a graph G contains two pendant cycles on a common vertex, the uniform distribution of pendant cycles decreases the largest eigenvalue of $A_f(G)$, when $\lambda_1(A_f(G)) > 2f(2, 2)$. The purpose of this paper is to unify the study of the graph operation on the largest eigenvalue for the degree-based weighted adjacency matrix.

Keywords: degree-based edge-weight, weighted adjacency matrix (largest eigenvalue); topological function index; Li-Feng transformation

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1 Introduction

Let $G = (V(G), E(G))$ be a finite, undirected, simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote by d_v the degree of the vertex v in G, $N_G(v)$ the set of neighbours of vertex v in G. An edge $e \in E(G)$ with end vertices u and v is usually denoted by uv.

In molecular graph theory, the topological indices of molecular graphs are used to reflect chemical properties of chemical molecules. The degree-based index $TI_f(G)$ of G with positive symmetric function $f(x, y)$ is defined as

$$TI_f(G) = \sum_{uv \in E(G)} f(d_u, d_v).$$

Gutman [11] collected many important and well-studied topological indices; see them in Table 1. Based on one concrete index, a molecular graph has a single number, obtained by summing up the edge-weights in a molecular graph with edge-weights defined by the function $f(x, y)$.

In algebraic graph theory, the study of matrices associated with a graph G is an important topic. One of the authors Li in [17] proposed that if we use a matrix to represent the structure of a molecular graph with weights separately on its pairs of adjacent vertices, it will keep more structural information of the graph than an index. For example, the Randić matrix [31, 32], the ABC matrix [5], the AG matrix [40] and the Sombor matrix [12] were considered separately. The weighted adjacency matrix $A_f(G)$ first appeared in [4], and it is defined as

$$A_f(G)_{uv} = \begin{cases} f(d_u, d_v), & uv \in E(G); \\ 0, & otherwise. \end{cases}$$

The largest eigenvalue of the weighted adjacency matrix $A_f(G)$ is $\lambda_1(A_f(G))$. If for $uv \in E(G)$, $f(d_u, d_v) = 1$ and 0 otherwise, then $A_f(G)$ is the adjacency matrix $A(G)$, we omit the writing of f.

As one can see that from each index in Table 1, one can get a weighted matrix defined by that index. There have been a lot of publications studying these indices and matrices one by one separately. However, the methods used in these publications are the same or similar. So in recent years, it is a trend to develop unified methods to deal with such degree-based indices and function-weighted adjacency matrices, see [2, 6, 7, 8, 14, 15, 19, 20, 21, 23, 24, 25, 26] and a survey paper [22]. About the weighted adjacency matrix, Li and Wang [23] first tried to find unified methods to study the extremal eigenvalues of $A_f(G)$. They obtained the trees with the largest

Edge-weight function f(x,y)	**The corresponding index**
$x+y$	first Zagreb index
xy	second Zagreb index
$(x+y)^2$	first hyper-Zagreb index
$(xy)^2$	second hyper-Zagreb index
$x^{-3}+y^{-3}$	modified first Zagreb index
$\|x-y\|$	Albertson index
$(x/y+y/x)/2$	extended index
$(x-y)^2$	sigma index
$1/\sqrt{xy}$	Randić index
$\sqrt{xy}$	reciprocal Randić index
$1/\sqrt{x+y}$	sum-connectivity index
$\sqrt{x+y}$	reciprocal sum-connectivity index
$2/(x+y)$	harmonic index
$\sqrt{(x+y-2)/(xy)}$	atom-bond-connectivity (ABC) index
$(xy/(x+y-2))^3$	augmented Zagreb index
x^2+y^2	forgotten index
$x^{-2}+y^{-2}$	inverse degree
$2\sqrt{xy}/(x+y)$	geometric-arithmetic (GA) index
$(x+y)/(2\sqrt{xy})$	arithmetic-geometric (AG) index
$xy/(x+y)$	inverse sum index
$x+y+xy$	first Gourava index
$(x+y)xy$	second Gourava index
$(x+y+xy)^2$	first hyper-Gourava index
$((x+y)xy)^2$	second hyper-Gourava index
$1/\sqrt{x+y+xy}$	sum-connectivity Gourava index
$\sqrt{(x+y)xy}$	product-connectivity Gourava index
$\sqrt{x^2+y^2}$	Sombor index

Table 1: Some well-studied topological indices

eigenvalue of $A_f(G)$ is star S_n or double star $S_{d,n-d}$, with the smallest eigenvalue of $A_f(G)$ is P_n, when $f(x,y)$ has some functional properties. Zheng et al. [39] added a property P* to $f(x,y)$ and studied the trees and unicyclic graphs with the largest and smallest weighted adjacency eigenvalues. Li and Yang [26] gave some lower and upper bounds of the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$. In [8], Gao and Yang got the gap between two largest eigenvalues: $\lambda_1(A_f(G))$, $\lambda_1(A_f(H))$, where H is obtained from G by some kinds of graph operations, including deleting vertices, deleting an edge and subdividing an edge. They also obtained some bounds for the largest weighted adjacency eigenvalue of irregular weighted graphs. Gao et al. [7] considered the unimodality of the eigenvector $\mathbf{x}$ on an induced path of G, and investigated how the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ changes when G is perturbed by vertex contraction or edge subdivision. In this paper, we are interested in the effects on the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ under two kinds of graph transformation.

Let v_0 be a vertex of a connected graph G. Attaching two new paths: $v_0v_1v_2\dots v_p$ and $v_0u_1u_2\dots u_q$ of length p and q, respectively, at v_0, we obtain the connected graph $G_{v_0}(p,q)$. It is clear that graph $G_{v_0}(p,q)$ contains two pendent paths. Li and Feng [18] introduced a transformation by changing the lengths of these two pendant paths and obtained the following result.

Theorem 1.1 *(Li-Feng transformation [18]) If $p \geq q+2 \geq 2$ in graph $G_{v_0}(p,q)$, then*

$$\lambda_1(A(G_{v_0}(p-1,q+1))) > \lambda_1(A(G_{v_0}(p,q))).$$

Since then, Theorem 1.1 has been extensively studied in spectral graph theory. Because this is a powerful tool to investigate the graph with maximum or minimum spectral radius among a given class of graphs, see [1, 9, 16, 27, 29, 30, 33, 34, 35, 36, 38]. Using Theorem 1.1, Berman and Zhang [1] studied the spectral radius of graphs with n vertices and k cut vertices and described the graph that has the maximal spectral radius in this class; Simić et al. [33] considered the set of caterpillars with a fixed order and diameter, or with a fixed degree sequence, whose spectral radius is maximal; Guo [9] determined graphs with the largest spectral radius among all the unicyclic and all the bicyclic graphs with n vertices and k pendant vertices, respectively; Stevanović and Hansen [35] obtained the minimum spectral radius of graphs with a given clique number. Furthermore, researchers considered the Li-Feng transformation for the largest eigenvalues of the Laplacian matrix [10], the signless Laplacian matrix [3], the A_α-matrix [28]. These results have been extensively proved to be efficient in ordering graphs by the largest eigenvalue, and will be very important for our future research. Here, we consider the Li-Feng transformation for the largest eigenvalues of the weighted adjacency matrix.

Moreover, in [37], Xue et al. considered the cycle version of Li-Feng transformation. Let v_0 be a vertex of a connected graph G. Attaching two new cycles: $v_0v_1v_2\dots v_{k-1}v_0$ and $v_0u_1u_2\dots u_{l-1}v_0$ of length k and l, respectively, at v_0, we obtain the connected graph $G^c_{v_0}(k,l)$. It is clear that graph $G^c_{v_0}(k,l)$ contains two pendent cycles. They obtained the following theorem.

Theorem 1.2 *[37] If $k \geq l \geq 4$ in graph $G^c_{v_0}(k,l)$, then*

$$\lambda_1(A(G^c_{v_0}(k+1,l-1))) > \lambda_1(A(G^c_{v_0}(k,l))).$$

The cycle version of Li-Feng transformation tells us that the uniform distribution of pendant cycles decreases the spectral radius. Hence, this result provides us a direction for future research on the extremal spectral radius of graphs containing cycles. This is an important graph operation in our study. Next, we also consider the cycle version of Li-Feng transformation for the largest eigenvalues of the weighted adjacency matrix.

2 Some preliminary results

In this section, we provide some preliminary results of matrix theory and weighted adjacency matrix that will be used in the subsequent sections.

Lemma 2.1 *[13] Let M be an $n \times n$ nonnegative and symmetric matrix. Then $\lambda_1(M) \geq \mathbf{y}^T M \mathbf{y}$ for any unit vector $\mathbf{y}$, and the equality holds if and only if $M\mathbf{y} = \lambda_1(M)\mathbf{y}$.*

Lemma 2.2 *(Perron–Frobenius [13]) Let M be a nonnegative irreducible square matrix. Then the largest eigenvalue $\lambda_1(M)$ is simple, with a corresponding eigenvector whose entries are all positive.*

We call such a positive unit eigenvector $\mathbf{x}$ corresponding to the largest eigenvalue of M is a principal eigenvector. In this paper, for a principal eigenvector $\mathbf{x}$ of $A_f(G)$, we use x_v to denote the entry of $\mathbf{x}$ corresponding to this vertex v. Note that in the remainder of this paper, we always assume that the edge-weight $f(d_u, d_v) > 0$ for any edge $uv \in E(G)$.

Lemma 2.3 *[7] Let $v_1v_2\dots v_{k-1}$ be an induced path of G such that $d_{v_i}=2$ for $1\le i\le k-1$, and $\mathbf{x}$ be a principal eigenvector of $\lambda_1(A_f(G))$. If $\lambda_1(A_f(G))>2f(2,2)$, then the following statements hold.*

(1) If $x_{v_1}=x_{v_{k-1}}$, then

$$x_{v_1}>x_{v_2}>\cdots>x_{v_{\lfloor\frac{k}{2}\rfloor}}=x_{v_{\lceil\frac{k}{2}\rceil}}<\cdots<x_{v_{k-2}}<x_{v_{k-1}}$$

and $x_{v_i}=x_{v_{k-i}}$ for $2\le i\le k-2$.

(2) If $x_{v_1}<x_{v_{k-1}}$, then there is an integer $1\le j\le\lfloor\frac{k}{2}\rfloor$ such that

$$x_{v_1}>x_{v_2}>\cdots>x_{v_j}\ge x_{v_{j+1}}<\cdots<x_{v_{k-2}}<x_{v_{k-1}}$$

or

$$x_{v_1}>x_{v_2}>\cdots>x_{v_j}<x_{v_{j+1}}<\cdots<x_{v_{k-2}}<x_{v_{k-1}}.$$

Moreover, $x_{v_i}<x_{v_{k-i}}$ for $2\le i\le\lceil\frac{k}{2}\rceil-1$.

Theorem 2.4 *[7] Let $G\ne C_n$ be a connected graph of order n. If $f(x,y)>0$ is increasing in variable x and G contains a cycle, then $\lambda_1(A_f(G))>2f(2,2)$.*

3 Main results

In this section, we first consider the cycle version of Li-Feng transformation with respect to the largest eigenvalue of $A_f(G)$, and then consider the Li-Feng transformation with respect to the largest eigenvalue of $A_f(G)$. Before proving Theorem 3.2, we first compare two entries x_{v_1}, x_{u_1} of a principal eigenvector $\mathbf{x}$ on two cycles in graph $G^c_{v_0}(k,l)$.

Lemma 3.1 *If $k\ge l\ge 4$ in graph $G^c_{v_0}(k,l)$ and $\lambda_1(A_f(G^c_{v_0}(k,l)))>2f(2,2)$, then*

$$x_{v_1}\le x_{u_1}.$$

In particular, $x_{v_1}<x_{u_1}$ if $k>l\ge 4$.

Proof. Let $\mathbf{x}$ be a principal eigenvector of $A_f(G^c_{v_0}(k,l))$. First, we consider the induced path $v_1v_2\dots v_{k-1}$, where $d_{v_i}=2$ for $1\le i\le k-1$. We have

$$\lambda_1(A_f(G^c_{v_0}(k,l)))x_{v_i}=f(2,2)x_{v_{i-1}}+f(2,2)x_{v_{i+1}},$$

for $2 \le i \le k-2$. Hence,

$$\frac{\lambda_1(A_f(G^c_{v_0}(k,l)))}{f(2,2)} x_{v_i} = x_{v_{i-1}} + x_{v_{i+1}},$$

for $2 \le i \le k-2$. This is a recurrence relation and the characteristic equation is

$$t^2 - \frac{\lambda_1(A_f(G^c_{v_0}(k,l)))}{f(2,2)} t + 1 = 0.$$

Because $\lambda_1(A_f(G^c_{v_0}(k,l))) > 2f(2,2)$, the equation above has two unequal real roots t_1 and t_2 such that

$$\begin{cases} t_1 + t_2 = \frac{\lambda_1(A_f(G^c_{v_0}(k,l)))}{f(2,2)}, \\ t_1 \cdot t_2 = 1. \end{cases}$$

Without loss of generality, we assume that $t_2 > 1 > t_1 > 0$. Let x_{v_1} and $x_{v_{k-1}}$ be the initial conditions. We obtain the solution of this linear homogeneous recurrence relation with constant coefficients is

$$x_{v_i} = \frac{1}{t_2^{k-1} - t_1^{k-3}}((x_{v_1} t_2^k - x_{v_{k-1}} t_2^2) t_1^i + (x_{v_{k-1}} - x_{v_1} t_1^{k-2}) t_2^i).$$

Since

$$\lambda_1(A_f(G^c_{v_0}(k,l))) x_{v_1} = f(d_{v_0}, 2) x_{v_0} + f(2,2) x_{v_2},$$

we have

$$\begin{aligned} x_{v_0} &= \frac{1}{f(d_{v_0},2)}(\lambda_1(A_f(G^c_{v_0}(k,l))) x_{v_1} - f(2,2) x_{v_2}) \\ &= \frac{1}{f(d_{v_0},2)}((t_1+t_2) f(2,2) x_{v_1} - f(2,2) x_{v_2}) \\ &= \frac{f(2,2)}{f(d_{v_0},2)}((t_1+t_2) x_{v_1} - \frac{1}{t_2^{k-1} - t_1^{k-3}}((x_{v_1} t_2^k - x_{v_{k-1}} t_2^2) t_1^2 + (x_{v_{k-1}} - x_{v_1} t_1^{k-2}) t_2^2)) \\ &= \frac{f(2,2)}{f(d_{v_0},2)} \left(\frac{t_2^k - t_1^{k-2}}{t_2^{k-1} - t_1^{k-3}} x_{v_1} + \frac{1 - t_2^2}{t_2^{k-1} - t_1^{k-3}} x_{v_{k-1}} \right). \end{aligned}$$

Recall that $x_{v_1} = x_{v_{k-1}}$. It follows that

$$x_{v_0} = \frac{f(2,2)}{f(d_{v_0},2)} \cdot \frac{t_2^k - t_1^{k-2} + 1 - t_2^2}{t_2^{k-1} - t_1^{k-3}} x_{v_1}. \tag{3.1}$$

If we consider the induced path $u_1 u_2 \dots u_{l-1}$, with the same method, then we obtain

$$x_{v_0} = \frac{f(2,2)}{f(d_{v_0},2)} \cdot \frac{t_2^l - t_1^{l-2} + 1 - t_2^2}{t_2^{l-1} - t_1^{l-3}} x_{u_1}. \tag{3.2}$$

Now, let

$$h(x) = \frac{t_2^x - t_1^{x-2} + 1 - t_2^2}{t_2^{x-1} - t_1^{x-3}}.$$

By calculating the first-order derivative of $h(x)$, we get

$$h^{'}(x) = \frac{t_2 \ln t_2 (t_2^{\frac{x-2}{2}} - t_1^{\frac{x-2}{2}})^2 (t_2^2 - 1)}{(t_2^{x-1} - t_1^{x-3})^2} > 0,$$

for $t_2 > 1$. This means that $h(x)$ is monotonically increasing in x.

From the definition of the graph $G^c_{v_0}(k, l)$, we know that v_0 is the common vertex of cycles $v_0v_1v_2 \ldots v_{k-1}v_0$ and $v_0u_1u_2 \ldots u_{l-1}v_0$. From equations (3.1), (3.2) and the monotonicity of $h(x)$, we can obtain our result directly. □

Now we give the cycle version of Li-Feng transformation with respect to the largest eigenvalue of $A_f(G)$. The following result means that if a graph contains two pendant cycles on a common vertex, the uniform distribution of pendant cycles decreases the largest weighted adjacency eigenvalue. This conclusion is the same as Theorem 1.2.

Theorem 3.2 *If $k \geq l \geq 4$ in graph $G^c_{v_0}(k, l)$ and $\lambda_1(A_f(G^c_{v_0}(k, l))) > 2f(2, 2)$, then*

$$\lambda_1(A_f(G^c_{v_0}(k + 1, l - 1))) > \lambda_1(A_f(G^c_{v_0}(k, l))).$$

Proof. Let $\mathbf{x}$ be a principal eigenvector of $A_f(G^c_{v_0}(k, l))$. Since $x_{v_1} = x_{v_{k-1}}$ and $x_{u_1} = x_{u_{l-1}}$, from Lemma 2.3 (1), we have

$$x_{v_1} > x_{v_2} > \cdots > x_{v_{\lfloor \frac{k}{2} \rfloor}}$$

and

$$x_{u_1} > x_{u_2} > \cdots > x_{u_{\lfloor \frac{l}{2} \rfloor}}.$$

Furthermore, we have

$$x_{v_i} = \frac{x_{v_1}}{t_2^{k-1} - t_1^{k-3}} (t_2^{k-i} - t_1^{i-2} + t_2^i - t_1^{k-2-i}),$$

for $1 \leq i \leq k - 1$, and

$$x_{u_i} = \frac{x_{u_1}}{t_2^{l-1} - t_1^{l-3}} (t_2^{l-i} - t_1^{i-2} + t_2^i - t_1^{l-2-i}),$$

for $1 \leq i \leq l - 1$. Next, we consider four cases.

Case 1. Suppose that k and l are both even. Set $k = 2p$ and $l = 2q$. It follows that

$$x_{v_p} = \frac{2}{t_2^{p-1} + t_1^{p-1}} x_{v_1}$$

and

$$x_{u_q} = \frac{2}{t_2^{q-1} + t_1^{q-1}} x_{u_1}.$$

Since $p \geq q$, from Lemma 3.1, we know $x_{v_1} \leq x_{u_1}$. And $\frac{2}{t_2^{p-1}+t_1^{p-1}} \leq \frac{2}{t_2^{q-1}+t_1^{q-1}}$, thus $x_{v_p} \leq x_{u_q}$. Recall that $x_{v_1} > x_{v_2} > \cdots > x_{v_p}$. We consider two subcases.

Subcase 1.1 There exists an integer $1 \leq i \leq p-1$ such that $x_{v_i} \geq x_{u_q} \geq x_{v_{i+1}}$.

Suppose that $G^c_{v_0}(k+1, l-1)$ is a graph obtained from $G^c_{v_0}(k, l)$ by deleting edges $u_q u_{q-1}, u_q u_{q+1}, v_i v_{i+1}$ and adding edges $u_{q-1}u_{q+1}, u_q v_i, u_q v_{i+1}$. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G^c_{v_0}(k+1, l-1))\mathbf{x} - \mathbf{x}^T A_f(G^c_{v_0}(k, l))\mathbf{x})$$

$$\begin{aligned}
&= f(d_{u_{q-1}}, d_{u_{q+1}})x_{u_{q-1}}x_{u_{q+1}} + f(d_{u_q}, d_{v_i})x_{u_q}x_{v_i} + f(d_{u_q}, d_{v_{i+1}})x_{u_q}x_{v_{i+1}} - \\
&f(d_{u_q}, d_{u_{q-1}})x_{u_q}x_{u_{q-1}} - f(d_{u_q}, d_{u_{q+1}})x_{u_q}x_{u_{q+1}} - f(d_{v_i}, d_{v_{i+1}})x_{v_i}x_{v_{i+1}} \\
&= f(2,2)(x_{u_{q-1}}x_{u_{q+1}} + x_{u_q}x_{v_i} + x_{u_q}x_{v_{i+1}} - x_{u_q}x_{u_{q-1}} - x_{u_q}x_{u_{q+1}} - x_{v_i}x_{v_{i+1}}) \\
&= f(2,2)(x^2_{u_{q-1}} + x_{u_q}x_{v_i} + x_{u_q}x_{v_{i+1}} - 2x_{u_q}x_{u_{q-1}} - x_{v_i}x_{v_{i+1}}) \\
&= f(2,2)((x_{u_q} - x_{u_{q-1}})^2 + (x_{v_i} - x_{u_q})(x_{u_q} - x_{v_{i+1}})) \\
&> 0.
\end{aligned}$$

From Lemma 2.1, thus $\lambda_1(A_f(G^c_{v_0}(k+1, l-1))) > \lambda_1(A_f(G^c_{v_0}(k, l)))$.

Subcase 1.2 $x_{u_q} > x_{v_1}$.

Suppose that $G^c_{v_0}(k+1, l-1)$ is a graph obtained from $G^c_{v_0}(k, l)$ by deleting edges $u_q u_{q-1}, u_q u_{q+1}, v_0 v_1$ and adding edges $u_{q-1}u_{q+1}, u_q v_0, u_q v_1$. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G^c_{v_0}(k+1, l-1))\mathbf{x} - \mathbf{x}^T A_f(G^c_{v_0}(k, l))\mathbf{x})$$

$$\begin{aligned}
&= f(d_{u_{q-1}}, d_{u_{q+1}})x_{u_{q-1}}x_{u_{q+1}} + f(d_{u_q}, d_{v_0})x_{u_q}x_{v_0} + f(d_{u_q}, d_{v_1})x_{u_q}x_{v_1} - \\
&f(d_{u_q}, d_{u_{q-1}})x_{u_q}x_{u_{q-1}} - f(d_{u_q}, d_{u_{q+1}})x_{u_q}x_{u_{q+1}} - f(d_{v_0}, d_{v_1})x_{v_0}x_{v_1} \\
&= f(2,2)x^2_{u_{q-1}} + f(2, d_{v_0})x_{u_q}x_{v_0} + f(2,2)x_{u_q}x_{v_1} - 2f(2,2)x_{u_q}x_{u_{q-1}} - f(2, d_{v_0})x_{v_0}x_{v_1} \\
&= f(2,2)(x_{u_q} - x_{u_{q-1}})^2 + (f(2, d_{v_0})x_{v_0} - f(2,2)x_{u_q})(x_{u_q} - x_{v_1}) \\
&> 0.
\end{aligned}$$

Because

$$\begin{aligned}
f(2, d_{v_0})x_{v_0} - f(2,2)x_{u_q} &= f(2, d_{v_0}) \cdot \frac{f(2,2)}{f(2, d_{v_0})} \cdot \frac{t_2^{2q} - t_1^{2q-2} + 1 - t_2^2}{t_2^{2q-1} - t_1^{2q-3}} x_{u_1} - f(2,2) \cdot \frac{2(t_2^q - t_1^{q-2})}{t_2^{2q-1} - t_1^{2q-3}} x_{u_1} \\
&= f(2,2) \frac{(t_2^{\frac{q}{2}} - t_1^{\frac{q}{2}})^2 (t_2^q - t_1^{q-2})}{t_2^{2q-1} - t_1^{2q-3}} x_{u_1} \\
&> 0,
\end{aligned}$$

the inequality above holds. From Lemma 2.1, thus $\lambda_1(A_f(G^c_{v_0}(k+1,l-1))) > \lambda_1(A_f(G^c_{v_0}(k,l)))$.

Case 2. Suppose that k is even and l is odd. Set $k=2p$ and $l=2q+1$. It follows that

$$x_{v_p} = \frac{2}{t_2^{p-1}+t_1^{p-1}} x_{v_1}$$

and

$$x_{u_q} = \frac{t_2+1}{t_2^q+t_1^{q-1}} x_{u_1}.$$

Since $k>l$, from Lemma 3.1, we know $x_{v_1} < x_{u_1}$. And $p \geq q+1$, we have

$$(t_2+1)(t_2^{p-1}+t_1^{p-1}) = t_2^p + t_1^{p-2} + t_2^{p-1} + t_1^{p-1} > 2(t_2^q + t_1^{q-1}).$$

Thus $x_{v_p} < x_{u_q}$. Recall that $x_{v_1} > x_{v_2} > \cdots > x_{v_p}$. We divided into two subcases.

Subcase 2.1 There exists an integer $1 \leq i \leq p-1$ such that $x_{v_i} \geq x_{u_q} \geq x_{v_{i+1}}$.

Suppose that $G^c_{v_0}(k+1,l-1)$ is a graph obtained from $G^c_{v_0}(k,l)$ by deleting edges $u_qu_{q-1}, u_qu_{q+1}, v_iv_{i+1}$ and adding edges $u_{q-1}u_{q+1}, u_qv_i, u_qv_{i+1}$. We have

$$\frac{1}{2}(\mathbf{x}^TA_f(G^c_{v_0}(k+1,l-1))\mathbf{x} - \mathbf{x}^TA_f(G^c_{v_0}(k,l))\mathbf{x})$$

$$\begin{aligned}
&= f(d_{u_{q-1}}, d_{u_{q+1}})x_{u_{q-1}}x_{u_{q+1}} + f(d_{u_q}, d_{v_i})x_{u_q}x_{v_i} + f(d_{u_q}, d_{v_{i+1}})x_{u_q}x_{v_{i+1}} - \\
&f(d_{u_q}, d_{u_{q-1}})x_{u_q}x_{u_{q-1}} - f(d_{u_q}, d_{u_{q+1}})x_{u_q}x_{u_{q+1}} - f(d_{v_i}, d_{v_{i+1}})x_{v_i}x_{v_{i+1}} \\
&= f(2,2)(x_{u_q}x_{v_i} + x_{u_q}x_{v_{i+1}} - x_{u_q}^2 - x_{v_i}x_{v_{i+1}}) \\
&= f(2,2)(x_{v_i} - x_{u_q})(x_{u_q} - x_{v_{i+1}}) \\
&\geq 0.
\end{aligned}$$

If $\frac{1}{2}(\mathbf{x}^TA_f(G^c_{v_0}(k+1,l-1))\mathbf{x} - \mathbf{x}^TA_f(G^c_{v_0}(k,l))\mathbf{x}) = 0$, because $x_{v_i} > x_{v_{i+1}}$, then either $x_{u_q} = x_{v_i}$ or $x_{u_q} = x_{v_{i+1}}$. This implies that

$$\lambda_1(A_f(G^c_{v_0}(k+1,l-1))) = \mathbf{x}^TA_f(G^c_{v_0}(k+1,l-1))\mathbf{x} = \mathbf{x}^TA_f(G^c_{v_0}(k,l))\mathbf{x} = \lambda_1(A_f(G^c_{v_0}(k,l))).$$

We assume that $x_{u_q} = x_{v_i}$, then it is not difficult for us to deduced that

$$\begin{cases} \lambda_1(A_f(G^c_{v_0}(k+1,l-1)))x_{v_i} = f(d_{v_{i-1}},2)x_{v_{i-1}} + f(2,2)x_{u_q}, \\ \lambda_1(A_f(G^c_{v_0}(k,l)))x_{v_i} = f(d_{v_{i-1}},2)x_{v_{i-1}} + f(2,2)x_{v_{i+1}}. \end{cases}$$

Because $x_{u_q} > x_{v_{i+1}}$, this is a contradiction. From Lemma 2.1, thus $\lambda_1(A_f(G^c_{v_0}(k+1,l-1))) > \lambda_1(A_f(G^c_{v_0}(k,l)))$.

Subcase 2.2 $x_{u_q} > x_{v_1}$.

Suppose that $G^c_{v_0}(k+1,l-1)$ is a graph obtained from $G^c_{v_0}(k,l)$ by deleting edges $u_qu_{q-1}, u_qu_{q+1}, v_0v_1$ and adding edges $u_{q-1}u_{q+1}, u_qv_0, u_qv_1$. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G^c_{v_0}(k+1,l-1))\mathbf{x} - \mathbf{x}^T A_f(G^c_{v_0}(k,l))\mathbf{x})$$

$$\begin{aligned}
&= f(d_{u_{q-1}}, d_{u_{q+1}})x_{u_{q-1}}x_{u_{q+1}} + f(d_{u_q}, d_{v_0})x_{u_q}x_{v_0} + f(d_{u_q}, d_{v_1})x_{u_q}x_{v_1} - \\
&f(d_{u_q}, d_{u_{q-1}})x_{u_q}x_{u_{q-1}} - f(d_{u_q}, d_{u_{q+1}})x_{u_q}x_{u_{q+1}} - f(d_{v_0}, d_{v_1})x_{v_0}x_{v_1} \\
&= f(2, d_{v_0})x_{u_q}x_{v_0} + f(2,2)x_{u_q}x_{v_1} - f(2,2)x^2_{u_q} - f(d_{v_0}, 2)x_{v_0}x_{v_1} \\
&= (f(2, d_{v_0})x_{v_0} - f(2,2)x_{u_q})(x_{u_q} - x_{v_1}) \\
&> 0.
\end{aligned}$$

Because

$$f(2, d_{v_0})x_{v_0} - f(2,2)x_{u_q}$$

$$\begin{aligned}
&= f(2, d_{v_0}) \cdot \frac{f(2,2)}{f(2, d_{v_0})} \cdot \frac{t_2^{2q+1} - t_1^{2q-1} + 1 - t_2^2}{t_2^{2q} - t_1^{2q-2}} x_{u_1} - f(2,2) \cdot \frac{t_2^{q+1} - t_1^{q-2} + t_2^q - t_1^{q-1}}{t_2^{2q} - t_1^{2q-2}} x_{u_1} \\
&= f(2,2)\frac{(t_2^q - 1)(t_2 - t_1^q)(t_2^q - t_1^{q-1})}{t_2^{2q} - t_1^{2q-2}} x_{u_1} \\
&> 0,
\end{aligned}$$

the inequality above holds. From Lemma 2.1, thus $\lambda_1(A_f(G^c_{v_0}(k+1,l-1))) > \lambda_1(A_f(G^c_{v_0}(k,l)))$.

Case 3. Suppose that k and l are both odd. Set $k = 2p+1$ and $l = 2q+1$. It follows that

$$x_{v_p} = \frac{t_2 + 1}{t_2^p + t_1^{p-1}} x_{v_1}$$

and

$$x_{u_q} = \frac{t_2 + 1}{t_2^q + t_1^{q-1}} x_{u_1}.$$

Since $k \geq l$, from Lemma 3.1, we know $x_{v_1} \leq x_{u_1}$. And $\frac{t_2+1}{t_2^p + t_1^{p-1}} \leq \frac{t_2+1}{t_2^q + t_1^{q-1}}$, thus $x_{v_p} \leq x_{u_q}$. Recall that $x_{v_1} > x_{v_2} > \cdots > x_{v_p}$. We also consider two subcases. Since the proof in this case is similar to subcases 2.1 and 2.2, we omit it for the sake of brevity.

Case 4. Suppose that k is odd and l is even. Set $k = 2p+1$ and $l = 2q$. It follows that

$$x_{v_p} = \frac{t_2 + 1}{t_2^p + t_1^{p-1}} x_{v_1}$$

and

$$x_{u_q} = \frac{2}{t_2^{q-1} + t_1^{q-1}} x_{u_1}.$$

Since $k > l$, from Lemma 3.1, we know $x_{v_1} < x_{u_1}$. And $p \geq q$, we have

$$2(t_2^p + t_1^{p-1}) > (t_2 + 1)(t_2^{p-1} + t_1^{p-1}) \geq (t_2 + 1)(t_2^{q-1} + t_1^{q-1}).$$

Thus $x_{v_p} < x_{u_q}$. Recall that $x_{v_1} > x_{v_2} > \cdots > x_{v_p}$. We can divided into two subcases. Because the proof in this case is similar to subcases 1.1 and 1.2, we omit it for the sake of brevity.

In each case, we obtain the largest weighted adjacency eigenvalue of $G^c_{v_0}(k+1, l-1)$ is greater than the largest weighted adjacency eigenvalue of $G^c_{v_0}(k, l)$. Hence, the proof of the theorem is complete. □

Remark 1. If $f(x, y) > 0$ is increasing in variable x, from Lemma 2.4, then we have $\lambda_1(A_f(G^c_{v_0}(k, l))) > 2f(2, 2)$. Thus Theorem 3.2 is hold, when the edge-weight functions $f(x, y)$ is increasing in variable x.

Next, we consider the Li-Feng transformation with respect to the largest eigenvalue of $A_f(G)$. We first give a result about pendant paths. Assume that $f(x, y) > 0$ is increasing in invariable x, the entries of a principle eigenvector $\mathbf{x}$ on an induced path have the monotone property.

Lemma 3.3 *Let $v_0v_1v_2 \ldots v_{p-1}v_p$ be a pendant path in graph G with $d_{v_0} \geq 3$. If $f(x, y) > 0$ is increasing in invariable x and $\lambda_1(A_f(G)) > 2f(2, 2)$, then $x_{v_1} > x_{v_2} > \cdots > x_{v_p}$.*

Proof. Let $\mathbf{x}$ be a principal eigenvector of $A_f(G)$. It is not difficult for us to have

$$\begin{cases} \lambda_1(A_f(G))x_{v_p} = f(2, 1)x_{v_{p-1}}, \\ \lambda_1(A_f(G))x_{v_{p-1}} = f(2, 1)x_{v_p} + f(2, 2)x_{v_{p-2}}. \end{cases}$$

If $x_{v_p} \geq x_{v_{p-1}}$, then $f(2, 1)x_{v_{p-1}} \geq f(2, 1)x_{v_p} + f(2, 2)x_{v_{p-2}}$. From Lemma 2.2, we know $x_{v_{p-2}} > 0$, this is a contradiction. Thus $x_{v_p} < x_{v_{p-1}}$.

Since $\lambda_1(A_f(G)) > 2f(2, 2)$, we have $\lambda_1(A_f(G))x_{v_{p-1}} = f(2, 1)x_{v_p} + f(2, 2)x_{v_{p-2}} > f(2, 2)x_{v_{p-1}} + f(2, 2)x_{v_{p-1}}$. Recall that $f(x, y)$ is increasing in invariable x and $x_{v_p} < x_{v_{p-1}}$. It follows that $f(2, 1)x_{v_p} < f(2, 2)x_{v_{p-1}}$. Hence $x_{v_{p-2}} > x_{v_{p-1}}$. From Lemma 2.3 (2), we obtain $x_{v_1} > x_{v_2} > \cdots > x_{v_p}$. □

In graph $G_{v_0}(p, q)$, if $f(x, y) > 0$ is increasing in variable x and $q \geq 2$, the uniform distribution of pendent paths increases the largest weighted adjacency eigenvalue. This conclusion is the same as Theorem 1.1.

Theorem 3.4 *Assume that $f(x,y)>0$ is increasing in variable x. If $p\geq q+2\geq 4$ in graph $G_{v_0}(p,q)$, then*

$$\lambda_1(A_f(G_{v_0}(p-1,q+1)))>\lambda_1(A_f(G_{v_0}(p,q))).$$

Proof. Let $\mathbf{x}$ be a principal eigenvector of $G_{v_0}(p,q)$. First, we claim that there is no integer $1\leq i\leq q-1$ such that $x_{u_i}=x_{v_{i+1}}$. By way of contradiction, suppose that there exists an integer i_0 such that $x_{u_{i_0}}=x_{v_{i_0+1}}$. Deleting edges $u_{i_0}u_{i_0+1}$ and $v_{i_0+1}v_{i_0+2}$ and adding edges $u_{i_0}v_{i_0+2}$ and $v_{i_0+1}u_{i_0+1}$ in graph $G_{v_0}(p,q)$, we obtain graph $G_{v_0}(p-1,q+1)$. It is not difficult for us to have

$$\frac{1}{2}(\mathbf{x}^TA_f(G_{v_0}(p-1,q+1))\mathbf{x}-\mathbf{x}^TA_f(G_{v_0}(p,q))\mathbf{x})$$

$$=f(2,d_{v_{i_0+2}})x_{u_{i_0}}x_{v_{i_0+2}}+f(2,d_{u_{i_0+1}})x_{v_{i_0+1}}x_{u_{i_0+1}}-f(2,d_{u_{i_0+1}})x_{u_{i_0}}x_{u_{i_0+1}}-f(2,d_{v_{i_0+2}})x_{v_{i_0+1}}x_{v_{i_0+2}}$$
$$=0.$$

This means that $\mathbf{x}$ is a principal eigenvector of $G_{v_0}(p-1,q+1)$. Thus

$$\begin{cases}\lambda_1(A_f(G_{v_0}(p,q)))x_{u_{i_0}}=f(2,d_{u_{i_0-1}})x_{u_{i_0-1}}+f(2,d_{u_{i_0+1}})x_{u_{i_0+1}},\\ \lambda_1(A_f(G_{v_0}(p-1,q+1)))x_{u_{i_0}}=f(2,d_{u_{i_0-1}})x_{u_{i_0-1}}+f(2,d_{v_{i_0+2}})x_{v_{i_0+2}}.\end{cases}$$

If $d_{u_{i_0+1}}=2$, since $p\geq q+2$, then we have $d_{v_{i_0+2}}=2$. Thus $x_{u_{i_0+1}}=x_{v_{i_0+2}}$. Repeat this process, we can get $x_{u_{q-1}}=x_{v_q}$. So

$$\begin{cases}\lambda_1(A_f(G_{v_0}(p,q)))x_{u_{q-1}}=f(2,d_{u_{q-2}})x_{u_{q-2}}+f(2,1)x_{u_q},\\ \lambda_1(A_f(G_{v_0}(p-1,q+1)))x_{u_{q-1}}=f(2,d_{u_{q-2}})x_{u_{q-2}}+f(2,2)x_{v_{q+1}}.\end{cases}$$

Because $f(x,y)$ is increasing in variable x, this means that $x_{v_{q+1}}\leq x_{u_q}$. However, recall that $\mathbf{x}$ is a principal eigenvector of $A_f(G_{v_0}(p,q))$, hence

$$\begin{cases}\lambda_1(A_f(G_{v_0}(p,q)))x_{u_q}=f(2,1)x_{u_{q-1}},\\ \lambda_1(A_f(G_{v_0}(p,q)))x_{v_{q+1}}=f(2,2)x_{v_q}+f(2,d_{v_{q+2}})x_{v_{q+2}}.\end{cases}$$

From Lemma 2.2, $x_{v_{q+2}}>0$ and $f(x,y)$ is increasing in variable x, we can deduced that $x_{v_{q+1}}>x_{u_q}$. This is a contradiction.

If $d_{u_{i_0+1}}=1$, then $x_{u_{i_0}}=x_{u_{q-1}}$, that is $x_{u_{q-1}}=x_{v_q}$. As we discussed above, we can also have a contradiction. Next, we consider two cases.

Case 1. $x_{u_1}<x_{v_2}$.

Subcase 1.1 $x_{v_0}\geq x_{v_1}$.

Suppose that $G_{v_0}(p-1,q+1)$ is a graph obtained from $G_{v_0}(p,q)$ by deleting edges v_1v_2 and v_0u_1 and adding edges v_1u_1 and v_0v_2. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,q+1))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,q))\mathbf{x})$$

$$\begin{aligned}
&= f(2,2)x_{v_1}x_{u_1} + f(d_{v_0},2)x_{v_0}x_{v_2} - f(2,2)x_{v_1}x_{v_2} - f(d_{v_0},2)x_{v_0}x_{u_1} \\
&= (f(d_{v_0},2)x_{v_0} - f(2,2)x_{v_1})(x_{v_2} - x_{u_1}).
\end{aligned}$$

Since $f(x,y)$ is increasing in variable x, we have $f(d_{v_0},2)x_{v_0} - f(2,2)x_{v_1} \geq 0$. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,q+1))) > \lambda_1(A_f(G_{v_0}(p,q)))$.

Subcase 1.2 $x_{v_0} < x_{v_1}$.

Suppose that $G_{v_0}(p-1,q+1)$ is a graph obtained from $G_{v_0}(p,q)$ by removing all neighbors of v_0 except for u_1, v_1, to v_1. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,q+1))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,q))\mathbf{x})$$

$$\begin{aligned}
&= \sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\backslash\{u_1,v_1\}} f(d_{v_0},d_{v_j})x_{v_j}x_{v_1} + f(2,2)x_{v_0}x_{u_1} + f(d_{v_0},2)x_{v_1}x_{v_2} - \\
&\quad \sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\backslash\{u_1,v_1\}} f(d_{v_0},d_{v_j})x_{v_j}x_{v_0} - f(2,d_{v_0})x_{v_0}x_{u_1} - f(2,2)x_{v_1}x_{v_2} \\
&= \sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\backslash\{u_1,v_1\}} f(d_{v_0},d_{v_j})x_{v_j}(x_{v_1} - x_{v_0}) + (f(2,d_{v_0}) - f(2,2))(x_{v_1}x_{v_2} - x_{v_0}x_{u_1}).
\end{aligned}$$

Since $f(x,y)$ is increasing in variable x, we have $f(2,d_{v_0}) - f(2,2) \geq 0$. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,q+1))) > \lambda_1(A_f(G_{v_0}(p,q)))$.

Case 2. $x_{u_1} > x_{v_2}$.

Subcase 2.1 $x_{u_{q-1}} > x_{v_q}$.

If $x_{u_q} > x_{v_{q+1}}$, we delete edge $v_{q+1}v_{q+2}$ and add edge u_qv_{q+2} in graph $G_{v_0}(p,q)$, to obtain graph $G_{v_0}(p-1,q+1)$. It is clearly that

$$\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,q+1))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,q))\mathbf{x})$$

$$\begin{aligned}
&= f(2,2)x_{u_{q-1}}x_{u_q} + f(2,d_{v_{q+2}})x_{u_q}x_{v_{q+2}} + f(2,1)x_{v_q}x_{v_{q+1}} - f(2,1)x_{u_{q-1}}x_{u_q} \\
&\quad - f(2,d_{v_{q+2}})x_{v_{q+1}}x_{v_{q+2}} - f(2,2)x_{v_q}x_{v_{q+1}} \\
&= (f(2,2) - f(2,1))(x_{u_{q-1}}x_{u_q} - x_{v_q}x_{v_{q+1}}) + f(2,d_{v_{q+2}})x_{v_{q+2}}(x_{u_q} - x_{v_{q+1}}).
\end{aligned}$$

Since $f(x,y)$ is increasing in variable x, we have $f(2,2)-f(2,1)\geq 0$. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,q+1)))>\lambda_1(A_f(G_{v_0}(p,q)))$.

If $x_{u_q}\leq x_{v_{q+1}}$, we delete edges $u_{q-1}u_q$ and v_qv_{q+1} and add edges $u_{q-1}v_{q+1}$ and u_qv_q in graph $G_{v_0}(p,q)$, to obtain graph $G_{v_0}(p-1,q+1)$. It is clearly that

$$\frac{1}{2}(\mathbf{x}^TA_f(G_{v_0}(p-1,q+1))\mathbf{x}-\mathbf{x}^TA_f(G_{v_0}(p,q))\mathbf{x})$$

$$\begin{aligned}&=f(2,2)x_{u_{q-1}}x_{v_{q+1}}+f(2,1)x_{u_q}x_{v_q}-f(2,1)x_{u_{q-1}}x_{u_q}-f(2,2)x_{v_q}x_{v_{q+1}}\\&=(f(2,2)x_{v_{q+1}}-f(2,1)x_{u_q})(x_{u_{q-1}}-x_{v_q}).\end{aligned}$$

Since $f(x,y)$ is increasing in variable x, we have $f(2,2)x_{v_{q+1}}-f(2,1)x_{u_q}\geq 0$. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,q+1)))>\lambda_1(A_f(G_{v_0}(p,q)))$.

Subcase 2.2 $x_{u_{q-1}}<x_{v_q}$.

If $q=3$, we delete edges u_1u_2 and v_2v_3 and add edges v_2u_2 and u_1v_3 in graph $G_{v_0}(p,3)$, to obtain $G_{v_0}(p-1,4)$. We have

$$\frac{1}{2}(\mathbf{x}^TA_f(G_{v_0}(p-1,4))\mathbf{x}-\mathbf{x}^TA_f(G_{v_0}(p,3))\mathbf{x})$$

$$\begin{aligned}&=f(2,2)x_{u_1}x_{v_3}+f(2,2)x_{u_2}x_{v_2}-f(2,2)x_{u_1}x_{u_2}-f(2,2)x_{v_2}x_{v_3}\\&=f(2,2)(x_{u_1}-x_{v_2})(x_{v_3}-x_{u_2})\\&>0.\end{aligned}$$

From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,4)))>\lambda_1(A_f(G_{v_0}(p,3)))$.

If $q\geq 4$, we consider two induced paths $u_1u_2\ldots u_{q-1}$ and $v_2v_3\ldots v_q$. Let $x_{u_1},x_{u_{q-1}}$ and x_{v_2},x_{v_q} be the initial conditions. We obtain

$$\begin{aligned}x_{u_i}&=\frac{1}{t_2^{q-1}-t_1^{q-3}}((x_{u_1}t_2^q-x_{u_{q-1}}t_2^2)t_1^i+(x_{u_{q-1}}-x_{u_1}t_1^{q-2})t_2^i)\\&=\frac{1}{t_2^{q-1}-t_1^{q-3}}((t_2^{q-i}-t_1^{q-2-i})x_{u_1}+(t_2^i-t_1^{i-2})x_{u_{q-1}})\end{aligned}$$

and

$$x_{v_{i+1}}=\frac{1}{t_2^{q-1}-t_1^{q-3}}((t_2^{q-i}-t_1^{q-2-i})x_{v_2}+(t_2^i-t_1^{i-2})x_{v_q}).$$

Suppose that

$$\begin{aligned}g(i)=(x_{v_{i+1}}-x_{u_i})(t_2^{q-1}-t_1^{q-3})&=(t_2^{q-i}-t_1^{q-2-i})(x_{v_2}-x_{u_1})+(t_2^i-t_1^{i-2})(x_{v_q}-x_{u_{q-1}})\\&=(t_2^{q-i}-t_2^{i+2-q})(x_{v_2}-x_{u_1})+(t_2^i-t_2^{2-i})(x_{v_q}-x_{u_{q-1}}).\end{aligned}$$

By calculating the first-order derivative of $g(i)$, since $x_{u_1} > x_{v_2}$ and $x_{u_{q-1}} < x_{v_q}$, we have

$$g^{'}(i) = (x_{u_1} - x_{v_2})(t_2^{q-i} \ln t_2 + t_2^{i+2-q} \ln t_2) + (x_{v_q} - x_{u_{q-1}})(t_2^{i} \ln t_2 + t_2^{2-i} \ln t_2) > 0.$$

Because $g(1) < 0, g(q-1) > 0$, it means that there exists an integer $1 \le i \le q-2$ such that $x_{u_i} > x_{v_{i+1}}$ and $x_{u_{i+1}} < x_{v_{i+2}}$. Deleting edges $u_i u_{i+1}$ and $v_{i+1}v_{i+2}$ and adding edges $u_i v_{i+2}$ and $v_{i+1}u_{i+1}$ in graph $G_{v_0}(p, q)$, we obtain graph $G_{v_0}(p-1, q+1)$. It follows that

$$\begin{aligned}
&\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1, q+1))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p, q))\mathbf{x}) \\
&= f(2,2)x_{u_i}x_{v_{i+2}} + f(2,2)x_{u_{i+1}}x_{v_{i+1}} - f(2,2)x_{u_i}x_{u_{i+1}} - f(2,2)x_{v_{i+1}}x_{v_{i+2}} \\
&= f(2,2)(x_{u_i} - x_{v_{i+1}})(x_{v_{i+2}} - x_{u_{i+1}}) \\
&> 0.
\end{aligned}$$

From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1, q+1))) > \lambda_1(A_f(G_{v_0}(p, q)))$.

In each case, we have the largest weighted adjacency eigenvalue of $G_{v_0}(p-1, q+1)$ is greater than the largest weighted adjacency eigenvalue of $G_{v_0}(p, q)$, when $f(x, y)$ is increasing in invariable x and $p \ge q+2 \ge 4$. This completes the proof. □

Remark 2. Theorem 3.4 is suitable for the edge-weight functions $f(x, y)$ from fourteen indices in Table 1, including the first Zagreb index, second Zagreb index, first hyper-Zagreb index, second hyper-Zagreb index, reciprocal sum-connectivity index, reciprocal Randić index, first Gourava index, second Gourava index, first hyper-Gourava index, second hyper-Gourava index, product-connectivity Gourava index, forgotten index, Sombor index and inverse sum index.

From Theorem 3.4, when $q \ge 2$, we have proved the Li-Feng transformation. In fact, there are still two cases: $q = 1$ and $q = 0$ that have to be considered. If $q = 0$, based on the condition"$f(x, y) > 0$ is increasing in invariable x", we add a condition "the second-order derivative $f_{xx} \ge 0$", then obtain a result as below.

Theorem 3.5 *If $f(x, y) > 0$ is increasing in invariable x, $f_{xx} \ge 0$ and $\lambda_1(A_f(G_{v_0}(p, 0)) > 2f(2, 2)$, then*

$$\lambda_1(A_f(G_{v_0}(p-1, 1))) > \lambda_1(A_f(G_{v_0}(p, 0))),$$

where $p \ge 3$.

Proof. Let $\mathbf{x}$ be a principal eigenvector of $A_f(G_{v_0}(p, 0))$. Now, we consider two cases.

Case 1. $x_{v_0} \ge x_{v_1}$.

Suppose that $G_{v_0}(p-1,1)$ is a graph obtained from $G_{v_0}(p,0)$ by deleting edge $v_p v_{p-1}$ and add edge $v_p v_0$. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,1))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,0))\mathbf{x})$$

$$= \sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} f(d_{v_0}+1, d_{v_j})x_{v_j}x_{v_0} + f(d_{v_0}+1,1)x_{v_0}x_{v_p} + f(d_{v_0}+1,2)x_{v_0}x_{v_1} + f(2,1)x_{v_{p-2}}x_{v_{p-1}}$$
$$- \sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} f(d_{v_0}, d_{v_j})x_{v_j}x_{v_0} - f(2,1)x_{v_p}x_{v_{p-1}} - f(d_{v_0},2)x_{v_0}x_{v_1} - f(2,2)x_{v_{p-1}}x_{v_{p-2}}$$
$$= \sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} (f(d_{v_0}+1, d_{v_j}) - f(d_{v_0}, d_{v_j}))x_{v_j}x_{v_0} + (f(d_{v_0}+1,1)x_{v_0} - f(2,1)x_{v_{p-1}})x_{v_p} +$$
$$(f(d_{v_0}+1,2) - f(d_{v_0},2))x_{v_0}x_{v_1} - (f(2,2) - f(2,1))x_{v_{p-1}}x_{v_{p-2}}.$$

From Lemma 3.3 and $x_{v_0} \geq x_{v_1}$, it follows that $x_{v_0} > x_{v_{p-1}}$ and $x_{v_1} \geq x_{v_{p-2}}$. Combine with $f(x,y) > 0$ is increasing in invariable x and $f_{xx} \geq 0$, from Lemma 2.1, we get $\lambda_1(A_f(G_{v_0}(p-1,1))) > \lambda_1(A_f(G_{v_0}(p,0)))$.

Case 2. $x_{v_0} < x_{v_1}$.

Subcase 2.1 $x_{v_0} \leq x_{v_2}$.

Suppose that $G_{v_0}(p-1,1)$ is a graph obtained from $G_{v_0}(p,0)$ by removing all neighbors of v_0 except for v_1, to v_1. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,1))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,0))\mathbf{x})$$

$$= \sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} f(d_{v_0}+1, d_{v_j})x_{v_j}x_{v_1} + f(d_{v_0}+1,1)x_{v_0}x_{v_1} + f(d_{v_0}+1,2)x_{v_1}x_{v_2} -$$
$$\sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} f(d_{v_0}, d_{v_j})x_{v_j}x_{v_0} - f(d_{v_0},2)x_{v_0}x_{v_1} - f(2,2)x_{v_1}x_{v_2}$$
$$\geq \sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} (f(d_{v_0}+1, d_{v_j})x_{v_1} - f(d_{v_0}, d_{v_j})x_{v_0})x_{v_j} + (f(d_{v_0}+1,2) - f(2,2))x_{v_1}x_{v_2} -$$
$$(f(d_{v_0},2) - f(1,2))x_{v_0}x_{v_1}.$$

Since $f(x,y) > 0$ is increasing in invariable x and $f_{xx} \geq 0$, the above inequality is greater than 0. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,1))) > \lambda_1(A_f(G_{v_0}(p,0)))$.

Subcase 2.2 $x_{v_0} > x_{v_2}$.

Suppose that $G_{v_0}(p-1,1)$ is a graph obtained from $G_{v_0}(p,0)$ by removing all neighbors of v_0 except for v_1, to v_1, and deleting edge v_2v_3 and adding edge v_0v_3. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,1))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,0))\mathbf{x})$$

$$= \sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} f(d_{v_0}+1, d_{v_j})x_{v_j}x_{v_1} + f(d_{v_0}+1,1)x_{v_1}x_{v_2} + f(d_{v_0}+1,2)x_{v_0}x_{v_1} + f(2,2)x_{v_0}x_{v_3}$$
$$- \sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} f(d_{v_0}, d_{v_j})x_{v_j}x_{v_0} - f(2,2)x_{v_1}x_{v_2} - f(d_0,2)x_{v_0}x_{v_1} - f(2,2)x_{v_2}x_{v_3}$$
$$\geq \sum_{v_j \in N_{G_{v_0}(p,0)}(v_0)\backslash\{v_1\}} (f(d_{v_0}+1, d_{v_j})x_{v_1} - f(d_{v_0}, d_{v_j})x_{v_0})x_{v_j} + (f(2,2)x_{v_0} - f(2,2)x_{v_2})x_{v_3} +$$
$$((f(d_{v_0}+1,2) - f(d_{v_0},2))x_{v_0} - (f(2,2) - f(2,1))x_{v_2})x_{v_1}.$$

Since $f(x,y) > 0$ is increasing in invariable x and $f_{xx} \geq 0$, the above inequality is greater than 0. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,1))) > \lambda_1(A_f(G_{v_0}(p,0)))$.

In each case, we have the largest weighted adjacency eigenvalue of $G_{v_0}(p-1,1)$ is greater than the largest weighted adjacency eigenvalue of $G_{v_0}(p,0)$, when $f(x,y)$ is increasing in invariable x and $f_{xx} \geq 0$. Our proof is completed. □

Remark 3. If $f(x,y) > 0$ is increasing in variable x and graph $G_{v_0}(p,0)$ is not a tree, from Lemma 2.4, then we have $\lambda_1(A_f(G_{v_0}(p,0))) > 2f(2,2)$. Thus, Theorem 3.5 is hold for the edge-weight functions $f(x,y)$ from ten indices in Table 1, including the first Zagreb index, second Zagreb index, first hyper-Zagreb index, second hyper-Zagreb index, first Gourava index, second Gourava index, first hyper-Gourava index, second hyper-Gourava index, forgotten index and Sombor index, when $f(x,y)$ is increasing in invariable x, $f_{xx} \geq 0$ and graph $G_{v_0}(p,0)$ is not a tree.

If $q = 1$, the result below tells us that the largest weighted adjacency eigenvalue of $G_{v_0}(p-1,2)$ is greater than the largest weighted adjacency eigenvalue of $G_{v_0}(p,1)$.

Theorem 3.6 *If $f(x,y) > 0$ is increasing in invariable x and $f(d_{v_0},2) - f(2,2) > f(d_{v_0},1) - f(2,1)$, then*

$$\lambda_1(A_f(G_{v_0}(p-1,2))) > \lambda_1(A_f(G_{v_0}(p,1))),$$

where $p \geq 3$.

Proof. Let $\mathbf{x}$ be a principal eigenvector of $A_f(G_{v_0}(p,1))$. Now, we consider two cases.

Case 1. $x_{u_1} < x_{v_2}$.

Subcase 1.1 $x_{v_0} \geq x_{v_1}$.

Suppose that $G_{v_0}(p-1,2)$ is a graph obtained from $G_{v_0}(p,1)$ by deleting edges v_1v_2, v_0u_1 and adding edges v_1u_1, v_0v_2. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,2))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,1))\mathbf{x})$$

$$
\begin{aligned}
&= f(2,1)x_{v_1}x_{u_1} + f(d_{v_0},2)x_{v_0}x_{v_2} - f(2,2)x_{v_1}x_{v_2} - f(d_{v_0},1)x_{v_0}x_{u_1}\\
&= x_{v_0}(f(d_{v_0},2)x_{v_2} - f(d_{v_0},1)x_{u_1}) - x_{v_1}(f(2,2)x_{v_2} - f(2,1)x_{u_1})\\
&\geq x_{v_0}((f(d_{v_0},2) - f(2,2))x_{v_2} - (f(d_{v_0},1) - f(2,1))x_{u_1}).
\end{aligned}
$$

Since $f(x,y) > 0$ is increasing in invariable x, thus $f(2,2)x_{v_2} - f(2,1)x_{u_1} > 0$, the inequality above is hold. We know that $f(d_{v_0},2) - f(2,2) > f(d_{v_0},1) - f(2,1)$, it follows that the inequality above is greater than 0. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,2))) > \lambda_1(A_f(G_{v_0}(p,1)))$.

Subcase 1.2 $x_{v_0} < x_{v_1}$.

Suppose that $G_{v_0}(p-1,2)$ is a graph obtained from $G_{v_0}(p,1)$ by removing all neighbors of v_0 except for u_1, v_1 to v_1. We have

$$
\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,2))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,1))\mathbf{x})
$$

$$
\begin{aligned}
&= \sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\setminus\{u_1,v_1\}} f(d_{v_0},d_{v_j})x_{v_j}x_{v_1} + f(2,1)x_{v_0}x_{u_1} + f(d_{v_0},2)x_{v_1}x_{v_2} -\\
&\sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\setminus\{u_1,v_1\}} f(d_{v_0},d_{v_j})x_{v_j}x_{v_0} - f(1,d_{v_0})x_{v_0}x_{u_1} - f(2,2)x_{v_1}x_{v_2}\\
&= \sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\setminus\{u_1,v_1\}} f(d_{v_0},d_{v_j})x_{v_j}(x_{v_1} - x_{v_0}) + (f(2,d_{v_0}) - f(2,2))x_{v_1}x_{v_2} -\\
&(f(1,d_{v_0}) - f(2,1))x_{v_0}x_{u_1}.
\end{aligned}
$$

Since $f(d_{v_0},2) - f(2,2) > f(d_{v_0},1) - f(2,1)$, the equality above is greater than 0. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,2))) > \lambda_1(A_f(G_{v_0}(p,1)))$.

Case 2. $x_{u_1} \geq x_{v_2}$.

Subcase 2.1 $x_{v_0} \geq x_{v_1}$.

Suppose that $G_{v_0}(p-1,2)$ is a graph obtained from $G_{v_0}(p,1)$ by deleting edge v_2v_3 and adding edge u_1v_3. We have

$$
\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,2))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,1))\mathbf{x})
$$

$$
\begin{aligned}
&= f(d_{v_0},2)x_{v_0}x_{u_1} + f(2,2)x_{u_1}x_{v_3} + f(2,1)x_{v_1}x_{v_2} - f(d_{v_0},1)x_{v_0}x_{u_1} - f(2,2)x_{v_2}x_{v_3} - f(2,2)x_{v_1}x_{v_2}\\
&= f(2,2)x_{v_3}(x_{u_1} - x_{v_2}) + (f(d_{v_0},2) - f(d_{v_0},1))x_{v_0}x_{u_1} - (f(2,2) - f(2,1))x_{v_1}x_{v_2}\\
&> 0.
\end{aligned}
$$

From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,2))) > \lambda_1(A_f(G_{v_0}(p,1)))$.

Subcase 2.2 $x_{v_0} < x_{v_1}$.

Suppose that $G_{v_0}(p-1,2)$ is a graph obtained from $G_{v_0}(p,1)$ by removing all neighbors of v_0 except for v_1, to v_1, and deleting edges v_1v_2, v_2v_3 and adding edges v_0v_2, u_1v_3. We have

$$\frac{1}{2}(\mathbf{x}^T A_f(G_{v_0}(p-1,2))\mathbf{x} - \mathbf{x}^T A_f(G_{v_0}(p,1))\mathbf{x})$$

$$= \sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\setminus\{u_1,v_1\}} f(d_{v_0}, d_{v_j})x_{v_j}x_{v_1} + f(2,1)x_{v_0}x_{v_2} + f(d_{v_0},2)x_{v_1}x_{u_1} + f(2,2)x_{u_1}x_{v_3} -$$
$$\sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\setminus\{u_1,v_1\}} f(d_{v_0}, d_{v_j})x_{v_j}x_{v_0} - f(2,2)x_{v_1}x_{v_2} - f(1,d_{v_0})x_{v_0}x_{u_1} - f(2,2)x_{v_2}x_{v_3}$$
$$= \sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\setminus\{u_1,v_1\}} f(d_{v_0}, d_{v_j})x_{v_j}(x_{v_1} - x_{v_0}) + f(2,2)x_{v_3}(x_{u_1} - x_{v_2}) +$$
$$x_{u_1}(f(2,d_{v_0})x_{v_1} - f(d_{v_0},1)x_{v_0}) - x_{v_2}(f(2,2)x_{v_1} - f(2,1)x_{v_0})$$
$$\geq \sum_{v_j \in N_{G_{v_0}(p,q)}(v_0)\setminus\{u_1,v_1\}} f(d_{v_0}, d_{v_j})x_{v_j}(x_{v_1} - x_{v_0}) + f(2,2)x_{v_3}(x_{u_1} - x_{v_2}) +$$
$$x_{u_1}((f(2,d_{v_0}) - f(2,2))x_{v_1} - (f(1,d_{v_0}) - f(2,1))x_{v_0})$$

Since $f(x,y) > 0$ is increasing in invariable x, thus $f(2,2)x_{v_1} - f(2,1)x_{u_0} > 0$, the inequality above is hold. Recall that $f(d_{v_0},2) - f(2,2) > f(d_{v_0},1) - f(2,1)$. It follows that the inequality above is greater than 0. From Lemma 2.1, thus $\lambda_1(A_f(G_{v_0}(p-1,2))) > \lambda_1(A_f(G_{v_0}(p,1)))$.

In each case, we have the largest weighted adjacency eigenvalue of $G_{v_0}(p-1,2)$ is greater than the largest weighted adjacency eigenvalue of $G_{v_0}(p,1)$, when $f(x,y)$ is increasing in invariable x and $f(d_{v_0},2) - f(2,2) > f(d_{v_0},1) - f(2,1)$. This completes the proof. □

Remark 4. Theorem 3.6 covers the edge-weight functions $f(x,y)$ from ten indices in Table 1, including the second Zagreb index, first hyper-Zagreb index, second hyper-Zagreb index, reciprocal Randić index, first Gourava index, second Gourava index, first hyper-Gourava index, second hyper-Gourava index, product-connectivity Gourava index and inverse sum index.

4 Concluding

In this paper, we mainly consider the Li-Feng transformation and the cycle version of Li-Feng transformation of the largest eigenvalue of $A_f(G)$, respectively. As we can see, if $k \geq l \geq 4$ and $\lambda_1(A_f(G^c_{v_0}(k,l))) > 2f(2,2)$, then $\lambda_1(A_f(G^c_{v_0}(k+1,l-1))) > \lambda_1(A_f(G^c_{v_0}(k,l)))$. This means that the uniform distribution of pendent cycles decreases the largest weighted adjacency eigenvalue. When $p-2 \geq q \geq 2$ and $f(x,y)$ is increasing in variable x, we have $\lambda_1(A_f(G_{v_0}(p-1,q+1))) > \lambda_1(A_f(G_{v_0}(p,q)))$. When $q = 0,1$ and $f(x,y)$ is increasing in variable x, we add some restrictions to $f(x,y)$ and have $\lambda_1(A_f(G_{v_0}(p-1,1))) > \lambda_1(A_f(G_{v_0}(p,0)))$, $\lambda_1(A_f(G_{v_0}(p-1,2))) > \lambda_1(A_f(G_{v_0}(p,1)))$, respectively. This means that the uniform distribution of pendent paths increases the largest weighted adjacency eigenvalue. As one can see, the graph operation that decreases or increases its largest eigenvalue is so obvious. In the future, these results will play an important role in the study of the largest eigenvalue of the weighted adjacency matrix. In fact, we can try to prove that $\lambda_1(A_f(G_{v_0}(p-1,q+1)))$ is greater than $\lambda_1(A_f(G_{v_0}(p,q)))$ when $p-2 \geq q \geq 0$ and $f(x,y)$ is increasing in variable x.

References

[1] A. Berman, X. Zhang, On the spectral radius of graphs with cut vertices, *J. Combin. Theory Ser. B* 83 (2001) 233–240.

[2] R. Cruz, J. Rada, W. Sanchez, Extremal unicyclic graphs with respect to vertex-degree-based topological indices, *MATCH Commun. Math. Comput. Chem.* 88 (2022) 481–503.

[3] D. Cvetković, S. Simić, Towards a spectral theory of graphs based on the signless Laplacian I, *Publ. Inst. Math.*(Beograd) 85 (2009) 19–33.

[4] K. Das, I. Gutman, I. Milovanović, E. Milovanović, B. Furtula, Degree-based energies of graphs, *Linear Algebra Appl.* 554 (2018) 185–204.

[5] E. Estrada, The ABC matrix, *J. Math. Chem.* 55 (2017) 1021–1033.

[6] W. Gao, Trees with maximum vertex-degree-based topological indices, *MATCH Commun. Math. Comput. Chem.* 88 (2022) 535–552.

[7] J. Gao, X. Li, N. Yang, The effect on the largest eigenvalue of degree-based weighted adjacency matrix by perturbations, *Bull. Malays. Math. Sci. Soc.* 47 (2024) 30.

[8] J. Gao, N. Yang, Some bounds on the largest eigenvalue of degree-based weighted adjacency matrix of a graph, *Discrete Appl. Math.* 356 (2024) 21–31.

[9] S. Guo, The spectral radius of unicyclic and bicyclic graphs with n vertices and k pendant vertices, *Linear Algebra Appl.* 408 (2005) 78–85.

[10] J. Guo, The effect on the Laplacian spectral radius of a graph by adding or grafting edges, *Linear Algebra Appl.* 413 (2006) 59–71.

[11] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* 86 (2021) 11–16.

[12] I. Gutman, Spectrum and energy of the Sombor matrix, *Mil. Tech. Cour.* 69 (2021) 551–561.

[13] R. Horn, C. Johnson, Matrix Analysis, Combridge University Press, New York, 2013.

[14] Z. Hu, L. Li, X. Li, D. Peng, Extremal graphs for topological index defined by a degree-based edge-weight function, *MATCH Commun. Math. Comput. Chem.* 88 (2022) 505–520.

[15] Z. Hu, X. Li, D. Peng, Graphs with minimum vertex-degree function-index for convex functions, *MATCH Commun. Math. Comput. Chem.* 88 (2022) 521–533.

[16] J. Huang, X. Geng, S. Li, Z, Zhou, On spectral extrema of graphs with given order and dissociation number, *Discrete Appl. Math.* 342 (2024) 368–380.

[17] X. Li, Indices, polynomials and matrices - a unified viewpoint, *Invited talk at the 8th Slovenia Conf. Graph Theory*, Kranjska Gora, June 21–27, 2015.

[18] Q. Li, K. Feng, On the largest eigenvalue of graphs (in Chinese), *Acta Math. Appl. Sin.* 2 (1979) 167–175.

[19] X. Li, Y. Li, J. Song, The asymptotic value of graph energy for random graphs with degree-based weights, *Discrete Appl. Math.* 284 (2020) 481–488.

[20] X. Li, Y. Li, Z. Wang, The asymptotic value of energy for matrices with degree-distance-based entries of random graphs, *Linear Algebra Appl.* 603 (2020) 390–401.

[21] X. Li, Y. Li, Z. Wang, Asymptotic values of four Laplacian-type energies for matrices with degree distance-based entries of random graphs, *Linear Algebra Appl.* 612 (2021) 318–333.

[22] X. Li, D. Peng, Extremal problems for graphical function-indices and f-weighted adjacency matrix, *Discrete Math. Lett.* 9 (2022) 57–66.

[23] X. Li, Z. Wang, Trees with extremal spectral radius of weighted adjacency matrices among trees weighted by degree-based indices, *Linear Algebra Appl.* 620 (2021) 61–75.

[24] X. Li, N. Yang, Some interlacing results on weighted adjacency matrices of graphs with degree-based edge-weights, *Discrete Appl. Math.* 333 (2023) 110–120.

[25] X. Li, N. Yang, Unified approach for spectral properties of weighted adjacency matrices for graphs with degree-based edge-weights, *Linear Algebra Appl.* 696 (2024) 46–67.

[26] X. Li, N. Yang, Spectral properties and energy of weighted adjacency matrix for graphs with a degree-based edge-weight function, *Acta Math. Sin. (Ser.En)*, 40(12) (2024) 3027–3042.

[27] W. Lin, X. Guo, Ordering trees by their largest eigenvalues, *Linear Algebra Appl.* 418 (2006) 450–456.

[28] H. Lin, X. Huang, J. Xue, A note on the α-spectral radius of graphs, *Linear Algebra Appl.* 557 (2018) 430–437.

[29] C. Liu, J. Li, Z. Xie, The minimum spectral radius of graphs with a given domination number, *Linear Algebra Appl.* 673 (2023) 233–258.

[30] R. Liu, M. Zhai, J. Shu, The least eigenvalue of unicyclic graphs with n vertices and k pendant vertices, *Linear Algebra Appl.* 461 (2009) 657–665.

[31] J. Rodríguez, A spectral approach to the Randić index, *Linear Algebra Appl.* 400 (2005) 339–344.

[32] J. Rodríguez, J. Sigarreta, On the Randić index and conditional parameters of a graph, *MATCH Commun. Math. Comput. Chem.* 54 (2005) 403–416.

[33] K. Simić, M. Li Marzi, F. Belardo, On the index of caterpillars, *Discrete Math.* 308 (2008) 324–330.

[34] H. Song, Q. Wang, L. Tian, New upper bounds on the spectral radius of trees with the given number of vertices and maximum degree, *Linear Algebra Appl.* 439 (2013) 2527–2541.

[35] D. Stevanović, P. Hansen, The minimum spectral radius of graphs with a given clique number, *Electron. J. Linear Algebra* 17 (2008) 110–117.

[36] B. Wu, E. Xiao, Y. Hong, The spectral radius of trees on k pendant vertices, *Linear Algebra Appl.* 395 (2005) 343–349.

[37] J. Xue, R. Liu, J. Shu, Unimodality of principal eigenvector and its applications, *Graph Combinator.* 36 (2020) 1177–1188.

[38] A. Yu, Ordering trees by their spectral radii, *Acta Math. Sin. (Ser.En)* 30 (2014) 1107–1112.

[39] R. Zheng, X. Guan, X. Jin, Extremal trees and unicyclic graphs with respect to spectral radius of weighted adjacency matrices with property P^*, *J. Appl. Math. Comput.* 69 (2023) 2573–2594.

[40] L. Zheng, G. Tian, S. Cui, On spectral radius and energy of arithmetic-geometric matrix of graphs, *MATCH Commun. Math. Comput. Chem.* 83 (2020) 635–650.