Some digraph classes that meet the directed path partition conjecture

Jiawen Bo*, Zhipeng Gao†, Xiaopan Lian‡, Jianing Liu§

February 21, 2025

Abstract

Let $\lambda(D)$ denote the order of a longest path in a digraph D. For disjoint $A, B \subseteq V(D)$, if $A \cup B = V(D)$, we say that (A, B) is a partition of D. The Directed Path Partition Conjecture (DPPC) states that for every digraph D and every integer q with $q < \lambda(D)$, there is a partition (A, B) of D such that $\lambda(D[A]) \leq q$ and $\lambda(D[B]) \leq \lambda(D) - q$.

Arroyo and Galeana-Sánchez [2] proved that every strong 3-quasi-transitive digraph satisfies the DPPC. They also showed that the DPPC holds for compositions over an acyclic digraph with digraphs that meet the DPPC. In the paper, we show that non-strong 3-quasi-transitive digraphs and strong 4-transitive digraphs also satisfy the DPPC. Additionally, we show that the DPPC holds for the compositions over a unicyclic digraph with digraphs that satisfy the DPPC and certain other conditions. Furthermore, by applying different arguments, we show that the compositions over a cycle with arbitrary digraphs meets the DPPC.

Keywords: longest path; 3-quasi-transitive; unicyclic; composition

1 Introduction

Given a digraph D, let $\lambda(D)$ denote the order of a longest path in D. A *partition* of D consists of a pair (A, B) of disjoint subsets of $V(D)$ such that $A \cup B = V(D)$. In this paper, all the paths and cycles are considered to be directed.

We consider the so-called Directed Path Partition Conjecture (DPPC), which states as follows:

Conjecture 1.1. ***DPPC:*** *For every digraph D and any choice of positive integer $q < \lambda(D)$, there exists a partition (A, B) of D such that $\lambda(D[A]) \leq q$ and $\lambda(D[B]) \leq \lambda(D) - q$.*

The analogous conjecture for undirected graphs, known as the Path Partition Conjecture (PPC), was initially formulated in 1981 and remains open to this day. Several results supporting the PPC

*School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P.R. China. Email: bojiawen@mail.nankai.edu.cn.

†School of Mathematics and Statistics, Xidian University, Xi'an 710126, P.R. China. Email: gaozhipeng@xidian.edu.cn.

‡Corresponding Author. Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, P.R. China. Email: Lian@nankai.edu.cn.

§School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P.R. China. Email: liujianing@mail.nankai.edu.cn.

can be obtained in a comprehensive survey in [4]. Over four decades old, the PPC has generated a host of other interesting conjectures, with the DPPC being a notable one. A vast array of literature delves deeply into the PPC and its associated problems. For a thorough examination of these topics, readers may refer to [1, 2, 3, 5, 8, 9].

A digraph D is *symmetric* if it can be obtained from an undirected graph G by replacing each edge uv by the pair (u,v), (v,u) of arcs. Note that the DPPC is equivalent to the PPC for symmetric digraphs. All these two conjectures are very difficult to attack for general digraphs (graphs), since very little can be said about the structure of longest paths in general digraphs (graphs).

Several publications, such as [2, 3], have successfully proven the DPPC for specific classes of digraphs, offering valuable insights and inspirations. Motivated by these achievements, this paper delves into several distinct classes of digraphs that adhere to the DPPC.

Denote by $u \to v$ the arc (u,v). A digraph D is *k-quasi-transitive* if for every pair of vertices u,v of D, the existence of a path from u to v of order $k+1$ in D implies that u and v are adjacent. For an integer n, without causing ambiguity, denote $\{1,\ldots,n\}$ by $[n]$. Also, for $i \le j$, denote $\{i, i+1, \ldots, j\}$ by $[i,j]$. Let H be a digraph with vertex set $\{u_1,\ldots,u_t\}$ and for every $i \in [t]$, let D_i be an arbitrary digraph. Then the *composition* over H with $D_1,\ldots,D_t$, denoted by $D = H[D_1,\ldots,D_t]$, is the digraph with vertex set $\bigcup_{i\in[t]} V(D_i)$ and arc set

$$A(D) = \left(\bigcup_{i\in[t]} A(D_i)\right) \cup \{(u,v) : u \in V(D_i) \text{ and } v \in V(D_j) \text{ with } (u_i,u_j) \in A(H), i,j \in [t]\}.$$

If each D_i is an independent set, then D is called an *extension* of H.

Arroyo and Galeana-Sánchez [2] proved the following results.

Theorem 1.2 ([2]). *Every strong 3-quasi-transitive digraph meets the DPPC.*

Theorem 1.3 ([2]). *Let $D = C_t[D_1,\ldots,D_n]$ where $C_t = v_1v_2\cdots v_tv_1$ is a cycle of order t and each D_i is an arbitrary digraph for $i \in [t]$. Suppose that both of D_1 and D_2 are independent sets, and D_i satisfies the DPPC for each $i \in [t]\setminus[2]$. Then D satisfies the DPPC.*

Motivated by the above work, now, we present our results, which also outlines the organization of the paper. In Section 2, we show that 3-quasi-transitive digraphs and strong 4-transitive digraphs meet the DPPC in Theorem 2.3 and Theorem 2.6, respectively. Since the family consisting of unicyclic digraphs contains cycles, in Section 3, we show in Theorem 3.3 that if D is the composition over a unicyclic digraph with some other digraphs which satisfy certain conditions, then D meets the DPPC. In Theorem 3.4, we show that as long as D is the composition over a cycle with arbitrary digraphs, D meets the DPPC.

Additional Notation.

Given a digraph $D = (V,A)$, denote by $|D|$ the number of vertices in D. A *k-path subdigraph* of D is a collection of k vertex-disjoint paths in D. Denote by $\lambda_k(D)$ the maximum order of a k-path subdigraph in D; clearly $\lambda(D) = \lambda_1(D)$. For $(u,v) \in A$, we say that u *dominates* v and v is *dominated by* u. For $X \subseteq V$, let $D[X]$ and $D - X$ be the subdigraphs of D induced by X and $V(D)\setminus X$, respectively. If $X = \{v\}$, then write $D - v$ for short. For $X,Y \subseteq V(D)$, denote by $X \mapsto Y$ if $(u,v) \in A(D)$ for each $u \in X$ and each $v \in Y$ and there is no arc from Y to X. If $X = \{v\}$ or $Y = \{v\}$, denote by $v \mapsto Y$ or $X \mapsto v$, respectively. For a path P in D, denote by $\mathrm{ini}(P)$ and $\mathrm{ter}(P)$ the initial vertex and terminal vertex of P, respcetively. We say that a path

P is appendable to a path Q if $V(P) \cap V(Q) = \emptyset$ and $\text{ter}(P) \rightarrow \text{ini}(Q)$. The path obtained by appending P to Q is denoted by PQ. If P or Q consists of a single vertex v, then denote by vQ or Pv, respectively. The *circumference* of D is the length of a longest cycle of D. A path is called *Hamiltonian* if it pass though every vertex of D. For a digraph which contains exactly one cycle, we call it *unicyclic*. A digraph is *strong* if for any ordered pair of vertices (u, v) there is path from u to v, otherwise, it is *non-strong*.

2 4(3-Quasi)-Transitive Digraph

2.1 3-Quasi-Transitive Digraph

To show Theorem 2.3, we need a result proved by Galeana-Sánchez, Goldfeder, and Urrutia. For an integer $n \geq 4$, let F_n be the digraph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and arc set $\{(x_1, x_2), (x_2, x_3), (x_3, x_1)\} \cup \{(x_1, x_{i+3}), (x_{i+3}, x_2) : i \in [n-3]\}$, see Figure 1. Then their result can be stated as follows.

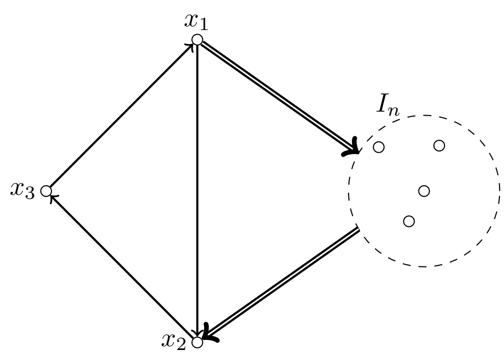

Figure 1: F_n where I_n is an independent set with vertex set $\{x_4, x_5, ..., x_n\}$

Theorem 2.1 ([6])**.** *Let D be a strong 3-quasi-transitive digraph of order n. Then D is a semicomplete digraph, or a semicomplete bipartite digraph, or isomorphic to F_n.*

For a non-strong 3-quasi-transitive digraph D, let $\mathscr{S}$, $\mathscr{B}$ and $\mathscr{F}$ be the set of all strong components of D that are semicomplete digraph of order at least 3, semicomplete bipartite digraph and isomorphic to F_n for some n, respectively. Note that by Theorem 2.1, for each strong component F of D, we have that $F \in \mathscr{S} \cup \mathscr{B} \cup \mathscr{F}$. The following helpful lemma is about the dominating relationship between strong components of D.

Lemma 2.2 ([9])**.** *Let D be a non-strong 3-quasi-transitive digraph with its distinct strong components D_1 and D_2. Suppose that there are $x \in V(D_1)$ and $y \in V(D_2)$ such that $(x, y) \in A(D)$. Then the following hold.*

(1) If $D_1 \in \mathscr{S} \cup \mathscr{F}$, then $V(D_1) \mapsto y$.

(2) If $D_2 \in \mathscr{S} \cup \mathscr{F}$, then $x \mapsto V(D_2)$.

(3) If $D_1 \in \mathscr{B}$ with $D_1 = (S_1, T_1)$ and $x \in S_1$, then $S_1 \mapsto y$.

(4) If $D_2 \in \mathscr{B}$ with $D_2 = (S_2, T_2)$ and $y \in S_2$, then $x \mapsto S_2$.

Theorem 2.3. *Every non-strong 3-quasi-transitive digraph satisfies the DPPC.*

Proof. Let D be a 3-quasi-transitive digraph with $D_1, \ldots, D_n$ being all its strong components. Furthermore, if $D_i \in \mathscr{B}$, we denote the partite sets of D_i by S_i and T_i. Observe that for each $i \in [n]$, the intersection of a path in D with D_i is either a path or empty.

For each $i \in [n]$, let $\mathcal{P}_i^{\text{in}}$ be the set of longest paths in $D - D_i$ that have an arc into D_i and let $\mathcal{P}_i^{\text{end}}$ be the set of longest paths in D that end in D_i. Denote by p_i^{in} and p_i^{end} the order of a path in $\mathcal{P}_i^{\text{in}}$ and $\mathcal{P}_i^{\text{end}}$, respectively. Observe that, for $D_i = (S_i, T_i) \in \mathscr{B}$ and for each $P \in \mathcal{P}_i^{\text{in}}$ where $i \in [n]$, by Lemma 2.2(4), we have that $\text{ter}(P) \mapsto S_i$ or $\text{ter}(P) \mapsto T_i$ or $\text{ter}(P) \mapsto V(D_i)$.

Let $I_0 = \{i \in [n] : D_i \in \mathscr{B}$ and there is a partite set of D_i such that there is no $P \in \mathcal{P}_i^{\text{in}}$ with $\text{ter}(P)$ dominating it$\}$. For $i \in I_0$, denote by S_i the partite set of D_i that is not dominated by ter(P) for all the $P \in \mathcal{P}_i^{\text{in}}$. Therefore, for each $i \in I_0$, each $P \in \mathcal{P}_i^{\text{in}}$ satisfies that $\text{ter}(P)$ dominates T_i.

Let q be a positive integer with $q < \lambda$.We define four useful sets to construct the desired partition (A, B) of D. Let

$$I := \left\{ i \in I_0 : D_i \in \mathscr{B} \text{ with } p_i^{\text{in}} = q \right\},$$
$$S := \left\{ i \in [n] \setminus I : p_i^{\text{end}} \leq q \right\},$$
$$R := \left\{ i \in [n] \setminus I : p_i^{\text{in}} \geq q \right\},$$
$$J := \left\{ i \in [n] \setminus I : p_i^{\text{in}} < q < p_i^{\text{end}} \right\}.$$

Note that I, S, R and J are pair-wise disjoint with $I \cup S \cup R \cup J = [n]$.

We proceed the proof by defining a partition (A_i, B_i) for D_i where $i \in I \cup J$.

For $i \in I$, let $A_i = S_i$ and $B_i = T_i$. Now, suppose that $i \in J$. Let $q_i = q - p_i^{\text{in}} > 0$. If $D_i \in \mathscr{S} \cup \mathscr{F}$, let (A_i, B_i) be the partition of D_i such that $\lambda(D[A_i]) \leq q_i$ and $\lambda(D[B_i]) \leq \lambda(D_i) - q_i$. If $D_i \in \mathscr{B}$, let B_i be the partite set of D_i that contains $\text{ter}(P)$ for some $P \in \mathcal{P}_i^{\text{end}}$, and A_i be the remaining partite set.

Let

$$A = \left(\bigcup_{i \in S} V(D_i) \right) \cup \left(\bigcup_{i \in I \cup J} A_i \right), \quad B = \left(\bigcup_{i \in R} V(D_i) \right) \cup \left(\bigcup_{i \in I \cup J} B_i \right).$$

Note that (A, B) is a partition of D.

We finish the proof by showing that $\lambda(D[A]) \leq q$ and $\lambda(D[B]) \leq \lambda(D) - q$.

Claim 1. $\lambda(D[A]) \leq q$.

Let α be a longest path in $D[A]$. Assume it terminates in D_i. Write $\alpha = Q\alpha_{D_i}$, where α_{D_i} is the subpath of α in D_i and Q is the remaining subpath of α in $D - D_i$.

Suppose that $i \in S$. Then by the definition of S, we have that $|\alpha| \leq p_i^{\text{end}} \leq q$. Hence, suppose that $i \in J$. If $D_i \in \mathscr{S} \cup \mathscr{F}$, then by Lemma 2.2(2), $\text{ter}(Q) \mapsto A_i$. So, $|\alpha| = |Q| + |\alpha_{D_i}| \leq |Q| + \lambda(D[A_i]) \leq p_i^{\text{in}} + q_i = q$. If $D_i \in \mathscr{B}$, then $|\alpha| = |Q| + 1 \leq p_i^{\text{in}} + 1 \leq q$. Therefore, we may assume that $i \in I$. By the definition of I_0, all the terminal vertices of the paths in $\mathcal{P}_i^{\text{in}}$ do not dominate $S_i = A_i$. We have that $|Q| \leq p_i^{\text{in}} - 1 = q - 1$. Thus, $|\alpha| = |Q| + 1 \leq p_i^{\text{in}} - 1 + 1 = q$.

Claim 2. $\lambda(D[B]) \leq \lambda(D) - q$.

Let β be a longest path in $D[B]$. Assume that it starts at D_i. Write $\beta = \beta_{D_i} Q$, where β_{D_i} is the subpath of β in D_i and Q is the remaining subpath of β in $D - D_i$.

Suppose that $i \in R$. If $D_i \in \mathscr{S} \cup \mathscr{F}$, then by Lemma 2.2(2), $\mathrm{ter}(P) \mapsto V(D_i)$ for each $P \in \mathcal{P}_i^{\mathrm{in}}$. Thus, we can append such P to β. So, $\lambda(D) \geq |P| + |\beta| = p_i^{\mathrm{in}} + |\beta| \geq |\beta| + q$. Hence, assume that $D_i \in \mathscr{B}$ and $\mathrm{ini}(\beta) \in S_i$. If there is a path $P \in \mathcal{P}_i^{\mathrm{in}}$ such that $\mathrm{ter}(P) \mapsto S_i$, then we can append P to β. Thus, $\lambda(D) \geq |P| + |\beta| = p_i^{\mathrm{in}} + |\beta| \geq |\beta| + q$. Hence, assume that $\mathrm{ter}(P) \mapsto T_i$ for all $P \in P_i^{\mathrm{in}}$. We claim that $p_i^{\mathrm{in}} \geq q + 1$. Suppose not, i.e., $p_i^{\mathrm{in}} = q$. By the definition of I, we have that $i \in I$, contradicting that $i \in R$ and $I \cap R = \emptyset$. Let $P \in \mathcal{P}_i^{\mathrm{in}}$ with $(x, \mathrm{ter}(P)) \in A(P)$ and $(\mathrm{ter}(P), y) \in A(D)$, where $y \in T_i$. Since $D_i \in \mathscr{B}$ is strong, there is a path P' of even order from y to $\mathrm{ini}(\beta)$. Then $x \to \mathrm{ini}(\beta)$ or $\mathrm{ini}(\beta) \to x$ by the fact that D is 3-quasi-transitive. We assert that $x \to \mathrm{ini}(\beta)$. Suppose that $\mathrm{ini}(\beta) \to x$. Then $P' \cup (\mathrm{ini}(\beta), x) \cup (x, \mathrm{ter}(P) \cup (\mathrm{ter}(P), y)$ is a cycle passing through at least two strong components of D, a contradiction. Thus, $(P - \mathrm{ter}(P))\beta$ is a path in D which implies that $\lambda(D) \geq |P| - 1 + |\beta| = p_i^{\mathrm{in}} - 1 + |\beta| \geq |\beta| + q$.

Suppose that $i \in J$. First, assume that $D_i \in \mathscr{S} \cup \mathscr{F}$. Let P be a path in $\mathcal{P}_i^{\mathrm{in}}$, by Lemma 2.2(2), $\mathrm{ter}(P) \mapsto V(D_i)$. Also by Lemma 2.2(1), $V(D_i) \mapsto \mathrm{ini}(Q)$. Hence, we can append P to any path in D_i, and then append the new path to Q. So, $\lambda(D) \geq p_i^{\mathrm{in}} + \lambda(D_i) + |Q|$. Because β_{D_i} is contained in B_i, we have that $|\beta| = |\beta_{D_i}| + |Q| \leq \lambda(D_i) - q_i + |Q| = \lambda(D_i) - q + p_i^{\mathrm{in}} + |Q| \leq \lambda(D) - q$. Therefore, assume that $D_i \in \mathscr{B}$. By the choice of B_i, there is a path $P \in \mathcal{P}_i^{\mathrm{end}}$ such that $\mathrm{ter}(P) \in B_i$. By Lemma 2.2(3), $B_i \mapsto \mathrm{ini}(Q)$. Thus, we can append P to Q which implies that $\lambda(D) \geq |P| + |Q| = p_i^{\mathrm{end}} + |Q| \geq q + 1 + |Q|$. Since $|\beta| = |Q| + 1$, we have that $|\beta| \leq \lambda(D) - q$.

Suppose that $i \in I$. Let P be a path in $\mathcal{P}_i^{\mathrm{in}}$. By the choice of B_i, it holds that $\mathrm{ter}(P) \mapsto B_i$. Therefore, we can append P to β which implies that $\lambda(D) \geq |P| + |\beta| = q + |\beta|$. □

Note that by Theorem 1.2 and Theorem 2.3, the following holds.

Corollary 2.4. *Every 3-quasi-transitive digraph meets the DPPC.*

2.2 Strong 4-transitive digraph

A digraph D is *cyclically k-partite* if there exists a partition $\{V_1, V_2, \ldots, V_{k-1}\}$ of $V(D)$ such that every arc of D is a $V_i V_{i+1}$-arc (modulo k). For integers m, n, let $K_{1,n}$ be the undirected graph with one vertex of degree n and all other vertices of degree 1. We call the undirected graph $D_{m,n}$ a *double star* which is obtained from $K_{1,m}$ and $K_{1,n}$ by adding an edge between the vertex of degree m in $K_{1,m}$ and the vertex of degree n in $K_{1,n}$.

Hernández-Cruz [7] fully characterized the structure of strong 4-transitive digraphs.

Theorem 2.5 ([7])**.** *Let D be a strong 4-transitive digraph. Then the following are all possible scenarios for D.*

(1) D is a complete digraph.

(2) D is a directed 3-cycle extension.

(3) D has circumference 3 which contains a directed 3-cycle extension as a spanning subdigraph with cyclic partition $\{V_0, V_1, V_2\}$. At least one symmetrical arc exists in D and for every symmetrical arc $(v_i, v_{i+1}) \in A(D)$ with $v_i \in V_i$ and $v_{i+1} \in V_{i+1}$(index under modulo 3), we have that $|V_i| = 1$ or $|V_{i+1}| = 1$.

(4) *D has circumference 3 and $UG(D)$ is not 2-edge-connected. Consider $S_1, S_2, \ldots, S_n$ the vertex set of the maximal 2-edge-connected subgraphs of $UG(D)$. Then $S_i = \{u_i\}$ for every $2 \le i \le n$, and $D[S_1]$ contains a directed 3-cycle extension as a spanning subdigraph with cyclic partition $\{V_0, V_1, V_2\}$. There exists a vertex $v_0 \in V_0$ such that $(v_0, u_j), (u_j, v_0) \in A(D)$ for every $2 \le j \le n$. Also, $|V_0| = 1$ and $D[S_1]$ has the structure described in (2) or (3), depending on the existence of symmetric arcs.*

(5) *D is the biorientation of a 5-cycle.*

(6) *D is the biorientation of a star $K_{1,n}$, $n \ge 3$.*

(7) *D is the biorientation of a double star $D_{n,m}$.*

(8) *D is a strong digraph of order at most 4 not included in the previous families of digraphs.*

Theorem 2.6. *Every strong 4-transitive digraph D satisfies the DPPC.*

Proof. Let q be a positive integer with $q < \lambda$. We discuss the cases according to the division in Theorem 2.5.

If D satisfies (1) or (5), then D has a Hamiltonian path. If D satisfies (2), then by Theorem 1.3 we are done. If D satisfies (6) or (7), let (A, B) be the bipartition of D, then sets A and B are both independent sets in D. Thus, $\lambda(D[A]) = \lambda(D[B]) = 1$.

If D satisfies (3) or (4), without loss of generality, we may assume that $|V_0| = 1$. By symmetry, it suffices to consider the case that $q \le \frac{\lambda(D)}{2}$. If there are no symmetric arcs in $D[V_1 \cup V_2]$, let $A = V_0$ and $B = V(D) \setminus A$, then we have that $\lambda(A) = 1 \le q$ and $\lambda(B) = 2 \le \lambda(D) - q$ (If $\lambda(D) = 3$, then $q = 1$. If $\lambda(D) > 3$, then $2 \le \frac{\lambda(D)}{2} \le \lambda(D) - q$). If there exists a symmetric arc in $D[V_1 \cup V_2]$, without loss of generality, assume that $|V_1| = 1$. Let $B = V_0 \cup V_1$ and $A = V(D) \setminus B$, then $\lambda(A) = 1 \le q$ and $\lambda(B) = 2 \le \lambda(D) - q$.

Therefore, suppose that D satisfies (8). Since D is strong, it is sufficient to consider that $|D| = 4$. Then D has a Hamiltonian path since D is not included in the previous families of digraphs. □

3 Unicyclic Composition

In Theorem 3.3, we prove that if D is the composition over a unicyclic digraph with some other digraphs that satisfy certain conditions, then D meets the DPPC. In Theorem 3.4, by applying a different argument, we claim that if D is the composition over a cycle with arbitrary other digraphs, then D meets the DPPC. Before devoted to the proof of these two main results, we present the following lemmas.

Lemma 3.1. *Let $D = C[D_1, D_2, \ldots, D_t]$, where $C = v_1 v_2 \ldots v_t v_1$ is a cycle and each D_i is an arbitrary digraph for $i \in [t]$. Suppose that S is a non-empty subset of $V(D_i)$ such that $|D_i - S| < |D_j|$ for some $i \in [t]$ and all $j \in [t]$. Then for any path P in $D[S]$ and any path Q in $D - S$ with $ter(Q) \in V(D_m)$ where $m \in [t]$, there exists a path P^* in D such that $ini(P^*) = ini(P)$, $ter(P^*) \in V(D_m)$ and $|P^*| \ge |P| + |Q|$.*

Proof. Observe that for each $j \in [t]$, the intersection of a path in D and D_j consists of vertex disjoint paths. Hence, for each $j \in [t]$, let $\mathcal{Q}_j$ be the set of all maximal subpaths of Q contained in D_j.

First, suppose that $\text{ini}(Q) \in V(D_i)$. Then $|\mathcal{Q}_j| \le |\mathcal{Q}_i|$ for each $j \in [t]\setminus\{i\}$. Since $|D_i - S| < |D_j|$, for each $j \in [t] \setminus \{i\}$, we have that $V(D_j) \setminus V(Q) \ne \emptyset$ or there is a path $Q_j \in \mathcal{Q}_j$ with $|Q_j| \ge 2$. Let $J_1 = \{j \in [t] \setminus \{i\} : \text{there is a } Q_j \in \mathcal{Q}_j \text{ with } |Q_j| \ge 2\}$ and $X_1 = \{\text{ini}(Q_j) : j \in J_1\}$. By the definition of D, there is a path Q' in $D[V(Q) \setminus X_1]$ satisfying that

$$V(Q') = V(Q) \setminus X_1,\ \text{ini}(Q') = \text{ini}(Q) \text{ and } \text{ter}(Q') = \text{ter}(Q).$$

For each $j \in [t] \setminus (\{i\} \cup J_1)$, let $x_j \in V(D_j) \setminus V(Q)$. Then $V(P) \cup X_1 \cup \{x_j : j \in [t] \setminus (\{i\} \cup J_1)\}$ induces a path P' in D with $|P'| = |P| + t - 1$, $\text{ini}(P') = \text{ini}(P)$ and $\text{ter}(P') \in V(D_{i-1})$. Then, $P^* := P'Q'$ is the desired path in D.

Now, suppose that $\text{ini}(Q) \in V(D_r)$ where $r \ne i$. Replacing $[t]$ in the above arguments by $[i, r-1]$ and $i-1$ by $r-1$ where index under modulo t, we conclude that D contains the desired path P^*. □

Lemma 3.2. *Let $D = C[D_1, D_2, \ldots, D_t]$ where $C = v_1v_2\ldots v_tv_1$ is a cycle and D_i is an arbitrary digraph with $|D_i| \ge 2$ for every $i \in [t]$. Suppose that $S = (\bigcup_{i=1}^{j-1} V(D_i)) \cup M_j$ where $j \in [2, n-1]$ and $\emptyset \ne M_j \subseteq V(D_j)$. Let P be a path in $D[S]$ and Q be a path in $D - S$ with ter$(Q) \in V(D_m)$ where $m \in [t]$. Then there exists a path P^* in D such that ini$(P^*) =$ ini(P), ter$(P^*) \in V(D_m)$ and $|P^*| \ge |P| + |Q|$.*

Proof. Let $\text{ter}(P) \in V(D_i)$ and $\text{ini}(Q) \in V(D_k)$ where $i, k \in [t]$. Suppose that $i \ne k$. Let $x_r \in V(D_r)$ where $r \in [i+1, k-1]$. Then the subdigraph induced by $V(P) \cup \{x_r : r \in [i+1, k-1]\} \cup V(Q)$ of D contains the desired path. Therefore, we may assume that $\text{ter}(P)$ and $\text{ini}(Q)$ are both in D_j which implies that $|M_j| < |D_j|$. Let P_j and Q_j be the subpath of P and Q in D_j, respectively.

Let $X = P$ for $r \in [j-1]$ and $X = Q$ for $r \in [t] \setminus [j]$. For $r \in [t] \setminus \{j\}$, if X intersects D_r, let X_r be the subpath of X in D_r. Otherwise, let X_r be the empty digraph. For $r \in [t] \setminus \{j\}$, if $|X_r| < |D_r|$, let $u_r \in V(D_r) \setminus V(X_r)$ and $P_r^* = X_r$; if $|X_r| = |D_r|$, let $u_r = \text{ter}(X_r)$ and $P_r^* = X_r - \text{ter}(X_r)$. Finally, let $P^* = P_1^* \cdots P_{j-1}^* Q_j u_{j+1} \cdots u_t u_1 \cdots u_{j-1} P_j P_{j+1}^* \cdots P_t^*$. □

For a digraph D and $q \in [\lambda(D) - 1]$, let $M_q(D)$ be a maximal subdigraph of D such that $\lambda(M_q(D)) = q$.

Theorem 3.3. *Let $D = T[D_1, D_2, \ldots, D_t, \ldots, D_n]$ where T is a unicyclic digraph with $C = v_1v_2\cdots v_tv_1$ being the unique cycle of T and each D_i is an arbitrary digraph for $i \in [n]$. Suppose that both of the following hold:*

(a) $|D_1| = \cdots = |D_t|$;

(b) each D_i where $i \in [n] \setminus [t]$ satisfies the DPPC.

Then D satisfies the DPPC.

Proof. Let $D' = C[D_1, \ldots, D_t]$ and $|D_1| = |D_2| = \cdots = |D_t| = k$.

For $i \in [n]$, define p_i^{in} as the order of a longest path in $D - D_i$ whose terminal vertex has an arc into D_i when $i \in [n] \setminus [t]$ and the order of a longest path in $D - D'$ whose terminal vertex has an arc into D_i when $i \in [t]$. For $i \in [n] \setminus [t]$, define p_i^{end} as the order of a longest path in D that ends in D_i.

Let q be a positive integer with $q < \lambda$. To construct the desired partition of $V(D)$, we define the following sets. Let

$$S = \left\{ i \in [n] \setminus [t] : p_i^{\text{end}} \le q \right\},$$
$$R = \left\{ i \in [n] \setminus [t] : p_i^{\text{in}} \ge q \right\},$$
$$J = \left\{ i \in [n] \setminus [t] : p_i^{\text{in}} < q < p_i^{\text{end}} \right\}.$$

Note that S, R and J are pair-wise disjoint with $S \cup R \cup J = [n] \setminus [t]$. For $i \in J$, let $q_i = q - p_i^{\text{in}}$. As assumed, there exists a partition (A_i, B_i) of $V(D_i)$ such that $\lambda(D[A_i]) \le q_i$ and $\lambda(D[B_i]) \le \lambda(D) - q_i$. Define

$$A'' = \left(\bigcup_{i \in S} V(D_i) \right) \cup \left(\bigcup_{i \in J} A_i \right) \quad \text{and} \quad B'' = \left(\bigcup_{i \in R} V(D_i) \right) \cup \left(\bigcup_{i \in J} B_i \right).$$

Then (A'', B'') is a partition of $D - D'$.

Suppose that (A, B) is a partition of $V(D)$ with $A'' \subseteq A$ and $B'' \subseteq B$. Let α be a longest path in $D[A]$ with $\text{ter}(\alpha) \in V(D_r)$ and β be a longest path in $D[B]$ with $\text{ini}(\beta) \in V(D_j)$ where $r, j \in [n]$. Note that if $r \in S$, then $|\alpha| \le p_r^{\text{end}} \le q$. If $r \in J$, then $|\alpha| \le p_r^{\text{in}} + \lambda(D[A_r]) \le p_r^{\text{in}} + q_r = q$ by the choice of A_r. If $j \in R$, then $\lambda(D) \ge p_j^{\text{in}} + |\beta| \ge q + |\beta|$. If $j \in J$, then $\lambda(D) \ge p_j^{\text{in}} + \lambda(D_j) + |Q| \ge p_j^{\text{in}} + \lambda(D[B_j]) + q - p_j^{\text{in}} + |Q| \ge |\beta| + q$ by the choice of B_j. Therefore, it suffices to construct a partition (A', B') of D' such that each of the following holds.

(1) $A = A' \cup A''$ and $B = B' \cup B''$.

(2) Let α be a longest path in $D[A]$. Then $\text{ter}(\alpha) \in A'$ or $A' = \emptyset$.

(3) Let β be a longest path in $D[B]$. Then $\text{ini}(\beta) \in B'$ or $B' = \emptyset$.

(4) $|\alpha| \le q$ and $|\beta| \le \lambda(D) - q$.

Now, let $\alpha = P\alpha'$, where P is the subpath of α contained in $D - D'$ and α' is the subpath of α contained in D'. Similarly, let $\beta = \beta' Q$, where Q is the subpath of β contained in $D - D'$ and β' is the subpath of β contained in D'. Let

$$U = \left\{ i \in [t] : p_i^{\text{in}} < q \right\}, \quad W = \left\{ i \in [t] : p_i^{\text{in}} \ge q \right\}.$$

We construct a desired partition (A', B') of $V(D')$ and finish the proof according to the following cases.

Case 1: $U = \emptyset$ i.e., $W = [t]$.

Let $A' = \emptyset$, $B' = V(D')$. Then, it suffices to consider that $\text{ini}(\beta) \in B'$. By the definition of W, we obtain that $\lambda(D) \ge p_i^{\text{in}} + |\beta| \ge q + |\beta|$.

Case 2: $U \ne \emptyset$ and $W \ne \emptyset$.

Suppose that $p_i^{\text{in}} + \lambda(D_i) \le q$ for all $i \in U$. Let $\ell \in [t]$ such that $\ell \in U$ and $\ell + 1 \in W$. Let

$$A' = V(D_\ell), \quad B' = V(D' - D_\ell).$$

Note that $|\alpha| \le p_\ell^{\text{in}} + \lambda(D_\ell) \le q$ by the assumption that $\text{ter}(\alpha) \in A' = V(D_\ell)$. Now, we claim that $\text{ini}(\beta) \in V(D_{\ell+1})$. Suppose for a contradiction that $V(\beta' \cap D_{\ell+1}) = \emptyset$. Then we can append a path starting from $D_{\ell+1}$ to β, contradicting the assumption that β is a longest path in $D[B]$. Since $p_{\ell+1}^{\text{in}} \ge q$, we have that $\lambda(D) \ge p_{\ell+1}^{\text{in}} + |\beta| \ge q + |\beta|$.

Therefore, assume that there exists an $\ell \in U$ such that $p_\ell^{\text{in}} + \lambda(D_i) > q$. Let $q_\ell = q - p_\ell^{\text{in}}$. Then define

$$A' = V(M_{q_\ell}(D_\ell)), \quad B' = V(D' - M_{q_\ell}(D_\ell)).$$

Since $\lambda(M_{q_\ell}(D_\ell)) = q_\ell$ by the definition of $M_{q_\ell}(D_\ell)$, we obtain that $|\alpha| \le p_\ell^{\text{in}} + \lambda(M_{q_\ell}(D_\ell)) = p_\ell^{\text{in}} + q_\ell = q$. Let P_ℓ be a path in $M_{q_\ell}(D_\ell)$ with order q_ℓ. Note that $|D_\ell - V(M_{q_\ell}(D_\ell))| < |D_i|$ for all $i \in [t]$, by Lemma 3.1, there is a path P^* in D' with $|P^*| \ge |P_\ell| + |\beta'|$ such that $\text{ini}(P^*) = \text{ini}(P_\ell)$ and $\text{ter}(P^*)$ is contained in the same subdigraph of D with β'. Thus, $\lambda(D) \ge p_\ell^{\text{in}} + |P^*| + |Q| \ge p_\ell^{\text{in}} + \lambda(M_{q_\ell}(D_\ell)) + |\beta'| + |Q| = p_\ell^{\text{in}} + \lambda(M_{q_\ell}(D_\ell)) + |\beta| = q + |\beta|$.

Case 3: $W = \emptyset$ i.e., $U = [t]$.

Let $m \in U$ such that $p_m^{\text{in}} = \max_{i \in U}\{p_i^{\text{in}}\}$. Without loss of generality, assume that $m = 1$.
Suppose that $p_1^{\text{in}} + \lambda(D') \le q$, let

$$A' = V(D'), \quad B' = \emptyset.$$

Then it suffices to show that $|\alpha| \le q$. By the maximum of p^{in}, $|\alpha| = |P| + |\alpha'| \le p_1^{\text{in}} + \lambda(D') \le q$. Therefore, in the following, we may assume that $p_1^{\text{in}} + \lambda(D') > q$.

If there exists an $\ell \in U$ such that $p_\ell^{\text{in}} + \lambda(D_\ell) \ge q$, let $q_\ell = q - p_\ell^{\text{in}}$. Then define

$$A' = V(M_{q_\ell}(D_\ell)), \quad B' = V(D' - M_{q_\ell}(D_\ell)).$$

By a similar argument to that in the final subcase of Case 2, we have that $\lambda(D[A]) \le q$ and $\lambda(D[B]) \le \lambda(D) - q$. Hence, we may assume that $p_i^{\text{in}} + \lambda(D_i) < q$ for all $i \in U$.

If $p_1^{\text{in}} + \sum_{i=1}^{t-1} \lambda(D_i) \ge q$, let $\ell \in U$ be the minimum integer such that $\lambda(D_\ell) \ge q - (p_1^{\text{in}} + \sum_{i=1}^{\ell-1} \lambda(D_\ell))$. Let $q_\ell = q - (p_1^{\text{in}} + \sum_{i=1}^{\ell-1} \lambda(D_i))$. Then define

$$A' = \left(\bigcup_{i=1}^{\ell-1} V(D_i)\right) \cup V(M_{q_\ell}(D_\ell)), \quad B' = V(D') \setminus A'.$$

By the definition of $M_{q_\ell}(D_\ell)$, we have that $|\alpha| \le p_1^{\text{in}} + \sum_{i=1}^{\ell-1} \lambda(D_i) + \lambda(M_{q_\ell}(D_\ell)) = q$. Observe that $\text{ini}(\beta) \in V(D_\ell)$ or $\text{ini}(\beta) \in V(D_{\ell+1})$ depending on whether B contains a vertex in D_ℓ. Recall that $|D_1| = |D_2| = \cdots = |D_t| = k$. If $k = 1$, then $\text{ini}(\beta') \in D_{\ell+1}$ and we can append α' to β'. If $k \ge 2$, then by Lemma 3.2, there is a path P^* with $|P^*| \ge |\alpha'| + |\beta'|$ such that $\text{ini}(P^*) \in D_1$ and $\text{ter}(P^*)$ is contained in the same subdigraph of D with $\text{ter}(\beta')$. Then PP^*Q is a path in D. Hence, in both cases $\lambda(D) \ge p_1^{\text{in}} + \sum_{i=1}^{\ell-1} \lambda(D_i) + \lambda(M_{q_\ell}(D_\ell)) + |\beta| = q + |\beta|$.

Therefore, it suffices to consider that $p_1^{\text{in}} + \sum_{i=1}^{t-1} \lambda(D_i) < q$. Let $W \subsetneq V(D_t)$ with $|W| = s < k$. For each $i \in [t-1]$, let $P_i^1, \ldots, P_i^{s+1}$ be vertex disjoint paths contained in D_i with $|\bigcup_{d \in [s+1]} V(P_i^d)| = \lambda_{s+1}(D_i)$. Then by the definition of D', we have that $D'[W \cup (\bigcup_{i \in [t-1]} V(P_i^1 \cup \cdots \cup P_i^{s+1}))]$ contains a Hamiltonian path P_W with $\text{ini}(P_W) \in V(D_1)$ and $\text{ter}(P_W) \in V(D_{t-1})$.

Hence, if there is a $W \subseteq V(D_t)$ with $|W| = s < k$ such that P_W is of order $q - p_1^{\text{in}}$, then let

$$A' = \left(\bigcup_{i \in [t-1]} V(D_i) \right) \cup W, \quad B' = V(D') \setminus A'.$$

Note that $|\alpha| \leq p_1^{\text{in}} + \sum_{i=1}^{t-1} \lambda_{s+1}(D_i) + s = p_1^{\text{in}} + |P| = q$. Since $\text{ter}(P) \in V(D_{t-1})$ and $V(B') \cap V(D') \subseteq V(D_t)$, the path P_W is appendable to β and P is appendable to P_W. Hence, $\lambda(D) \geq |P| + |P_W| + |\beta| = q + |\beta|$. Therefore, suppose that there is no such W.

Recall that $p_1^{\text{in}} + \lambda(D') > q$ and D' has a Hamiltonian path. By the above arguments, there is an integer s with $s \leq k - 2$ such that

- for each $W_1 \subseteq V(D_t)$ with $|W_1| = s$, we have that $|P_{W_1}| < q - p_1^{\text{in}}$, and
- for each $W_2 \subseteq V(D_t)$ with $|W_2| = s + 1$, we have that $|P_{W_2}| > q - p_1^{\text{in}}$.

Let $S(D_t) \subseteq V(D_t)$ with $|S(D_t)| = s$. Then define

$$A' = \left(\bigcup_{i=1}^{t-1} V(D_i) \right) \cup S(D_t), \quad B' = V(D_t) - S(D_t).$$

Note that $|\alpha| \leq p_1^{\text{in}} + \sum_{i=1}^{t-1} \lambda_{s+1}(D_i) + s = p_1^{\text{in}} + |P_{S(D_t)}| \leq q$. Now, let $r = k - s$, we have that $|\beta| = |\beta'| + |Q| \leq r + |Q|$. Since

$$\begin{aligned}
\lambda(D) &\geq p_1^{\text{in}} + \lambda(D') + |Q| \\
&\geq p_1^{\text{in}} + \sum_{i=1}^{t-1} \lambda_{s+2}(D_i) + k + |Q| \\
&= p_1^{\text{in}} + \sum_{i=1}^{t-1} \lambda_{s+2}(D_i) + s + 1 - 1 + r + |Q|,
\end{aligned}$$

by the choice of s, we have that

$$p_1^{\text{in}} + \sum_{i=1}^{t-1} \lambda_{s+2}(D_i) + s + 1 \geq q + 1.$$

Thus, $\lambda(D) \geq q + 1 - 1 + r + |Q| = q + r + |Q|$. Hence, $|\beta| = |\beta'| + |Q| \leq r + |Q| \leq \lambda(D) - q$.

Therefore, we finish the proof. □

Theorem 3.4. *Let $C = v_1 v_2 \dots v_t v_1$ be a cycle and $D = C[D_1, D_2, \dots D_t]$, where each D_i is an arbitrary digraph. Then D satisfies the DPPC.*

Proof. Let q be a positive integer with $q < \lambda$. Without loss of generality, we may assume that $|D_1| = \min_{i \in [t]} |D_i|$. By symmetry, it is sufficient to consider the case that $q \leq \frac{\lambda(D)}{2}$. Denote $\lambda(D_i)$ by λ_i for which $i \in [t]$. We distinguish between two cases depending on q and $\lambda_1, \dots, \lambda_t$.

Case 1: $q \geq \lambda_1$.

1. When $q \geq \sum_{i=1}^{t} \lambda_i$:

   Let $A = V(D_1)$ and $B = V(D) \setminus A$. Note that

$$\lambda(D[A]) = \lambda_1 \leq \sum_{i=1}^{t} \lambda_i \leq q,$$

$$\lambda(D[B]) = \sum_{i=2}^{t} \lambda_i \leq q \leq \lambda(D) - q.$$

2. When $q < \sum_{i=1}^{t} \lambda_i$:

   Let s be the smallest index such that $\sum_{i=1}^{s} \lambda_i \geq q$.

   Suppose that $s \leq t-1$. Then there is a path P of order q in $D[\bigcup_{i \in [s]} V(D_i)]$. Let $A = (\bigcup_{i=1}^{s-1} V(D_i)) \cup V(P)$ and $B = V(D) \setminus A$. Then $\lambda(D[A]) = q$. Next, we show that $\lambda(D[B]) \leq \lambda(D) - q$. Let α be a longest path in $D[A]$ and β be a longest path in $D[B]$. Note that $\text{ini}(\alpha) \in V(D_1)$ and $\text{ter}(\beta) \in V(D_t)$. Since $(v_t, v_1) \in A(C)$, we could append β to α. Hence, $|\beta| + |\alpha| = |\beta| + q \leq \lambda(D)$.

   Therefore, we may assume that $s = t$. If $\lambda_t \leq \lambda(D) - q$, then let $A = \bigcup_{i=1}^{t-1} V(D_i)$ and $B = V(D) \setminus A$. We have that $\lambda(D[A]) = \sum_{i=1}^{t-1} \lambda_i \leq q$ and $\lambda(D[B]) = \lambda_t \leq \lambda(D) - q$. If $\lambda_t > \lambda(D) - q \geq q$, then we can find a path P of order $q - \sum_{i=1}^{t-2} \lambda_i$ in D_t. Let $A = (\bigcup_{i=1}^{t-2} V(D_i)) \cup V(P)$ and $B = V(D) \setminus A$. Then $\lambda(D[A]) = q$. Similar to the case of $s \leq t-1$, by appending a longest path β in $D[B]$ to a longest path α in $D[A]$, we have that $\lambda(D[B]) \leq \lambda(D) - q$.

Case 2: $q < \lambda_1$.

Let P be a path of order q in D_1. Then let $A = V(P)$ and $B = V(D) \setminus A$. Note that $\lambda(D[A]) = |P| = q$. Let β be a longest path in $D[B]$. Since $|D_1| = \min_{i \in [t]} |D_i|$, we have that $|D_1 - P| < |D_i|$ for all $i \in [t]$. Then by Lemma 3.1, there is a path P^* in D of order at least $|P| + |\beta|$ which implies that $q + |\beta| = |P| + |\beta| \leq |P^*| \leq \lambda(D)$. □

4 Acknowledgements

Dedicated to Professor Fuji Zhang on the occasion of his 88th birthday. Zhipeng Gao is supported by the Fundamental Research Funds for the Central Universities(No. XJSJ24049).

References

[1] J. Ai, S. Gerke, G. Gutin, and Y. Zhou. Extended path partition conjecture for semicomplete and acyclic compositions. *Discrete Mathematics*, 345(11):113019, 2022.

[2] A. Arroyo and H. Galeana-Sánchez. The path partition conjecture is true for some generalizations of tournaments. *Discrete Mathematics*, 313(3):293–300, 2013.

[3] J. Bang-Jensen, M. Nielsen, and A. Yeo. Longest path partitions in generalizations of tournaments. *Discrete Mathematics*, 306(16):1830–1839, 2006.

[4] M. Frick. A survey of the path partition conjecture. *Discussiones Mathematicae Graph Theory*, 33(1):117–131, 2013.

[5] M. Frick, S. van Aardt, G. Dlamini, J. Dunbar, and O. Oellermann. The directed path partition conjecture. *Discussiones Mathematicae Graph Theory*, 25(3):331–343, 2005.

[6] H. Galeana-Sánchez, I. Goldfeder, and I. Urrutia. On the structure of strong 3-quasi-transitive digraphs. *Discrete Mathematics*, 310(19):2495–2498, 2010.

[7] C. Hernández-Cruz. 4-transitive digraphs I: The structure of strong 4-transitive digraphs. *Discussiones Mathematicae Graph Theory*, 33(2):247–260, 2013.

[8] J.M. Laborde, C. Payan, and N.H. Xuong. Independent sets and longest directed paths in digraphs. *Graphs and other combinatorial topics (Prague, 1982)*, 59:173–177, 1983.

[9] S. Wang and R. Wang. Independent sets and non-augmentable paths in arc-locally in-semicomplete digraphs and quasi-arc-transitive digraphs. *Discrete Mathematics*, 311(4):282–288, 2011.