A new perspective from hypertournaments to tournaments

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Abstract: A k-hypertournament H on n vertices is a pair (V, A) for $2 \leq k \leq n$, where $V(H)$ is a set of vertices, and $A(H)$ is a set of all possible k-tuples of vertices, such that for any k-subset S of V, $A(H)$ contains exactly one of the $k!$ possible permutations of S. In this paper, we investigate the relationship between a hyperdigraph and its corresponding normal digraph. Particularly, drawing on a result from Gutin and Yeo, we establish an intrinsic relationship between a strong k-hypertournament and a strong tournament, which enables us to provide an alternative (more straightforward and concise) proof for some previously known results and get some new results.

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1 Introduction

A hyperdigraph H is a pair $(V(H), A(H))$, where $V(H)$ is a set of vertices, and $A(H)$ is a set of ordered tuples of vertices in $V(H)$, called hyperarcs of H, such that no hyperarc contains any vertex twice. A k-hyperdigraph H on n vertices is a hyperdigraph on n vertices such that each hyperarc of H is a k-tuple of vertices, for $2 \leq k \leq n$. For

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a hyperarc $a = (x_1x_2 \ldots x_k)$, we say x_i precedes x_j if $i < j$, and write as x_iax_j. A 2-hyperdigraph is merely a digraph. A path P of length k in H is an alternating sequence $P = x_1a_1x_2a_2 \ldots x_ka_kx_{k+1}$ of distinct vertices x_i and distinct hyperarcs a_j such that $x_ia_ix_{i+1}$ for every $1 \leq i \leq k$. We call P a cycle of length k if $x_{k+1} = x_1$. A path (cycle) is Hamiltonian if it contains all vertices of H. A k-hyperdigraph H is strong if there is a path from u to v for each ordered pair (u, v), where u, v are distinct vertices in H. A vertex (a hyperarc) of H is pancyclic, if it is contained in an l-cycle for all $l \in \{3, \ldots, n\}$. A k-hyperdigraph H is vertex (hyperarc)-pancyclic if all of its vertices (hyperarcs) are pancyclic.

Tournaments are the widely studied class in digraphs. We call H a k-hypertournament if H is a k-hyperdigraph and each k-subset of $V(H)$ has exactly one permutation that belongs to $A(H)$. A tournament is a 2-hypertournament.

Given a digraph D, a hyperdigraph H is a *Berge-D* if there exists a bijection $\phi : A(D) \to A(H)$ such that for each arc $e \in A(D)$ we have $V(e) \subset V(\phi(e))$ and v_i precedes v_j in $\phi(e)$. Clearly, a path or a cycle in the hyperdigraph H is a Berge directed path or Berge directed cycle.

Let H be a k-hypertournament with n vertices where $3 \leq k \leq n-2$ and T a tournament with $V(T) = V(H)$. For a hyperarc a of $A(H)$ and an arc $e = (v_i, v_j)$ of $A(T)$, we call e is generated by a if $v_i, v_j \in V(a)$ and v_i precedes v_j. Moreover, we say a tournament T is generated by a k-hypertournament H if $V(T) = V(H)$ and each arc of T can be generated by a hyperarc of H such that any two arcs of T can be generated by different hyperarcs of H. We denote by $\mathcal{T}_H$ the set of all tournaments generated by H.

In [7], Gutin and Yeo proved Theorem 1.1, an extension of Redei's theorem and Camion's theorem to hypertournaments, which are the most basic results on tournaments. They showed every tournament contains a Hamiltonian path, and every strong tournament has a Hamiltonian cycle. Moreover, they proposed the following interesting question: Is a strong k-hypertournament pancyclic or vertex-pancyclic?

Theorem 1.1. *[7] For $k \geq 3$, every k-hypertournament on $n \geq k+1$ vertices has a Hamiltonian path, and every strong k-hypertournament on $n \geq k+2$ vertices contains a Hamiltonian cycle.*

Petrovic and Thomassen [13] proved that a k-hypertournament H on n vertices is vertex-pancyclic if and only if H is strong for sufficiently large n, which is an extension of Moon's theorem [12].

Yang [16] improved their result as follows.

Theorem 1.2. *[16] Let H be a k-hypertournament on n vertices. When* *(i)* $k = 3$ *and* $n \geq 15$*,*

(ii) $k = 4$ *and* $n \geq 11$*,*

(iii) $k \geq 5$ *and* $n \geq k + 4$*, or*

(iv) $k \geq 8$ *and* $n \geq k + 3$*,*

H is vertex-pancyclic if and only if H is strong.

In 2013, Li, Li, Guo, and Surmacs [10] solved this problem completely.

Theorem 1.3. *[10] When $k \geq 3$ and $n \geq k + 2$, an n-vertex k-hypertournament H is vertex-pancyclic if and only if H is strong.*

Besides Hamiltonian property and vertex-pancyclicity, there are some other properties of hypertournaments being studied in a large number of papers, see [1, 4, 8, 11, 14].

Petrovic, Thomassen [13], and Yang [16] constructed a certain strong semicomplete digraph D_H from a given strong k-hypertournament H to prove that H is vertex-pancyclic. In other words, they gave another proof of Theorem 1.1 for some conditions of k, n. With the help of Theorem 1.1 [7], we find a deeper relationship between a strong k-hypertournament and a corresponding strong tournament, which can imply some known results immediately.

Theorem 1.4. *For $n \geq 11$ and $3 \leq k \leq n - 2$, there is a strong tournament $T \in \mathcal{T}_H$ where H is a strong k-hypertournament on n vertices. Moreover, when $3 \leq k \leq n - 3$, the result holds for $n \geq 7$.*

The outline of the rest of the paper is as follows. We give some definitions and lemmas which will be used in the following proofs. In Section 3, we prove Theorem 1.4 and give a more straightforward proof for some previously known properties about strong k-hypertournaments. We devote Section 4 to investigating the connection between k-hyperdigraphs and their corresponding digraphs, and extend some results to k-hyperdigraphs.

2 Preliminary

The terminology not introduced in this paper can be found in [2]. For $k \geq 3$ and $n \geq k + 2$, let H be a strong k-hypertournament on n vertices. Note that H contains a Hamiltonian cycle C by Theorem 1.1, which implies that the hyperarcs of C can generate a Hamiltonian cycle of a strong tournament T. If other arcs of T can be generated by the remaining hyperarcs of H, then $T \in \mathcal{T}_H$. To achieve that, we need the following lemmas.

Lemma 2.1. *Let G be a bipartite graph with partite sets U and W, and p a positive integer. If $d(u) \geq p$ for each $u \in U$ and $d(w) \leq p$ for each $w \in W$, then G has a matching covering U.*

Proof. Let S be a subset of U and let E be the set of edges of G between S and $N(S)$. Since $d(u) \geq p$ for each $u \in U$ and $d(w) \leq p$ for each $w \in W$, we have

$$p\,|N(S)| \geq |E| \geq p\,|S|\,.$$

Since $p \geq 1$, it follows that $|N(S)| \geq |S|$. By Hall's theorem [9], there is a matching of G covering U. □

Lemma 2.2. *Let H be a 3-hypertournament on n vertices and C a Hamiltonian cycle of H. Then every pair of nonconsecutive vertices in C can be contained in at most four hyperarcs of C. More precisely, when $n = 8$ there are at most two nonadjacent pairs, each pair being contained in four hyperarcs of C. When $n = 7$, there are no two nonadjacent pairs, each pair being contained in four hyperarcs of C, and at most two nonadjacent pairs, each pair being contained in at least three hyperarcs of C. If there is one nonadjacent pair contained in four hyperarcs of C and one in three hyperarcs, then the other pairs are contained in at most one hyperarc.*

Proof. Let $C = v_1a_1v_2\dots a_{n-1}v_na_nv_1$ and $i, j \in [n]$ where $1 < j - i < n - 1$. Since $k = 3$ and i, j are not consecutive, the hyperarcs of C containing both v_i and v_j must contain exactly one vertex of $\{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\}$ (mod n). So the number of hyperarcs that contain v_i and v_j is at most four. If the number is exactly four, we have $2 < j - i < n - 2$.

When $n = 8$, assume the nonadjacent pair (v_i, v_j) is contained in four hyperarcs of C. By the above discussions, the four hyperarcs are $\{v_{i-1}, v_i, v_j\}$, $\{v_i, v_{i+1}, v_j\}$, $\{v_i, v_{j-1}, v_j\}$, $\{v_i, v_j, v_{j+1}\}$ (mod n). If there is another pair contained in four hyperarcs, it must be (v_{i+2}, v_{j+2}) (mod n) by $2 < j - i < n - 2$. For these two pairs, note that the corresponding hyperarc sets are disjoint. Since when $n = 8$, C contains eight hyperarcs, there are at most two nonadjacent pairs contained in four hyperarcs of C, and when $n = 7$, there are no two nonadjacent pairs contained in four hyperarcs of C.

When $n = 7$, C contains seven hyperarcs. For any two nonadjacent pairs contained in at least three hyperarcs of C, if their corresponding hyperarc sets are disjoint, then there are at most two nonadjacent pairs contained in at least three hyperarcs of C. If not, then there is at most one hyperarc containing them, and two pairs have exactly one common vertex. Assume the pairs are (v_i, v_j) and (v_i, v_ℓ), and there is a hyperarc $\{v_i, v_j, v_\ell\}$ in C containing them. Since v_i is not adjacent to v_j, v_l and $\{v_i, v_j, v_\ell\} \in E(C)$, without loss of generality, assume $\ell = j-1$ (module n). Some hyperarcs in C are $\{v_i, v_\ell, v_j\}$, $\{v_i, v_j, v_{j+1}\}$, $\{v_{i-1}, v_i, v_j\}$, $\{v_i, v_\ell, v_{i+1}\}$, $\{v_i, v_{\ell-1}, v_\ell\}$, where $\ell = j-1$. Suppose that there is another nonadjacent pair contained in at least three hyperarcs of C, so the pair is contained in some hyperarc which contains (v_i, v_j) or (v_i, v_ℓ). Since (v_i, v_j) and (v_i, v_ℓ) occur three times, the pair can not contain v_i. Then the pair must have one vertex of $\{v_j, v_\ell\}$, and the other vertex is adjacent to v_i. Without loss of generality, assume the pair is (v_{i-1}, v_j). All possible hyperarcs in C which can contain (v_{i-1}, v_j) are $\{v_{i-2}, v_{i-1}, v_j\}$, $\{v_{i-1}, v_i, v_j\}$, $\{v_{i-1}, v_{j-1}, v_j\}$, $\{v_{i-1}, v_j, v_{j+1}\}$. Since $\{v_i, v_{j-1}, v_j\}$, $\{v_i, v_j, v_{j+1}\} \in E(C)$, there are no three hyperarcs in C containing (v_{i-1}, v_j), a contradiction. Hence there are at most two nonadjacent pairs contained in at least three hyperarcs of C.

By symmetry, we can assume the nonadjacent pair (v_1, v_5) contained in four hyperarcs of C. If there is a nonadjacent pair contained in three hyperarcs of C, then the pair has a vertex that is not adjacent to v_1, v_5, which is v_3. The pair is (v_3, v_6) or (v_3, v_7). By symmetry, assume it is (v_3, v_7), then the hyperarcs are $\{v_7, v_1, v_5\}$, $\{v_1, v_2, v_5\}$, $\{v_1, v_4, v_5\}$, $\{v_1, v_5, v_6\}$, $\{v_3, v_6, v_7\}$, $\{v_2, v_3, v_7\}$, $\{v_3, v_4, v_7\}$, and observe that other pairs occur at most once. Otherwise, all pairs occur at most twice except (v_1, v_5). □

Lemma 2.3. *Let H be a 4-hypertournament on 7 vertices and C a Hamiltonian cycle of H. Then any two different nonadjacent pairs can be contained in at most four same hyperarcs of C.*

Proof. Let (v_i, v_j) and (v_k, v_ℓ) be two nonadjacent pairs. If v_i, v_j, v_k, v_ℓ are four distinct vertices, then there is at most one hyperarc containing all of them. Otherwise $|\{v_i, v_j, v_k, v_\ell\}| = 3$, then there are at most four hyperarcs containing $\{v_i, v_j, v_k, v_\ell\}$ as $n = 7$. □

We give the following lemma based on Lemma 8 in Yang [16].

Lemma 2.4. *If*

(i) $k = 3$ *and* $n \geq 9$,

(ii) $k = 4$ *and* $n \geq 8$,

(iii) $k \geq 5$ *and* $n \geq k + 3$,

then

$$\binom{k}{2} \leq \binom{n-2}{k-2} - 4 \quad \text{for } k = 3$$

and

$$\binom{k}{2} \leq \binom{n-2}{k-2} - n \quad \text{for } k \geq 4 \tag{1}$$

Proof. Since $\binom{k}{2} = \binom{3}{2} = 3 \leq \binom{n-2}{k-2} - 4 = n - 6$ when $k = 3$ and $n \geq 9$, we only need to check the case $k \geq 4$. If $n = k + 3$, rewrite the inequality (1) as

$$k^2 + k + 6 \leq 2\binom{k+1}{k-2} = \frac{1}{3}(k^3 - k).$$

Then $k^3 - 3k^2 - 4k - 18 \geq 0$ holds for $k \geq 5$. For $k = 4$, (1) holds with $n = 8$. Since increasing n by 1 increases the right-hand side $\binom{n-2}{k-2} - n$ by $\binom{n-1}{k-2} - \binom{n-2}{k-2} - 1 = \binom{n-2}{k-3} - 1$ which is non-negative for $k \geq 4, n \geq k+1$, and the left-hand side $\binom{k}{2}$ is independent of n, by applying an induction on n, the inequality (1) holds for $k \geq 5, n \geq k+3$ and $k = 4, n \geq 8$. □

Lemma 2.5. *Let H be a k-hypertournament on n vertices where $k = n-2$. Assume that S is a subset of $A(H)$ such that the union of unordered pairs (of vertices) in the vertex set of each hyperarc in S forms a proper subset of the set of unordered pairs in $V(H)$. Then $|S| \leq 3n - 6$.*

Proof. We may assume that $|S| \geq 1$. Set $a_0 \in S$ and $v_{n-1}, v_n \notin V(a_0)$. Then each of the other hyperarcs a in S must belong to one of the following kinds of hyperarcs:

1. $V(a) = (V(a_0) \cup \{v_{n-1}\}) \setminus \{v_t\}$, $t \in [n-2]$;
2. $V(a) = (V(a_0) \cup \{v_n\}) \setminus \{v_t\}$, $t \in [n-2]$;
3. $V(a) = (V(a_0) \cup \{v_{n-1}, v_n\}) \setminus \{v_{t_1}, v_{t_2}\} = V(D) \setminus \{v_{t_1}, v_{t_2}\}$, $t_1, t_2 \in [n-2]$ and $t_1 \neq t_2$.

If there is at least one hyperarc in S which is of the form 3, without loss of generality, we assume $V(a_1) = V(D) \setminus \{v_{n-2}, v_{n-3}\}$ and $a_1 \in S$. Then the set of unordered pairs in $V(a_0)$ and in $V(a_1)$ will contain all unordered pairs in $V(H)$, except $\{v_{n-2}, v_{n-1}\}, \{v_{n-3}, v_{n-1}\}, \{v_{n-2}, v_n\}, \{v_{n-3}, v_n\}$. In this case, the set of hyperarcs which are both in S and of the form 3 must be the subset of one of the following hyperarc sets:

(i) $\{a_1\}$;

(ii) $\{a_1\} \cup \{a \in A(D) \mid V(a) = V(D) \setminus \{v_{n-2}, v_{t_3}\}, t_3 \in [n-4]\}$;

(iii) $\{a_1\} \cup \{a \in A(D) \mid V(a) = V(D) \setminus \{v_{n-3}, v_{t_3}\}, t_3 \in [n-4]\}$.

For otherwise, there is a hyperarc $a_2 \in S$ such that $V(a_2) = V(D) \setminus \{v_{t_3}, v_{t_4}\}$, where $t_3, t_4 \in [n-4]$ and $t_3 \neq t_4$. Then the set of unordered pairs in $V(a_0)$, in $V(a_1)$, and in $V(a_2)$, will be the set of all unordered pairs in $V(H)$, which is a contradiction. By accounting the number of hyperarcs in the three kinds of hyperarcs, we know $|S| \leq 1 + n - 2 + n - 2 + n - 3 = 3n - 6$.

□

Remark Let H be a k-hypertournament on n vertices where $k = n-2$. By Lemma 2.5, for every subset S of $A(D)$ satisfying $|S| > 3n-6$, the union of unordered pairs (of vertices) in the vertex set of each hyperarc in S forms the set of unordered pairs in $V(H)$. This is the main tool to prove the case when $n \geq 11$ and $k = n-2$ in Theorem 1.4. However, it does not assist in handling the condition $n \leq 10$ and $k = n-2$. Furthermore, when $k = 3$ and $n = 5, 6$, since $\binom{n-2}{k-2} - 4 \leq 0$, there exists a strong 3-hypertournament H such that there is no strong tournament $T \in \mathcal{T}_H$.

3 Strong k-hypertournament

With the lemmas proved in the previous section, we are ready to prove Theorem 1.4. Moreover, it gives a more straightforward proof of the vertex-pancyclicity and arc-pancyclicity for strong hypertournaments.

Proof of Theorem 1.4. We divide the proof into two parts. Firstly, let H be a strong k-hypertournament on n vertices where $3 \leq k \leq n-3$ and $n \geq 7$, and let K_n be a complete graph with the same vertex set as H. By Theorem 1.1, we can assume that $C = v_1 b_1 v_2 \dots b_{n-1} v_n b_n v_1$ is a Hamiltonian cycle in H. Consider a bipartite graph G with partite sets $A = E(K_n) \setminus \{v_1v_2, v_2v_3, \dots, v_nv_1\}$ and $B = A(H) \setminus \{b_1, \dots, b_n\}$. For every $a \in A$ and $b \in B$, G has an edge ab if $a \subset \bar{b}$, where $\bar{b}$ denotes the set of vertices of b. We aim to prove that G has a matching covering A. If such a matching exists, then K_n has an orientation $T \in \mathcal{T}_H$ such that T is strong.

Notice that the degree of any vertex in B is at most $\binom{k}{2}$. When $k = 3$, by Lemma 2.2, the degree of any vertex in A is at least $\binom{n-2}{k-2} - 4$; when $k \geq 4$, the degree of a vertex in A is at least $\binom{n-2}{k-2} - n$. By Lemma 2.1 and Lemma 2.4, we have that G has a matching covering A except for three cases when $k = 3, n = 7$ or $n = 8$, and $k = 4, n = 7$. Suppose G has no matching covering A for the three remaining cases. By Hall's theorem, there is a subset $S \subseteq A$ such that $|N_G(S)| \leq |S| - 1$. Let E be the set of edges between S and $N_G(S)$.

Case 1: $k = 3$ and $n = 7$. According to Lemma 2.2, we consider the following three subcases. Recall that in this case, the degree of any vertex in B is at most $\binom{k}{2} = 3$. So, we have that $3|S| - 3 \geq 3|N_G(S)| \geq |E|$.

Subcase 1: There is a vertex in A with degree $\binom{n-2}{k-2} - 4$, a vertex in A with degree $\binom{n-2}{k-2} - 3$ and the other vertices in A with degree at least $\binom{n-2}{k-2} - 1$ in G.

We have

$$|E| \geq \left(\binom{n-2}{k-2} - 4\right) + \left(\binom{n-2}{k-2} - 3\right) + \left(\binom{n-2}{k-2} - 1\right)(|S| - 2) = 4|S| - 5.$$

When $|S| \geq 3$, we have $|E| \geq 3|S| - 2$, a contradiction. When $|S| = 1$ or 2, considering the degree of vertices in A, we have $|N(S)| \geq |S|$, a contradiction.

Subcase 2: There is a vertex in A with degree $\binom{n-2}{k-2} - 4$ and the other vertices in A with degree at least $\binom{n-2}{k-2} - 2$ in G.

We have

$$|E| \geq \left(\binom{n-2}{k-2} - 4\right) + \left(\binom{n-2}{k-2} - 2\right)(|S| - 1) = 3|S| - 2.$$

It contradicts to the fact that $3|S| - 3 \geq |E|$.

Subcase 3: There are at most two vertices in A with degree $\binom{n-2}{k-2} - 3$ and the other vertices in A with degree at least $\binom{n-2}{k-2} - 2$ in G.

We have

$$|E| \geq 2 \cdot \left(\binom{n-2}{k-2} - 3\right) + \left(\binom{n-2}{k-2} - 2\right)(|S| - 2) = 3|S| - 2.$$

Similarly, it is a contradiction.

Case 2: $k = 3$ and $n = 8$. By Lemma 2.2, there are at most two vertices in A with degree $\binom{n-2}{k-2} - 4$ and the other vertices in A with degree at least $\binom{n-2}{k-2} - 3$ in G. Therefore,

$$|E| \geq 2 \cdot \left(\binom{n-2}{k-2} - 4\right) + \left(\binom{n-2}{k-2} - 3\right)(|S| - 2) = 3|S| - 2.$$

On the other hand, we have

$$|E| \leq \binom{k}{2}|N_G(S)| = 3|N_G(S)| \leq 3|S| - 3.$$

Hence, $3|S| - 2 \leq |E| \leq 3|S| - 3$, a contradiction.

Case 3: $k = 4$ and $n = 7$. Since every four-tuple of $\{v_1, \ldots, v_7\}$ will contain a pair of consecutive vertices, the degree of any vertex in B is at most $\binom{k}{2} - 1$ in G. And by Lemma 2.3, either there are a vertex in A with degree $\binom{n-2}{k-2} - 7$ and the other vertices in A with degree at least $\binom{n-2}{k-2} - 4$ in G, or there are at most one vertex in A with degree $\binom{n-2}{k-2} - 6$ and the other vertices in A with degree at least $\binom{n-2}{k-2} - 5$. Therefore,

$$|E| \geq \left(\binom{n-2}{k-2} - 7\right) + \left(\binom{n-2}{k-2} - 4\right)(|S| - 1) = 6|S| - 3.$$

or

$$|E| \geq \left(\binom{n-2}{k-2} - 6\right) + \left(\binom{n-2}{k-2} - 5\right)(|S| - 1) = 5|S| - 1.$$

On the other hand,

$$|E| \leq \left(\binom{k}{2} - 1\right)|N_G(S)| = 5|N_G(S)| \leq 5|S| - 5.$$

Hence, $\min\{5|S|-1, 6|S|-3\} \leq |E| \leq 5|S|-5$, a contradiction.

Secondly, let H be a strong k-hypertournament on n vertices where $k = n-2$ and $n \geq 11$. Similarly, let K_n be a complete graph with the same vertex set as H. By Theorem 1.1, we can assume that $C = v_1 b_1 v_2 \ldots b_{n-1} v_n b_n v_1$ is a Hamiltonian cycle in H. Consider a bipartite graph G with partite sets $A = E(K_n) \setminus \{v_1v_2, v_2v_3, \ldots, v_nv_1\}$ and $B = A(H) \setminus \{b_1, \ldots, b_n\}$. For every $a \in A$ and $b \in B$, G has an edge ab if $a \subset \bar{b}$, where $\bar{b}$ denotes the set of vertices of b. It is sufficient to prove that G has a matching covering B. Since $|B| = \binom{n}{n-2} - n = \binom{n}{2} - n = |A|$, the matching is a perfect matching of G, and then K_n has an orientation $T \in \mathcal{T}_H$ such that T is strong.

For every hyperarc $a \in A(H)$, we may assume $V(D) \setminus V(a) = \{v_i, v_j\}$ and $i < j$. Since we removed at least three edges which incident with v_i or v_j in $E(K_n)$, and none of them are contained in $V(a)$, then the degree of a in G is at least $\binom{k}{2} - (n-3)$. So the degree of any vertex in B is at least $\binom{k}{2} - (n-3)$. For every $S \subseteq B$, if $|S| \leq \binom{k}{2} - n + 3$, then $|N(S)| \geq |S|$. If $|S| \geq \binom{k}{2} - n + 3 + 1 = \binom{n-2}{2} - n + 4$, when $n \geq 11$, we have

$$\binom{n-2}{2} - n + 4 - (3n-6) \geq 2.$$

By Lemma 2.5, we know $|N(S)| = |A| = |B| \geq |S|$. Moreover, according to the Hall's Theorem, we know G has a perfect matching.

□

In the following, we use Theorem 1.4 to give some immediate results which have been proved before by different and independent methods [8, 10, 13, 16].

Theorem 3.1. *[8] Let T be a strong tournament and C a Hamiltonian cycle in T. Then C contains at least three pancyclic arcs.*

Theorem 3.2. *If H is a strong k-hypertournament on n vertices for $n \geq 11$ and $3 \leq k \leq n-2$, H has following properties:*

(i) H is vertex-pancyclic.

(ii) If C is a Hamiltonian cycle in H, then C contains at least three pancyclic hyperarcs.

Proof. By Theorem 1.4, there exists a strong tournament $T \in \mathcal{T}_H$. We obtain that T is vertex-pancyclic and any Hamiltonian cycle C in T contains at least three pancyclic arcs by the Moon theorem and Theorem 3.1. The corresponding vertices and hyperarcs in H have the same property, then H is vertex-pancyclic and the corresponding Hamiltonian cycle in H contains at least three pancyclic hyperarcs. □

4 Concluding remarks

Let us show another relationship between hypertournament H and its corresponding tournament in $\mathcal{T}_H$, and generalize some results on tournaments to hypertournaments.

Lemma 4.1. *If H is a k-hypertournament on n vertices with $3 \leq k \leq n-2$, there is a tournament $T \in \mathcal{T}_H$.*

Proof. Consider a bipartite graph G with partite sets $A = E(K_n)$ and $B = A(H)$. For every $a \in A$ and $b \in B$, G has an edge $e = ab$ if $a \subset \bar{b}$, where $\bar{b}$ denotes the set of vertices of b. The degree of b is $\binom{k}{2}$, and the degree of a is $\binom{n-2}{k-2}$. Since $\binom{n-2}{k-2} \geq \binom{k}{2}$, then by Lemma 2.1, G has a matching saturating A. Hence, there is a tournament $T \in \mathcal{T}_H$. $\square$

Theorem 4.1. [3] *There is some constant C such that every $(k + C\ell)$-vertex tournament contains a copy of any k-edge oriented tree with ℓ leaves.*

Theorem 4.2. *For any t-edge oriented tree T with ℓ leaves, there is some constant C such that every k-hypertournament H on $t + C\ell$ vertices with $3 \leq k \leq t + C\ell - 2$ contains a copy of Berge-T.*

Proof. For any t-edge oriented tree T with ℓ leaves, by Theorem 4.1, there is some constant C such that every $(t + C\ell)$-vertex tournament contains a copy of T. By Lemma 4.1, there is a tournament $T_H \in \mathcal{T}_H$ such that $|V(T_H)| = t + C\ell$. Hence T_H contains a copy of T, and then, H contains a copy of Berge-T by the generated relationship. $\square$

The ℓ-th power of the directed path $P_t = x_1x_2 \ldots x_tx_{t+1}$ of length t is the digraph P_t^ℓ on the same vertex set containing the arc x_ix_j if and only if $i < j \leq i + \ell$.

Theorem 4.3. [5] *For $n \geq 2$, every n-vertex tournament contains the ℓ-th power of a directed path of length $n/2^{4\ell+6}\ell$.*

Theorem 4.4. *For $n \geq 2$, every n-vertex k-hypertournament H with $3 \leq k \leq n-2$ contains a copy of Berge-P_t^ℓ where $t = n/2^{4\ell+6}\ell$.*

Proof. Since H is an n-vertex k-hypertournament, there is a tournament $T_H \in \mathcal{T}_H$ such that $|V(T)| = n$ by Lemma 4.1. By the Theorem 4.3, there is an ℓ-th power of a directed path of length $n/2^{4\ell+6}\ell$. Moreover, H contains a copy of Berge-P_t^ℓ where $t = n/2^{4\ell+6}\ell$. $\square$

This perspective on hyperdigraphs allows us to address certain problems effectively. As an illustration, we will establish an extension of the Gallai-Milgram theorem to hyperdigraphs.

The *path covering number* of a hyperdigraph H denoted by $pc(H)$, is the minimum positive integer m such that there are m disjoint paths covering the vertex set of H. An independent set I of H is a set of vertices such that the induced sub-hyperdigraph of I has no hyperarcs. The *independence number* of H, $\alpha(H)$, is the maximum integer m such that H has an independent set of size m.

Theorem 4.5 (Gallai-Milgram Theorem)**.** *[6] For every digraph D, the path covering number is at most its independence number, that is $pc(D) \leq \alpha(D)$.*

Theorem 4.6. *For every hyperdigraph H, $pc(H) \leq \alpha(H)$.*

Proof. Construct a digraph D with $V(D) = V(H)$. Let each hyperarc of H generate an arc of D, and delete the parallel arcs. We call such a digraph D generated by H. For any path P in D, there is a path P' in H such that every arc of P is generated by hyperarcs of P'. Hence, we have $pc(H) \leq pc(D)$. By Theorem 4.5, we know $pc(D) \leq \alpha(D)$. On the other hand, if I is an independent set of D, then it is also an independent set of H. Otherwise, there is a hyperarc e in $H[I]$. By the definition of D, there must be an arc e' in $D[I]$ generated by e which contradicts that I is an independent set of D. Thus,

$$pc(H) \leq pc(D) \leq \alpha(D) \leq \alpha(H).$$

□

As we can see above, it is not hard to extend a property of digraphs to hyperdigraphs, it is natural to ask what else we can do. Based on the main result of this paper we could always find a degenerated strong tournament from a strong k-hypertournament, we could extend many properties of strong tournaments to k-hypertournaments besides what we give in Section 3. For example, it is well-known that there are at least three 2-kings in a strong tournament, and so is a strong k-hypertournament.

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